Inertial Instability of Horizontally Sheared Flow away from the Equator

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ABSTRACT

We investigate the temporal and spatial characteristics of unstable normal modes in a horizontally sheared flow on a sphere using the shallow water equations. Both inertial and barotropic instabilities are identified in cases where the appropriate necessary conditions are satisfied.

A primary focus is determining what conditions favor asymmetric modes of inertial instability rather than symmetric modes. With the Bickley jet profile, the region of instability \(|f + \tilde{f}| \approx 0\) is confined to the anticyclonic side of the jet in a limited region. We find that symmetric instability is preferred only for modes of very small vertical scale, for which the pressure gradient force is secondary. Relatively small dissipation is needed to stabilize these modes. With deeper vertical scales, asymmetric instabilities are preferred in which the zonal scale of the instability is comparable to the width of the unstable region.

This study extends previous results for linear shear on an equatorial beta plane to the midlatitude jet case. Our results suggest that deep atmospheric circulations in spatially confined regions of negative potential vorticity may develop as asymmetric rather than symmetric instabilities.

1. Introduction

Inertial instability, which occurs when the potential vorticity of a geophysical fluid is of opposite sign to the Coriolis parameter, is often referred to as symmetric instability. This identification implies that the most unstable mode of instability should occur for symmetric perturbations, i.e., for perturbations that display no structure in the direction of the basic state flow. By limiting a study to symmetric perturbations, one can often simplify the analysis. Emanuel (1979) used the symmetry property to reduce the set of governing equations in a sheared flow basic state to a high order equation in the streamfunction for the "meridional" (i.e., perpendicular to the basic state flow) perturbation circulation. In considering horizontally sheared flow near the equator, Dunkerton (1981) and Stevens (1983) reduced the governing equations to a single equation with exact analytic solutions on the equatorial beta plane, for both linear and quadratic shear.

Recently Boyd and Christidis (1982) and Dunkerton (1983) have shown that asymmetric modes of instability may be preferred in a basic state zonal flow near the equator with linear latitudinal shear. Generally these asymmetric instabilities are preferred if the vertical scale of the perturbation is sufficiently large. These modes will then be of physical relevance when dissipation acts to stabilize the smaller vertical scales of motion.

Inertial instability is also an important dynamical mechanism away from the equator where the Coriolis parameter does not change sign. This instability has been considered in studies of the anticyclonic upper
tropospheric outflow regions of hurricanes (Alaka, 1961; Black and Anthes, 1971; Anthes, 1972). Emanuel (1979, 1982) has suggested inertial instability for vertically sheared flow as a mechanism for organizing mesoscale rainbands in middle latitudes. Synoptic maps sometimes contain limited regions of negative absolute vorticity which would be susceptible to instability.

In this paper we investigate both symmetric and asymmetric forms of instability for horizontally sheared flow away from the equator. Section 2 describes the governing equations and the numerical model used to investigate the linear instabilities. Section 3 presents the growth rates, phase speeds and horizontal structures of the eigensolutions. Section 4 discusses and generalizes the numerical results with a focus on determining the conditions under which asymmetric instability would be preferred over symmetric instability.

2. Model description

The governing equations in pressure coordinates for linearized hydrostatic perturbations in a horizontally sheared basic state zonal flow on a sphere are

\[
\begin{align*}
\frac{\partial}{\partial t} + \tilde{U} \frac{\partial}{\partial x} u' - \tilde{v}' + \frac{\partial \psi'}{\partial x} &= 0 \\
\frac{\partial}{\partial t} + \tilde{U} \frac{\partial}{\partial x} v' + \tilde{u}' + \frac{\partial \psi'}{\partial y} &= 0 \\
\frac{\partial u'}{\partial x} + \frac{\partial (v' \cos \theta)}{\partial y} + \frac{\partial \phi'}{\partial \rho} &= 0
\end{align*}
\]

(2.1a)

(2.1b)

(2.1c)

