The Stokes Drift due to Vertically Propagating Internal Gravity Waves in a Compressible Atmosphere

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ABSTRACT

Vertically propagating, compressible, internal gravity waves are shown to have a vertical Stokes drift which is proportional to the vertical wave energy flux. In regions of the atmosphere dominated by upward propagating waves, such as the summer mesosphere, this Stokes drift will be upward. For the Lagrangian mean parcel motion to be small, a downward mean Eulerian velocity must exist to largely oppose the upward Stokes drift. These results may explain the downward mean Eulerian velocity observed at Poker Flat, Alaska in the summer mesosphere.

1. Introduction

Recently, attempts have been made to measure time-averaged vertical velocities in the mesosphere with vertically pointing radars (Balsley and Riddle, 1984). While these results are still somewhat controversial, it seems useful to discuss the interpretation of the results at this time since vertically pointing radars will be used to measure tropospheric, stratospheric, and mesospheric mean winds in the near future.

The vertical velocities measured by Balsley and Riddle (1984) were puzzling because they differed greatly from what is needed for the large-scale radiative–dynamical balance. In the summer mesosphere, Balsley and Riddle measured monthly mean vertical velocities on the order of 10 cm s^{-1} downward, while an upward mean vertical velocity on the order of 1 cm s^{-1} is necessary to maintain the very cold summer mesopause temperatures. While the Balsley and Riddle measurements were only from a single location, they were averaged over a long time period (1 month), so that their vertical velocities should be representative of the zonal mean at 65°N, and therefore a comparison with the zonally averaged radiative–dynamical vertical velocity is justified. The idea that the difference between the two velocities could be due to a Stokes drift was mentioned by Balsley and Riddle and others, but no estimates of the Stokes drift due to summertime mesospheric waves were made. The purpose of this paper is to make such an estimate and examine whether or not the Balsley and Riddle measurements are consistent with this explanation. The question of the accuracy of the measurements was discussed in detail by Balsley and Riddle and will not be considered here.

One difficulty in interpreting time-averaged vertical velocities comes about when waves are present. In this case the time-averaged velocity (i.e., the Eulerian average) is not directly related to the averaged parcel motion (the Lagrangian average), in that the second-order wave corrections (the Stokes drift) must be taken into account. The general definitions of Eulerian and Lagrangian averages were developed by Andrews and McIntyre (1978), and the differences between the Eulerian and Lagrangian circulations due to planetary-scale Rossby and inertial gravity waves have been discussed by a number of authors (McIntyre, 1980; Dunkerton, 1980; Andrews, 1980; Edmon et al., 1980, to name a few). These planetary waves will have a vertical Stokes drift whenever there is a horizontal heat-flux convergence or divergence due to meridional variations in wave amplitudes. In the present paper, a different aspect of Stokes drift, the vertical Stokes drift due to compressibility, will be considered.

It is well known that a steady wave source has an

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associated mean circulation that depends on the boundary conditions and wave dissipation. The simplest example is a two-dimensional plane oscillating sinusoidally that acts as a source of one-dimensional sound waves (Andrews and McIntyre, 1978; see footnote on page 630). In this case, the boundary condition and equation of mass conservation require that the mean Lagrangian circulation is zero. However, the sinusoidal wave solution possesses a Stokes drift away from the source, and this average drift must be balanced by an Eulerian mean velocity toward the source. To the extent that the troposphere acts as the source of mesospheric waves, and that vertically propagating internal gravity waves also have a vertical Stokes drift, an analogous result should hold. That is, a downward mean Eulerian velocity should exist in the mesosphere to balance the upward Stokes drift due to the waves so that the mean vertical Lagrangian motion is zero. Of course, in the mesosphere there is a nonzero Lagrangian motion forced by diabatic heating and horizontal variations in wave amplitudes, but if this Lagrangian motion is small compared to the local vertical Stokes drift in the mesosphere, then the mean Eulerian vertical velocity and the vertical Stokes drift should be approximately equal and opposite.

While all upward propagating waves (acoustic, gravity, and planetary) are shown to contribute to the vertical Stokes drift, only internal gravity waves are considered here due to their observed large amplitudes and significant effects at mesospheric heights. The plan of the paper is as follows. In section 2 a formula is derived for the vertical Stokes drift associated with a plane, slowly varying, internal gravity wave. The Stokes drift for a single adiabatic wave is evaluated in section 3. Section 4 describes the physical mechanism responsible for the vertical Stokes drift. Finally, section 5 examines some effects of dissipation on the Stokes drift.

2. Vertical Stokes drift

In this section a formula for the vertical Stokes drift is derived. This formula differs from expressions commonly used for atmospheric waves in that the effects of compressibility are included. It is shown that compressibility is necessary to obtain the correct internal gravity wave Stokes drift. Also, the derivation given here differs from other derivations by allowing for finite amplitude waves. While most derivations of the Stokes drift make use of a small amplitude assumption, such an assumption may be overly restrictive in some cases. In particular, for slowly varying transverse waves, finite amplitude waves can be considered because the nonlinear advection terms will be small. Considering finite amplitude effects can be important in the case of internal gravity waves, since internal gravity waves in the atmosphere are often observed to be near the "breaking" amplitude, where the potential temperature surfaces overturn, forming a convectively unstable region. At this amplitude the vertical parcel displacements are on the order of the vertical wavelength divided by 2π.

Following Andrews and McIntyre (1978), the vertical Stokes drift is defined as the difference between the Lagrangian and Eulerian averaged vertical velocities,

$$\bar{w}^S \equiv \bar{w}^L - \bar{w}, \tag{1}$$

where the Lagrangian average is defined in terms of the displacement vector $\boldsymbol{\xi} = (\zeta', \eta', \xi')$ as,

$$\bar{w}^L \equiv \overline{w^\xi} \tag{2}$$

with,

$$w^\xi = w(\mathbf{x} + \boldsymbol{\xi}, t). \tag{3}$$

Expanding w^ξ in a Taylor series about $\mathbf{x}$ gives,

$$w^\xi = w + (\boldsymbol{\xi}\cdot\nabla)w + \frac{1}{2}\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla w + \cdots. \tag{4}$$

Averaging (4) and subtracting $\bar{w}$ then gives,

$$\bar{w}^S = \overline{(\boldsymbol{\xi}\cdot\nabla)w} + \frac{1}{2}\overline{\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla w} + \cdots. \tag{5}$$

For small amplitude waves, (5) is usually approximated by retaining only terms of second order in wave amplitude. However, for a slowly varying, transverse wave the series in (5) can be truncated without assuming small amplitude. Consider a solution of the form $w = \bar{w} + w'$, $w' = w_0 e^{i\phi}$, with the wavenumber $\boldsymbol{k} = \nabla\phi$. Then (5) can be written as,

$$\bar{w}^S = \overline{e^{i\phi}\boldsymbol{\xi}\cdot\nabla w_0} + \overline{i(\boldsymbol{\xi}\cdot\boldsymbol{k})w'} + \frac{1}{2}\overline{\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla\bar{w}}$$
$$+ \frac{1}{2}\overline{e^{i\phi}\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla w_0} + \overline{e^{i\phi}i(\boldsymbol{\xi}\cdot\boldsymbol{k})(\boldsymbol{\xi}\cdot\nabla w_0)}$$
$$-\frac{1}{2}\overline{(\boldsymbol{\xi}\cdot\boldsymbol{k})^2 w'} + \cdots, \tag{6}$$

where complex conjugates are assumed to be appropriately taken when forming the averages. Also, for simplicity, the wavenumber $\boldsymbol{k}$ has been taken to be constant, but it could be considered to be slowly varying without affecting the results below.

Equation (6) can now be regarded as an expansion in two small parameters: $\epsilon_1 = |\boldsymbol{\xi}\cdot\boldsymbol{k}|$, and $\epsilon_2 = |\nabla w_0||\boldsymbol{\xi}|/w_0 \approx |\nabla\bar{w}||\boldsymbol{\xi}|/|\bar{w}|$. The first parameter is small if the wave is transverse, while the second parameter is small for slowly varying waves (and mean flows). The lowest order approximation to the vertical Stokes drift (assuming $\epsilon_1 \approx \epsilon_2 \ll 1$) is then,

$$\bar{w}^S = \overline{e^{i\phi}\boldsymbol{\xi}\cdot\nabla w_0} + \overline{i(\boldsymbol{\xi}\cdot\boldsymbol{k})w'} \tag{7}$$

or

$$\bar{w}^S = \overline{\boldsymbol{\xi}\cdot\nabla w'}. \tag{8}$$

Equation (8) is the standard expression for the Stokes drift. The present derivation has emphasized the validity of (8) for a finite amplitude, transverse, slowly varying wave. Thus, (8) should be a good approximation to the Stokes drift for a finite-amplitude internal gravity wave (or Rossby wave), but represents a small amplitude formula for an acoustic wave (or nonslowly varying waves). Obviously, equations analogous to (8) hold for the other velocity components.