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\[
\frac{\partial \Phi}{\partial t} + \frac{RT'}{p} = 0 \quad (2.1d)
\]

\[
\left( \frac{\partial}{\partial t} + \vec{U} \frac{\partial}{\partial x} \right) T - \frac{H^2 N^2 \omega'}{R} = 0 \quad (2.1e)
\]

where \( p \) is pressure, \( \phi \) is longitude, \( \theta \) is latitude, \( a \) is the earth's radius, \( dx = a \cos \phi \, d\phi, \, dy = a \, d\theta \), \( H = RT / g \) is the scale height, \( N \) is the Brunt–Väisälä frequency, \((u', v', \omega', \Phi', T')\) have the usual meanings and SI units; \( f = 2\Omega \sin \theta \) is the Coriolis parameter,

\[
\tilde{\eta} = f - \frac{d(\tilde{U} \cos \theta)}{\cos \theta dy}, \quad \tilde{f} = f + \frac{2 \tilde{U}}{a} \tan \theta
\]

are the absolute vorticity and modified Coriolis parameter, respectively.

The system of equations (2.1) is separable in its vertical dependence. Therefore we specify the separation constant \( h \) which is contained in the equation for the vertical structure function \( G(p) \) of \((u', v', \Phi')\)

\[
\frac{d}{dp} \left( \frac{d^2 G}{H^2 N^2 dp} + \frac{G}{gh} \right) = 0
\]

and in the equations for the horizontal structure function \((u, v, \Phi)\) of \((u', v', \Phi')\)

\[
\left( \frac{\partial}{\partial t} + \vec{U} \frac{\partial}{\partial x} \right) \Phi + gh \left[ \frac{\partial u}{\partial x} + \frac{\partial (v \cos \theta)}{\cos \theta dy} \right] = 0. \quad (2.2c)
\]

Since all coefficients are independent of time and longitude, we seek normal mode solutions of the form
\[
\exp[i(s \phi - \sigma t)],
\]
where \( s \) is the (integral) longitudinal wavenumber and \( \sigma \) is the complex frequency. It is useful to nondimensionalize the horizontal structure equations using the time scale \((2\Omega)^{-1}\), length scale \( a \), velocity scale \( 2\Omega a \) and geopotential scale \( 4\Omega^2 a^2 \). The complex \( \gamma \)-dependence of the horizontal structure functions is governed by

\[
-i \left( \sigma - \frac{s\tilde{U}}{\cos \theta} \right) u - \tilde{\eta} v + \frac{is}{\cos \theta} \Phi = 0 \quad (2.3a)
\]

\[
-i \left( \sigma - \frac{s\tilde{U}}{\cos \theta} \right) v + \tilde{f}u + \frac{d\Phi}{dy} = 0 \quad (2.3b)
\]

\[
-i \left( \sigma - \frac{s\tilde{U}}{\cos \theta} \right) \Phi + E^{-1} \left[ \frac{isu}{\cos \theta} + \frac{d(v \cos \theta)}{\cos \theta dy} \right] = 0. \quad (2.3c)
\]

Both dependent and independent variables are nondimensional in (2.3). \( E = 4\Omega^2 a^2 / gh \) is Lamb's parameter.

For an ideal gas, the dimensional vertical wavenumber \( \lambda \) is related to the equivalent depth by the relation

\[
\lambda^2 H^2 = \frac{N^2 H^2}{gh} \frac{1}{4}
\]

In the troposphere, with a typical value for \( N^2 \) \((10^{-4} \text{ s}^{-2})\), a vertically propagating wave has a vertical wavelength

\[
L_z = 2\pi / \lambda \approx (2 \text{ km}) / h / (1 \text{ m})
\]

which holds for \( h \ll 4N^2 H^2 / g \approx 2500 \text{ m} \). This relation calibrates the equivalent depths we analyze with dimensional vertical scales in the atmosphere. In the ocean, the external mode and first internal mode with a typical thermocline stratification have equivalent depths on the order of 5 km and 1 m, respectively; for the higher-order \( n \)th internal mode, equivalent depth is approximately inversely proportional to \( n^2 \) (Gill, 1982, p. 161).

For a given mean zonal flow \( \vec{U} \), equivalent depth \( h \) and zonal wavenumber \( s \), we obtain a set of complex eigenvalues \( \sigma_n \) and corresponding eigenfunctions \((u_n, v_n, \Phi_n)\). A positive imaginary part \((\sigma_I)\) indicates an unstable mode; \( \sigma_n \) is the growth rate. Negative \( \sigma_n \) indicates a stable, decaying mode. The real part \( \sigma_R \) of the eigenfunction is related to the dimensional zonal phase speed \( c_R \), by

\[
c_R = 2\Omega a (\sigma_R \cos \theta) / s
\]

such that \( c_R > 0 \) implies an eastward propagating mode.

If \([u](y), v(y), \Phi(y); \sigma \) is an eigensolution for a given \( s \), then \([u^\ast](y), v^\ast(y), \Phi^\ast(y); -\sigma^\ast \) is an eigensolution for wavenumber \(-s\). Thus, an unstable mode with growth rate \( \sigma_n \), phase speed \( c_R \) and wavenumber \( s \) can be represented identically by wavenumber \(-s\), growth rate \( -\sigma_n \), phase speed \( -c_R \) and the complex conjugate of \( u, v, \Phi \). We can therefore limit our study to positive \( s \), knowing that complete information is simultaneously obtained for negative \( s \).

As shown by Stevens (1983), for symmetric modes \((s = 0)\) all frequency eigenvalues are either pure real or pure imaginary. Those modes with real \( \sigma = \sigma_R \) are neutral inertia–gravity waves for \( \sigma \neq 0 \) and nondiagnostic, geostrophic modes for \( \sigma_R = 0 \). As \( s \) is gradually increased, some of the latter modes become baroclinically unstable. In this study we focus on the instabilities. The stable modes are not considered further; however, they are required to form a complete set of normal modes for an initial value problem.

To investigate the behavior of localized phenomena, we incorporate an arbitrary stretching of the latitudinal coordinate \( \theta \). The stretching formula used is given by

\[
T(\theta) = 0.8 \left[ \tan \left( \frac{\theta - \theta_C}{(2\Delta \theta)_H} \right) + \frac{\theta}{2\pi} \right]. \quad (2.5)
\]

The results cited in this paper for inertial instabilities were computed with \( \theta_C = (\theta_N + \theta_S) / 2 \), where \( \theta_N \) and \( \theta_S \) are the latitudes that bound the unstable region (cf. Table 1), and \( T_H = 3/4 \). This value of \( T_H \) was chosen so that roughly half the grid points are contained in
this unstable region. At the poles, the following boundary conditions on geopotential are applied: for \( s = 0 \), \( \partial \Phi / \partial \theta = 0 \); for \( s \neq 0 \), \( \Phi = 0 \). A staggered finite-difference grid with constant grid spacing \( \Delta T \) is employed with \( \Phi \) evaluated at the poles and \( u, v \) located midway between \( \Phi \) grid points in the stretched coordinate. The \( u, v \) and \( \Phi \) equations above are then evaluated at the corresponding \( u, v \) and \( \Phi \) interior grid points, yielding a standard matrix eigenvalue problem which is solved using a “canned” eigensolver package. The results shown in the following section are computed using either 41 or 61 \( \Phi \) grid points.

3. Numerical results

In this section we apply our model with basic state flows consisting of jet profiles given by

\[
\bar{U}(\theta) = \frac{U_0}{2\Omega a} \text{sech}^2\left(\frac{\theta - \theta_0}{\theta_1}\right).
\]

(3.1)

In this mean flow, known as the Bickley jet, \( U_0 \) gives the magnitude of the jet (\( U_0 > 0 \) for westerly jets), \( \theta_0 \) is the central latitude of the jet, and \( \theta_1 \) is the jet half-width. In particular we examine the instabilities of a near equatorial jet (Profiles IA and IB) centered at 4°N and a midlatitude jet (Profiles IIA and IIB) centered at 45°N. The parameters for these jets are listed in Table 1. Zonal jets with other values of \( U_0, \theta_0, \theta_1 \) were examined in this study, however their results are qualitatively consistent with the cases in Table 1, and thus are not presented here.

For inertial instability to occur, the absolute vorticity \( \bar{\eta} \) and the modified Coriolis parameter \( \bar{f} \) have to be of opposite sign (Stevens, 1983; Ripa, 1983). We define \( \alpha = (\bar{\eta}/\bar{f})_{\text{min}} \), where \( \alpha < 0 \) is a necessary condition for inertial instability. The \( \alpha \) measures the magnitude of the absolute vorticity relative to the local Coriolis parameter.