While (8) is the basic expression for the vertical Stokes drift, it can be rewritten as

$$\bar{w}^S = \nabla \cdot (\overline{\boldsymbol{\xi} w'}) - \overline{(\nabla \cdot \boldsymbol{\xi}) w'}. \tag{9}$$

Equation (9) expresses the Stokes drift in terms of the divergence of a wave-averaged quantity and the effects of compressibility. The compressibility term, involving $\nabla \cdot \boldsymbol{\xi}$, can be further examined by using the continuity equation.

Starting with the definition of $\boldsymbol{\xi}$ from Andrews and McIntyre (1978),

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{u}}^L \cdot \nabla\right) \boldsymbol{\xi} = \mathbf{u}^{\xi} - \bar{\mathbf{u}}^L, \tag{10}$$

and the continuity equation,

$$\frac{1}{\rho}\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \rho = -\nabla \cdot \mathbf{u}, \tag{11}$$

one can see that it is reasonable to expect a simple relationship between $\rho'/\bar{\rho}$ and $\nabla \cdot \boldsymbol{\xi}$. Assuming $|\rho'/\bar{\rho}| \ll 1$, the perturbation continuity equation can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)(\rho'/\bar{\rho}) - \overline{\mathbf{u}' \cdot \nabla(\rho'/\bar{\rho})} + [(\mathbf{u}' \cdot \nabla\bar{\rho})/\bar{\rho}] = -\nabla \cdot \mathbf{u}'. \tag{12}$$

The assumption that $|\rho'/\bar{\rho}| \ll 1$ is valid for most finite amplitude transverse waves in the atmosphere (including internal gravity waves at the breaking amplitude, provided the vertical wavelength is not too large). The nonlinear advection terms in (12) are small for transverse, slowly varying waves, and also small for small amplitude waves, and therefore may be neglected. While neglecting the nonlinear advection terms is in keeping with the present description of finite amplitude, transverse waves, and small amplitude, longitudinal waves, it should be noted that these terms will probably be important in a local, wave breaking region, where a breakdown of the simple transverse wave structure is expected. Neglecting nonlinear advection, (12) becomes,

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla\right)(\rho'/\bar{\rho}) + [(\mathbf{u}' \cdot \nabla\bar{\rho})/\bar{\rho}] = -\nabla \cdot \mathbf{u}'. \tag{13}$$

It may be tempting at this point to neglect the advection of the mean stratification, since $\bar{\rho}$ is presumably slowly varying. However, retaining this term does not complicate what follows, and turns out to be of some interest.

Going back to (10) and expanding $\mathbf{u}^{\xi} - \bar{\mathbf{u}}^L$ using the two small parameters gives

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{u}}^L \cdot \nabla\right) \boldsymbol{\xi} = \mathbf{u}' \tag{14}$$

to lowest order. Taking the divergence of (14) and neglecting derivatives of the slowly varying $\bar{\mathbf{u}}^L$ field gives

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{u}}^L \cdot \nabla\right) \nabla \cdot \boldsymbol{\xi} = \nabla \cdot \mathbf{u}'. \tag{15}$$

From (7) it is seen that the Stokes drift is of order ϵ_1, ϵ_2. That is, even for finite amplitude transverse waves, the Stokes drift is a small quantity. This allows the Stokes drift to be neglected in advection terms, so that $\bar{\mathbf{u}}^L$ can be replaced by $\bar{\mathbf{u}}$ in (14) and (15). Then, assuming $\nabla\bar{\rho}/\bar{\rho}$ is a constant (or slowly varying) and combining (13), (14), and (15) gives

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla\right)\{(\rho'/\bar{\rho}) + \nabla \cdot \boldsymbol{\xi} + [(\boldsymbol{\xi} \cdot \nabla\bar{\rho})/\bar{\rho}]\} = 0. \tag{16}$$

Equation (16) is just the perturbation equation of mass conservation, written in terms of the displacement vector instead of velocity. Integrating (16) along mean trajectories gives

$$\nabla \cdot \boldsymbol{\xi} = -(\rho'/\bar{\rho}) - [(\boldsymbol{\xi} \cdot \nabla\bar{\rho})/\bar{\rho}] + \text{constant}, \tag{17}$$

where the constant depends on the initial conditions. Substituting (17) into (9) results in

$$\bar{w}^S = (\overline{\rho' w'}/\bar{\rho}) + (1/\bar{\rho})\nabla \cdot (\bar{\rho}\overline{w' \boldsymbol{\xi}}). \tag{18}$$