Ripa (1983) showed that the maximum growth rate is given by \( (\bar{\eta}/\bar{f})^{1/2} \) when \( \alpha < 0 \). The \( \epsilon = (\bar{\eta}/\bar{f})^{1/2} \) is a nondimensional parameter which represents the maximum possible growth rate (nondimensionalized by twice the earth’s rotation rate). The \( \epsilon = 1 \) indicates an unstable \( e \)-folding time scale of 1.9 hours, while \( \epsilon = 0.1 \) corresponds to 19 hours. For the type A profiles listed in Table 1, the maximum growth rate parameter \( \epsilon \) is approximately double the corresponding value for type B profiles. The sensitivity to the strength of the instability parameter is obtained by varying the jet amplitude \( U_0 \) while keeping the central latitude \( \theta_0 \) and width parameter \( \theta_1 \) fixed.

The necessary condition for barotropic instability (\( \partial \bar{\eta} / \partial y = 0 \) somewhere in the fluid) is automatically satisfied for a Bickley jet which is inertially unstable. The barotropic stability problem for nondivergent flow has been previously studied by Kuo (1949) and Lipps (1962), and for divergent flow by Lipps (1963).

It is implicitly assumed that no other stabilizing processes act faster than the instabilities under consideration to eliminate the strong horizontal shear of the imposed basic state flow. We investigate the time scales and horizontal and vertical space scales of the instabilities, as well as their dynamical character and structure.

a. Equatorial jets

We first consider the model results for the narrow tropical jet (profile IA) shown in Fig. 1a. The conditions for barotropic and inertial instability are seen in the profiles (Fig. 1b) of the \( \bar{f} \) parameter and the absolute vorticity (\( \bar{\eta} \)) associated with this jet. For this basic state \( \bar{U} \) field, the computed growth rates (\( \epsilon \)) and phase speeds (\( c_\Phi \)) as a function of zonal wavenumber (\( s \)) are shown in Figs. 2a and 2b, respectively, for several values of equivalent depth (\( h \)). The growth rate curves maximize at different values of \( s \) as \( h \) varies. In particular, the zonally symmetric mode is most unstable only for the smallest equivalent depth (\( h = 0.01 \) m) examined here, and even in this case the decrease in growth rate with \( s \) is relatively slow. Thus, for “white noise” initial conditions, the symmetric instability will not predominate until many \( e \)-folding periods have elapsed, at which time a realistic instability may no longer be in the linear regime.

As the vertical scale increases, the growth rate of the symmetric (\( s = 0 \)) instability decreases rapidly. Stevens (1983) has shown that this stabilization occurs because a meridional pressure gradient is established which tends to oppose the inertial instability. However, for asymmetric (\( s \neq 0 \)) modes, in which the zonal pressure gradient is nonnegligible, this stabilization process is

<table>
<thead>
<tr>
<th>Profile</th>
<th>( \theta_0 ) (°N)</th>
<th>( \theta_1 ) (deg)</th>
<th>( U_0 ) (m s(^{-1}))</th>
<th>( \alpha )</th>
<th>( \epsilon )</th>
<th>( \theta_\parallel ) (°N)</th>
<th>( \theta_\perp ) (°N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>-2.28</td>
<td>0.0879</td>
<td>2.028</td>
<td>3.859</td>
</tr>
<tr>
<td>IB</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>-0.64</td>
<td>0.0463</td>
<td>2.566</td>
<td>3.706</td>
</tr>
<tr>
<td>IIA</td>
<td>45</td>
<td>3</td>
<td>75</td>
<td>-0.58</td>
<td>0.596</td>
<td>41.071</td>
<td>44.173</td>
</tr>
<tr>
<td>IIB</td>
<td>45</td>
<td>3</td>
<td>50</td>
<td>-0.10</td>
<td>0.239</td>
<td>42.176</td>
<td>43.578</td>
</tr>
</tbody>
</table>

TABLE 1. Parameters used in (3.1) for defining jet profiles examined in section 3. \( \alpha \) and \( \epsilon \) are nondimensional parameters which indicate the potential for inertial instability and the maximum possible growth rate, respectively. See text for details. \( \theta_\parallel \) and \( \theta_\perp \) are the latitudes that bound the inertially unstable region: i.e., \( \bar{\eta}(\theta_\parallel) = \bar{\eta}(\theta_\perp) = 0 \).
rather ineffective. Indeed, for very large $s$, the fractional change in growth rate with vertical scale is asymptotically small. As a result, for all vertical scales but the shallowest, the preferred inertial instability occurs in asymmetric modes.