(Analogous formulas hold for the horizontal components of the Stokes drift.) In (18), the effect of the mean density gradient has been incorporated into the divergence term. The Stokes drift produced by the time change of $\rho'/\bar{\rho}$ in response to $\nabla \cdot \mathbf{u}'$ is given by the first term on the right hand side of (18). Equation (18) shows that unlike the familiar example of water waves, a Stokes drift can be produced without gradients in the wave amplitude if compressibility is allowed. Also, it is important to note that the compressible Stokes drift is generally different from the incompressible Stokes drift (where $\nabla \cdot \boldsymbol{\xi} = 0$), and, that the Stokes drift obtained from the quasi-compressible assumption ($\bar{\rho}\nabla \cdot \boldsymbol{\xi} = -\boldsymbol{\xi} \cdot \nabla\bar{\rho}$) does not correctly approximate the compressible Stokes drift.

For internal gravity waves (18) can be simplified. The second term on the right-hand side of (18) is important for planetary waves, where $\overline{w'\eta'}$ tends to be nonzero [and approximately equal to $\overline{\theta' v'}/(d\bar{\theta}/dz)$ in many cases]. However, for internal gravity waves, $\overline{w'\boldsymbol{\xi}} \approx 0$. In particular, $\overline{w'\boldsymbol{\xi}'} = 0$, for a steady wave solution, in which the mean Lagrangian advection is small (compared to the group velocity), so that for two-dimensional internal gravity waves only the $\overline{\rho' w'}/\bar{\rho}$ term contributes significantly to the vertical Stokes drift.

Thus, for a finite amplitude, internal gravity wave (and also small amplitude, steady, acoustic waves) the vertical Stokes drift can be approximated as

$$\bar{w}^S = \overline{\rho' w'}/\bar{\rho}. \quad (19)$$

Equation (19) will be used as the formula for the vertical Stokes drift in the remainder of this paper. Note that no assumption about wave damping was needed in deriving (19), so that (19) should be valid for damped as well as undamped waves.

3. Adiabatic internal gravity waves

In this section $\overline{\rho' w'}$ will be evaluated for adiabatic waves. The most straightforward approach would be to solve the equations of motion for the ρ' and w' fields associated with a plane wave; however, a slightly more intuitive approach will be used in this paper. For $|\theta'/\bar{\theta}|$, $|\rho'/\bar{\rho}|$, and $|P'/\bar{P}| \ll 1$ the equation of state can be approximated as

$$\frac{\theta'}{\bar{\theta}} = -(\rho'/\bar{\rho}) + \gamma^{-1}(P'/\bar{P}), \quad (20)$$

where $\gamma = c_p/c_v$ and P is pressure. For most atmospheric problems involving stable stratification

$$|\theta'/\bar{\theta}| \approx |\rho'/\bar{\rho}| \gg |P'/\bar{P}|, \quad (21)$$

so that (20) is often approximated as

$$\theta'/\bar{\theta} = -\rho'/\bar{\rho}. \quad (22)$$

(For acoustic waves, of course, $|\rho'/\bar{\rho}| \approx |P'/\bar{P}|$.) However some care must be taken when quadratic quantities, which depend on the wave phases, are formed. If (20) is multiplied by w' and averaged, then

$$\overline{w'\rho'}/\bar{\rho} = \gamma^{-1}(\overline{w'P'}/\bar{P}), \quad (23)$$

since $\overline{w'\theta'} = 0$ for plane conservative waves.

For vertically propagating waves $\overline{w'P'}$ is just the vertical wave energy flux, which for a single wave is equal to $C_g E$, where C_g is the vertical group velocity and E is the wave energy density. (Andrews, 1980, derives this explicitly for inertial gravity waves.) Thus, (23) can be rewritten as

$$\frac{\overline{w'\rho'}}{\bar{\rho}} = \frac{C_g E}{\bar{\rho} c_s^2}, \quad (24)$$

where c_s is the speed of sound. According to (24), $\overline{w'\rho'}$ is proportional to the vertical group velocity. Since the vertical group velocity for planetary waves is much less than for gravity waves, $\overline{w'\rho'}$ in (18) is less important for planetary waves than for gravity waves provided that the gravity wave energy is sufficiently large. Combining (24) with (19) gives an expression for the vertical Stokes drift:

$$\bar{w}^S = \frac{C_g E}{\bar{\rho} c_s^2}. \quad (25)$$

Note that all upward propagating waves contribute to an upward Stokes drift regardless of their horizontal direction of propagation.