For sufficiently large vertical scale ($h \gg 1.0$ m), two distinct modes are observed for a given $h$. The more unstable set is characterized by phase speeds that increase with $s$. These modes are the analytic continuation of the inertial instabilities at smaller vertical scales. Computations for $h > 100$ m produce analogous results to the $h = 100$ m case. Unlike the inertial instabilities, the second set of unstable modes is characterized by phase speeds which decrease with $s$, growth rates that increase as the vertical scale increases, and confinement to smaller wavenumbers. These modes, labeled with a

---

**Fig. 1.** (a) Near equatorial Bickley jet in (m s$^{-1}$) for profile IA: $U_0 = 4$ m s$^{-1}$, $\theta_0 = 4^\circ$N, $\theta_i = 1^\circ$. (b) Nondimensional parameters associated with profile IA: $\mathcal{f}$ (dashed), $\mathcal{g}$ (solid). For definitions see text. Here the condition for inertial instability (i.e., $\alpha < 0$) is satisfied where $\mathcal{g}$ is negative. The necessary condition for barotropic instability is satisfied because $\partial \mathcal{g} / \partial y = 0$.

---

**Fig. 2a.** Nondimensional growth rates ($\sigma$) for jet profile IA as a function of wavenumber ($\beta$) for several values of equivalent depth. Value along a curve represents equivalent depth in meters. Scale to the right of the plot gives the relative growth rate ($\sigma^*$), that is, in comparison with the maximum possible growth rate $\epsilon$ (where $\epsilon = 8.79 \times 10^{-3}$). B signifies barotropic instability, whereas the dashed curve, labeled with an $N$, represents numerical modes. Other curves are inertial instability modes.

---

**Fig. 2b.** Phase speeds ($c_p$) in (m s$^{-1}$) corresponding to growth rate curves in Fig. 2a.
B, will be shown later in this section to be a manifestation of barotropic instability.

The dashed curve labeled with an $N$ in Fig. 2a gives the apparent growth rates for a set of numerical modes. The curve shown here was computed using 40 grid points; with 20 grid points the growth rates are higher, while for 60 grid points the instability curve disappears off the graph. The existence of this curve with low and intermediate resolution suggests that high resolution is required in order to avoid spurious computational instabilities. The other growth rate curves presented in this paper are considered numerically stable, that is, when the model resolution is increased from 40 to 60 points, the computed values change only in the third or fourth significant digit.

In jet IB we decrease the amplitude of the jet to 2 m s$^{-1}$ in order to investigate the situation with smaller potential for instability. Figure 3 displays the computed growth rates ($\sigma_j$) as a function of $s$ for several equivalent depths. We note that the growth rates are significantly reduced in absolute magnitude; but the relative growth rates [in comparison with the maximum possible growth rate ($\epsilon$)] are comparable. Hence the growth rates for the inertial instabilities scale with $\epsilon$.

The results from jet IB provide a clue to the atmosphere's response shortly after the onset of inertial instability. With the reduced instability potential, asymmetric instability is preferred at even smaller vertical scales than in jet IA. At $h = 0.01$ m, the $s = 25$ mode has a growth rate approximately 30% higher than that of the symmetric mode. This occurs at a vertical wavelength of 200 meters using a tropospheric stability! In order to have maximum instability in a symmetric overturning, $h$ must be reduced below 0.0014 m; i.e., a vertical wavelength less than 75 meters. As pointed out by Dunkerton (1981), a scale-selective dissipation process will act to stabilize these short vertical scales.

b. Midlatitude jets

We now consider the instabilities associated with two midlatitude jets. The stronger jet (profile IIA) is shown in Fig. 4a along with its associated parameters $\bar{f}$ and $\bar{n}$ (Fig. 4b). For this jet the computed growth rates ($\sigma_j$) and phase speeds ($c_p$) as a function of $s$ are shown in Figs. 5a and 5b, respectively, for several values of $h$. Likewise, the computed growth rates for the weaker midlatitude jet (profile IIB) are shown in Fig. 6. The results presented in these figures are confined to a more limited range of wavenumbers (0–15) than the tropical jet cases in Figs. 2 and 3. This is due to the effects of critical latitudes, which become more pronounced when $\sigma_R/\sigma_I$ gets much larger than 10 (as is the case for profiles IIA and IIB for $s > 20$); under such conditions the eigenvectors ($u_n, v_n, \Phi_n$) become noisy with 40 grid points and 120 degrees of freedom. In the tropical jet case, the effects of critical latitudes are not observed even out to $s = 80$ where the ratio $\sigma_R/\sigma_I$ is still less than 10. Several grid points are required in the vicinity of the critical latitude (where $\sigma_R - s\bar{U}/\cos \theta = 0$), either with damping or with a finite growth rate removing the singularity at the critical latitude. This internal boundary layer, within which $|\sigma_j| > |\sigma_R - s\bar{U}/\cos \theta|$, must be resolved in order for the finite difference scheme to simulate the continuous solution. Unfortunately, the required resolution can only be obtained a posteriori once $\sigma_R$ and $\sigma_I$ are known at least approximately.

The results for the midlatitude jets shown in Figs. 5 and 6 reflect the same basic conclusions as stated earlier for the tropical jets. In particular, we again note that the zonally symmetric mode is most unstable only for the shallowest vertical scale. Although the growth rates for the inertial instabilities associated with the midlatitude jets are nearly an order of magnitude larger than for the equatorial jets in Figs. 2 and 3, the relative growth rates (i.e., in comparison to $\epsilon$) are comparable.

In Fig. 7 selected eigenfunctions are displayed over one wavelength (2\pi/s) for several equivalent depths (0.1 m, 10 m, 100 m, 5 km), respectively, from Fig. 5. These eigenfunctions correspond to the most unstable modes at their respective equivalent depths. An alternate way to view the eigenfunctions is shown in Fig. 8, where amplitude and phase plots are used to depict the same basic information as contained in Fig. 7b. Viewing the eigenfunctions of these instabilities collectively, we note that the latitudinal extent of the instabilities tends to increase as $h$ increases. The symmetric instability mode (Fig. 7a), which has the largest growth rate and the smallest equivalent depth (0.1 m), is very similar in structure to the eigenmode found by Stevens (1983) with linear shear at the equator. As $h$ increases, the meridional flow in the unstable region stays approximately the same, with a single maximum

![Fig. 3. Nondimensional growth rates same as in Fig. 2a except for jet profile IB ($\epsilon = 4.63 \times 10^{-2}$).](image-url)
centered at the latitude where the condition for inertial instability is largest. On the other hand, the zonal flow perturbation develops strong reversal in direction. This latter feature is best observed with the format of Fig. 8, where nearly discontinuous phase changes separate the amplitude peaks in the \( u \) eigenfunction.

These tendencies in the structure of the eigenfunctions hold for all the modes in Figs. 2–5 except those labeled with a B; in these latter modes the meridional flow near the jet axis shows a strong reversal with a single peak in the geopotential field located near a latitude where the condition for barotropic instability is satisfied. The balance of terms for the modes labeled with a B are nearly geostrophic, in contrast to the other modes in which the pressure gradient terms are insignificant in comparison to the Coriolis force. In addition, the correlation of the \( u \) and \( v \) eigenfunctions for the B labeled modes is consistent with a growing disturbance from the energy equation for a barotropic instability (Haltiner and Williams, 1980; p. 74). Based on the above observations, the modes labeled with a B are identified as barotropic instabilities\(^1\), while the rest are inertial instabilities.

It is of interest to note in Fig. 7d that the barotropic instability is located predominantly on the poleward side of the jet, even though the basic state velocity field (Fig. 4a) is symmetric in latitude about the jet axis. Hartmann (1983) identified the equatorial region of negative vorticity, located at 42°N for profile IIA, as dynamically important for midlatitude instability. However, in contrast to the present study the condition for inertial instability was not present in jet profiles he examined. We speculate that an “orthogonality principle” may be operating which prevents substantial amplitude in the barotropic instability mode on the anticyclonic side of the jet, where inertial instabilities at smaller vertical scales (cf. Fig. 5a) are present with larger growth rates.