Equation (25) can be evaluated for different waves by using appropriate approximations for the vertical group velocity. For example, for a vertically propagating sound wave $C_g = c_s$ and $E = \rho \overline{w'^2}$ (assuming the average kinetic and potential energies are equal) giving

$$\bar{w}^S = \frac{\overline{w'^2}}{c_s}. \quad (26)$$

For a two-dimensional vertically propagating, hydrostatic internal gravity wave,

$$C_g = \frac{k}{N}(C - \bar{U})^2, \quad (27)$$

where k is the horizontal wavenumber, N the buoyancy frequency, C the horizontal phase speed, and $\bar{U}$ the mean wind in the direction of wave propagation. The total energy in this case is

$$E = \rho \overline{u'^2} \quad (28)$$

and (25) gives

$$\bar{w}^S = \frac{k}{N c_s^2}(C - \bar{U})^2 \overline{u'^2} \quad (29)$$

for internal gravity waves. Defining a nondimensional wave amplitude parameter,

$$\alpha^2 = \frac{2\overline{u'^2}}{(C - \bar{U})^2}, \quad (30)$$

so that $\alpha = 1$ corresponds to the wave breaking amplitude, allows (29) to be rewritten as

$$\bar{w}^S = \frac{1}{2}\frac{k}{N c_s^2}(C - \bar{U})^4 \alpha^2. \quad (31)$$

Taking $\alpha = 1$, Fig. 1 plots $\bar{w}^S$ as a function of $(C - \bar{U})$ for several values of k. The buoyancy frequency and speed of sound were set at typical mesospheric values. The wave breaking amplitude (where $|u'| = |C - \bar{U}|$ so that the potential temperature surface is vertical at one phase of the wave), was used in Fig. 1 because wave breaking, followed by wave saturation, is expected to be the process which limits wave amplitudes in the mesosphere (see the review by Fritts, 1984). Because of the fourth-power dependence, $\bar{w}^S$ can take on a wide range of values. Nevertheless, vertical Stokes drifts on the order of 10 cm s^{-1} appear to be consistent with what is currently known about internal gravity waves in the summer mesosphere. (For example, taking $2\pi/k = 50$ km, and $C - \bar{U} = 40$ m s^{-1} gives $\bar{w}^S \sim 10$ cm s^{-1}.)

There is still much uncertainty about the nature of mesospheric gravity waves. However, recent observations by Vincent (1984) show that the dominant contribution to the vertical wave energy flux comes from

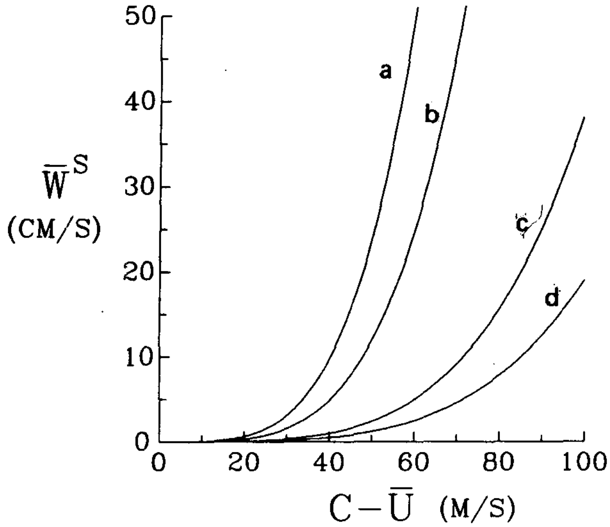

FIG. 1. Vertical Stokes drift (31) as a function of the intrinsic phase speed for $\alpha = 1$, $N = 0.02\ s^{-1}$, $H = 6$ km, and a horizontal wavelength of (a) 50 km, (b) 100 km, (c) 500 km and (d) 1000 km.

relatively high-frequency waves (with horizontal scales of 30–150 km, and $C - \bar{U} > 30\ m\ s^{-1}$). Many of these high-frequency gravity waves are near the breaking amplitude. (Vincent reports an rms velocity of $\sim$30 m s^{-1}.) In addition Vincent estimates that $\overline{w'P'} \sim 2.9 \times 10^{-2}\ W\ m^{-2}$ in the summer mesopause over Poker Flat. Taking $\bar{P} = 0.2\ N\ m^{-2}$ at the mesopause (from a standard atmosphere table) once again gives $\bar{w}^S \sim 10$ cm s^{-1}. Since the Lagrangian mean vertical motion due to the diabatic heating is on the order of 1 cm s^{-1}, it can be neglected in (1), making the Eulerian mean vertical velocity approximately equal to minus the Stokes drift. Thus, for adiabatic waves, a downward Eulerian mean vertical velocity should occur in the summer mesosphere.

4. Physical mechanism

The physical mechanism of the Stokes drift, which appears in the linear wave solution, can be understood in terms of the parcel motion. Consider first a vertically propagating sound wave. In this case, the phase propagation is upward so that when the parcel motion is upward it is moving in the direction of phase propagation and when the parcel motion is downward it is moving against the direction of phase propagation. Consequently, the parcel spends more time in the region of upward motion than in the region of downward motion, and therefore the parcel experiences a net upward Stokes drift, as given by (26). Note that this mechanism for Stokes drift is different from that due to surface wave motions in that no gradients of wave amplitude are required.

Unlike sound waves, the parcel motions for internal gravity waves are mainly along the phase lines. However, if the fluid is compressible then there exists a small velocity component that is perpendicular to the phase lines. Because the phase lines move downward for an upward propagating internal gravity wave, a small velocity component in the direction of the phase propagation will occur in the upward velocity region of the wave to produce an upward Stokes drift. This can be verified in the linear wave solution by considering the relation between u' and w'.

The relationship between u' and w' can be found from the linearized continuity equation,

$$\rho'_t + \bar{\rho}(u'_x + w'_z) + w'\frac{d\bar{\rho}}{dz} = 0, \tag{32}$$

where the mean density $\bar{\rho}$ is assumed to be a function only of z. Also, for convenience, the mean wind has been taken to be zero. Using the linearized equation of state (20) and the linearized thermodynamic equation,

$$\theta'_t + w'\frac{d\bar{\theta}}{dz} = 0, \tag{33}$$

(32) becomes,

$$\frac{1}{\bar{\rho}c_s^2}p'_t + u'_x + w'_z + w'\left[(1/\bar{\theta})\frac{d\bar{\theta}}{dz} + (1/\bar{\rho})\frac{d\bar{\rho}}{dz}\right] = 0. \tag{34}$$

Assuming sinusoidal motion, the horizontal momentum equation can be written as

$$\bar{\rho}Cu' = P', \tag{35}$$

where C is the horizontal phase speed of the wave. Using (35), (34) becomes,

$$\frac{C}{c_s^2}u'_t + u'_x + w'_z + w'\left((1/\bar{\theta})\frac{d\bar{\theta}}{dz} + (1/\bar{\rho})\frac{d\bar{\rho}}{dz}\right) = 0. \tag{36}$$

The partial derivatives can be evaluated by considering solutions of the form

$$u', w' \propto e^{z/2H}e^{i\phi}, \tag{37}$$

where $\phi = kx - mz - \omega t$.

The solution for u' will consist of a component in phase with w', as well as a component 90 degrees out of phase with w'. The out-of-phase component produces an elliptical motion which does not contribute to the vertical Stokes drift. The ratio of the in-phase components follows from (36) with

$$\frac{w_0}{u_0} = \frac{k}{m}(1 - C^2/c_s^2), \tag{38}$$

where w_0 and u_0 are the amplitudes of the in phase velocity components. Equation (38) shows that as $c_s \rightarrow \infty$, $w_0/u_0 \rightarrow k/m$ and no vertical Stokes drift is expected. Including compressibility, however, causes the velocity vectors to be more horizontal than the phase lines, as shown in Fig. 2. Thus, upward motion will be correlated with motion in the direction of phase propagation, and downward motion will be correlated with motion opposite the direction of phase propagation. These correlations produce an upward Stokes drift, as discussed earlier in this section. Of course the deviation of the velocity field from the phase lines is greatly exaggerated in Fig. 2. For typical gravity wave parameters the deviation of the velocity field from the phase lines will be on the order of a degree or less. This is why the incompressible gravity wave solution is often an excellent approximation to the compressible solution.

Finally, while no amplitude gradients are needed to produce a vertical Stokes drift, vertical gradients do sometimes play a role in the horizontal Stokes drift. For example, Nakamura (1976) considered the horizontal Stokes drift due to incompressible gravity waves. In this case vertical gradients of wave amplitude are necessary to obtain a horizontal Stokes drift.