4. Interpretation and discussion

Estimates on the maximum growth rates and corresponding zonal scales for inertial and barotropic instabilities of zonal jets can be compared with the numerical cases of the previous section. For symmetric instability at vanishing vertical scale, the maximum nondimensional growth rate is \( \epsilon \), as demonstrated analytically by Ripa (1983). Table 2 shows that the symmetric growth rate approaches \( \epsilon \) as the equivalent depth approaches zero.

At larger vertical scales, the numerical computations of section 3 indicate that the maximum growth rate for asymmetric inertial/shear instability is somewhat less than \( \epsilon \). Table 2 shows that these growth rates are all bounded by \( \frac{2}{3} \epsilon \), which we suggest tentatively as an upper limit. Below we will demonstrate that the boundary between symmetric and asymmetric instability occurs when the symmetric growth rate \( \sigma_I (s = 0) \approx \frac{2}{3} \epsilon \).

The semicircle (SC) theorem for barotropic instability provides an estimate on growth rate. Empirically from Table 2, the maximum instability occurs at zonal wavenumber \( (s_3) \) such that the zonal length scale \( (L_a/2\pi = a \cos \theta_0/s_3) \) approximates the jet width \( (2a \theta_1) \). According to Pedlosky (1979, p. 449), barotropic growth rate is limited by

\[
\sigma_I^2 \leq \sigma_{I,SC}^2 = \left( \frac{s_3}{2 \cos \theta_0} \frac{U_0}{2 \Omega \alpha} \right)^2 + \left( \frac{\cos \theta_0}{2} \frac{U_0}{2 \Omega \alpha} \right),
\]

\(^1\) To adequately resolve the barotropic modes, the cases where these instabilities were detected with the stretched grid described in section 2 were recomputed with broader stretching and a grid centered on the central latitude of the jet [i.e., \( T_y = 2 \) and \( \theta_C = \theta_0 \) in (2.5)].
We now address the question of identifying inertial instability as symmetric instability. First we obtain an estimate of the critical equivalent depth \( h_{\text{crit}} \) which separates the vertical scales with symmetric instability preferred \( (h < h_{\text{crit}}) \) from the vertical scale in which asymmetric instability is preferred \( (h > h_{\text{crit}}) \). In addition, we calculate the "flatness" of the growth rate curves. Following the lead of Dunkerton (1981), we determine the magnitude of eddy viscosity which would stabilize the smallest scales so that asymmetric instability would predominate. Finally, we consider a general relationship for the zonal wavenumber of greatest inertial instability as a function of vertical scale. In the following discussion we will refer to the first and second derivatives of \( \sigma \) with respect to \( s \) at \( s = 0 \) as \( \sigma'(0) \) and \( \sigma''(0) \), respectively.

Table 3 displays growth rates at \( s = 0 \) for three values of equivalent depth for each jet profile; these \( h \) values are chosen in order to bracket \( h_{\text{crit}} \). The growth rate divided by the maximum possible growth rate, \( \dot{\sigma}_1 = \sigma_1/\epsilon \), is also listed. Near \( s = 0 \) the \( \sigma_1 \) curve slopes upward with increasing \( s \) for \( h > h_{\text{crit}} \), while the \( \sigma_1 \) curves slope downward for \( h < h_{\text{crit}} \); cf. Figs. 2, 3, 5 and 6. Due to the symmetry of \( \sigma_1 \) about \( s = 0 \), \( \sigma''(0) = 0 \). Table 3 gives an estimate of \( \sigma''(0) \) for each value of \( h \), based on a standard fourth-order finite difference approximation with \( \sigma_1 \) evaluated at \( s = 0, 2, 4 \). For each jet profile we fit a parabola to the three estimates of \( \sigma''(0) \) to obtain \( h_{\text{crit}} \), which is defined as the value of \( h \) where \( \sigma''(0) = 0 \). In a similar fashion, a parabola is fit to the three values of \( \sigma_1(0) \) listed in Table 3 for each jet profile which results in a relationship between \( \sigma_1(0) \) and \( \log(h) \). Using \( h_{\text{crit}} \) in this relationship, we compute \( \sigma_1(0)_{\text{crit}} \) (i.e., the symmetric growth rate at \( h_{\text{crit}} \)). The dynamically normalized values \( \dot{\sigma}_1(0)_{\text{crit}} \), listed in Table