5. Effects of dissipation

A number of processes can affect the simple results of sections 2 and 3: the breakdown of the slowly varying transverse wave assumptions, the interaction of a number of waves, wave breaking, turbulent diffusion, and other wave dissipation processes. In this section the effect of eddy diffusion on the vertical Stokes drift will be examined. The eddies are assumed to be characterized by a much smaller scale than the scale of the gravity waves being considered, so the wave-parcel displacement vector is still well defined, and the Lagrangian averages used in section 2 remain valid. The situation envisioned is one in which a plane wave is propagating through a region characterized by a small eddy diffusion coefficient, D. This dissipation will cause a change in the slope of the velocity vector, and thus will cause a change in the vertical Stokes drift.

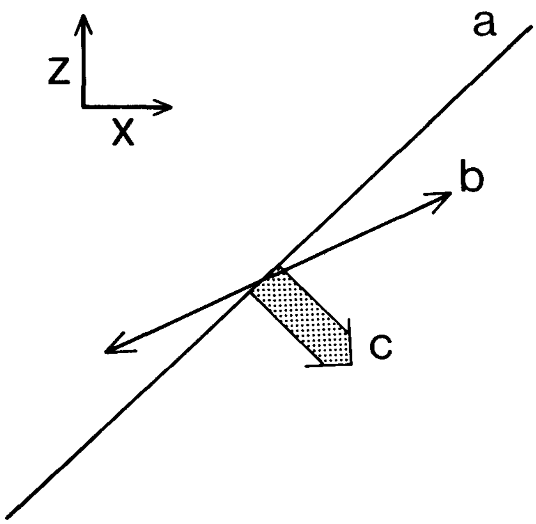

FIG. 2. Sketch of a plane, two-dimensional, compressible gravity wave showing (a) the slope of the phase lines, (b) the slope of the velocity vector, and (c) the direction of phase propagation, for an upward propagating wave.

Adding a small amount of damping to the equations of motion shifts the phases of the wave variables. In particular, $\overline{\theta' w'}$ is now nonzero, as can be seen directly from the thermodynamic equation. Starting with

$$\theta'_t + w'\frac{d\bar{\theta}}{dz} = D\theta'_{zz}, \tag{39}$$

multiplying by θ', and averaging gives

$$\frac{1}{2}(\overline{\theta'^2})_t + \overline{\theta' w'}\frac{d\bar{\theta}}{dz} = D\overline{\theta'\theta'_{zz}}. \tag{40}$$

For steady waves the time derivative in (40) is zero. Equation (40) can then be rewritten as

$$\overline{\theta' w'}\frac{d\bar{\theta}}{dz} = D(\overline{\theta'\theta'_z})_z - D\overline{(\theta'_z)^2}. \tag{41}$$

Following Fritts and Dunkerton (1985), the first term on the right-hand side of (41) will be neglected, since θ' and θ'_z are close to 90 degrees out of phase. Then (41) becomes

$$\overline{\theta' w'} = -D\overline{(\theta'_z)^2}\Big/\frac{d\bar{\theta}}{dz}, \tag{42}$$

which can be rewritten as

$$\overline{\theta' w'} = -\frac{D}{2}\alpha^2\frac{d\bar{\theta}}{dz} \tag{43}$$

since the nondimensional wave amplitude parameter defined by (30) can be approximated as

$$\alpha^2 = 2\frac{\overline{(\theta'_z)^2}}{(d\bar{\theta}/dz)^2}, \tag{44}$$

so that $\alpha = 1$ again corresponds to the wave breaking amplitude. The equation of state gives

$$\frac{\overline{w'\rho'}}{\bar{\rho}} = \frac{1}{\gamma}\frac{\overline{w'P'}}{\bar{P}} - \frac{\overline{w'\theta'}}{\bar{\theta}}, \tag{45}$$

and the expression for the vertical Stokes drift becomes

$$\bar{w}^S = \frac{C_g E}{\bar{\rho} c_s^2} + D\frac{\alpha^2}{\bar{\theta}}\frac{d\bar{\theta}}{dz}, \tag{46}$$

or

$$\bar{w}^S = \left[\frac{C_g(C-\bar{U})^2}{c_s^2} + \frac{D}{g}N^2\right]\frac{\alpha^2}{2}. \tag{47}$$

For typical values of D and N^2 (D = 100 m^2 s^{-1}, N^2 = 4 × 10^{-4} s^{-2}) the contribution of damping to the vertical Stokes drift (DN^2/g = 0.4 cm s^{-1}) will be relatively small, except for waves with very small $C - \bar{U}$ values. Since the waves with large $C - \bar{U}$ contribute the most to the vertical Stokes drift, the damping term can probably be neglected. However, some theories of internal wave breaking predict that large values of D may be necessary to limit wave growth with height. While not justified by the small dissipation assumption made in this paper, the amplitude limiting value of D will be considered as an extreme case.

In order to limit the exponential growth with height, the diffusive time scale must equal the exponential growth time scale:

$$\frac{1}{m^2 D} = \frac{2H}{C_g}; \tag{48}$$

where m is the vertical wavenumber. Using the hydrostatic gravity wave dispersion relation,

$$m^2 = \frac{N^2}{(C - \bar{U})^2}, \tag{49}$$

(48) can be solved for D:

$$D = \frac{C_g}{2H}\frac{(C - \bar{U})^2}{N^2}. \tag{50}$$

Equation (50) is similar to expressions in Hodges (1969), Lindzen (1981), Holton (1982), Schoeberl et al. (1983), and Fritts (1984). Using (50) in (47) gives

$$\bar{w}^S = \frac{C_g (C - \bar{U})^2}{c_s^2}[1 + \gamma/2]\frac{\alpha^2}{2}. \tag{51}$$

Since $\gamma = 1.4$, Eq. (51) shows that the dissipation can increase the vertical Stokes drift by ~70%. It should be emphasized again that (50) can lead to large values of D. For example, taking $C_g = 10$ m s^{-1}, $C - \bar{U} = 30$ m s^{-1}, $H = 6000$ m and $N^2 = 4 \times 10^{-4}$ s^{-2} gives $D = 1900$ m^2 s^{-1}. However, it is questionable whether such extreme values of eddy diffusion actually occur in the atmosphere.

6. Conclusions

If the mesosphere is disturbed by waves which have propagated from the lower atmosphere and if the Lagrangian mean vertical velocity is small, then the mesosphere should possess a downward time mean Eulerian velocity. This downward Eulerian mean velocity is required to assure that no net vertical parcel motion is produced by the wave-induced Stokes drift. While only internal gravity waves were discussed in this paper, all upward propagating waves should contribute to the downward mean Eulerian velocity. This downward mean Eulerian velocity may be the explanation for the mean downward velocity observed in the summer mesosphere by Balsley and Riddle (1984).

In this paper only a single, plane, internal gravity wave with small thermal damping was considered in calculating the Stokes drift. In the real atmosphere the situation is much more complex. There is generally a superposition of many waves, and these waves may be interacting or breaking nonlinearly. More work is needed to determine the vertical Stokes drift in such complicated systems. The purpose of the present paper was to present a simple physical picture of the origin of a downward mean Eulerian velocity due to the presence of upward propagating internal gravity waves.

One major problem remaining is the upward mean velocities observed by Balsley and Riddle (1984) in the winter mesosphere. Rather than speculate on possible explanations it will just be pointed out that the characteristics of the mesospheric echoes above Poker Flat differ considerably from summer to winter (Balsley et al., 1983). During summer the echoes between 75 and 90 km are nearly continuous in time allowing the internal gravity waves to be easily seen. In winter, however, most of the echoes above 75 km come from very sporadic meteors which do not yield a high enough time resolution to see gravity waves. Thus, the wintertime wave structure above 75 km is not as well known as the summertime wave structure at the same heights.

Current numerical models of the middle atmosphere do not need to consider explicitly the vertical Stokes drift because consideration of the vertical Stokes drift would merely involve a change in variables, replacing $\bar{\rho}\,\bar{w}$ terms with $\bar{\rho}\,\bar{w} + \overline{\rho' w'}$. The results here do change slightly the interpretation of model results since the $\bar{w}$ calculated in numerical models may not equal the Eulerian velocity in the atmosphere. Also, it should be mentioned that this Stokes drift mechanism should not be important in the troposphere where large-amplitude, high phase-speed gravity waves are not observed.

Finally, measuring the time mean vertical velocity in the summer mesosphere appears to give more information about the waves (in particular, the vertical wave energy flux), than about vertical velocities associated with mean meridional circulations. While the vertical wave energy flux may not be a useful quantity to measure by itself, when combined with measurements of other mesospheric gravity wave parameters, measurements of the time mean vertical velocity may aid in understanding the average properties of mesospheric gravity waves.

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