We find in Table 2 that the semicircle theorem bounds overestimate the numerical maximum growth rates by about an order of magnitude. The other two bounds given by Pedlosky, involving the maximum shear and the Fjortoft theorem parameter, provide no better estimates. The growth rates from the numerical computations and from the related estimates indicate that a jet which satisfies the necessary conditions for both inertial and barotropic instability will break down first by inertial instability.
3, show that the boundary between symmetric and asymmetric instabilities occurs when \( \delta_A(0)_{\text{crit}} \approx 4/5 \).

For small vertical scales, \( h < h_{\text{crit}} \), we obtain a measure of the flatness of the growth rate curves by using a Taylor series expansion for \( \sigma_t \) and terminating the series at the quadratic term:

\[
\sigma_A(s) \approx \sigma_A(0) + \sigma_A'(0)s + \frac{\sigma_A''(0)}{2}s^2.
\]

Since \( \sigma_A'(0) = 0 \), \( \sigma_t \) falls to half the symmetric value \( \sigma_A(0) \) at the half-point zonal wavenumber

\[
s_{1/2} = \left[ \frac{-\sigma_A(0)}{\sigma_A'(0)} \right]^{1/2}.
\]

The values of \( s_{1/2} \) in Table 3 show that the growth rates even at the smallest vertical scales are comparable to the symmetric rate for many finite zonal wavenumbers.
Based on the results for the four jet profiles (Figs. 2, 3, 5 and 6), Fig. 9 summarizes the relationship between maximum growth rate and vertical scale. In this figure, \( \sigma_1 \) is normalized by the maximum possible growth rate \( \epsilon \), and \( h \) is normalized by \( h_{crm} \). Also plotted is a representative curve depicting the dependence of linear dissipation \( (\gamma) \) on vertical scale for fixed eddy viscosity \( (\nu) \), where \( \gamma \) and \( \nu \) are related by

\[
\nu \frac{\partial^2 u'}{\partial z^2} \sim -\nu \lambda^2 u' \sim -\frac{\nu \lambda^2}{gh} u' \sim -\gamma u'.
\]  

(4.1)

Since the slope of the dissipation curve is greater than the slope of the instability curves, a fixed eddy viscosity will stabilize the small \( h \) fast growth rate modes, leaving the asymmetric, intermediate-vertical scale instabilities to effectively oppose the unstable shear.

<table>
<thead>
<tr>
<th>Profile</th>
<th>( \epsilon )</th>
<th>( \sigma_{11} )</th>
<th>( h_1 (m) )</th>
<th>( s_1 )</th>
<th>( \gamma_1 \epsilon )</th>
<th>( \sigma_{12} )</th>
<th>( h_2 (m) )</th>
<th>( s_2 )</th>
<th>( \sigma_{30} )</th>
<th>( \sigma_{13} )</th>
<th>( h_3 (m) )</th>
<th>( s_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA</td>
<td>( 8.79 \times 10^{-2} )</td>
<td>( 8.26 \times 10^{-2} )</td>
<td>( 1.1 \times 10^{-3} )</td>
<td>( 0 )</td>
<td>( 5.86 \times 10^{-2} )</td>
<td>( 4.15 \times 10^{-2} )</td>
<td>( 1.1 \times 10^{-3} )</td>
<td>( 20 )</td>
<td>( 7.11 \times 10^{-2} )</td>
<td>( 8.90 \times 10^{-3} )</td>
<td>( 100 )</td>
<td>( 25 )</td>
</tr>
<tr>
<td>IB</td>
<td>( 4.63 \times 10^{-1} )</td>
<td>( 4.41 \times 10^{-2} )</td>
<td>( 1.1 \times 10^{-4} )</td>
<td>( 0 )</td>
<td>( 3.09 \times 10^{-2} )</td>
<td>( 2.70 \times 10^{-2} )</td>
<td>( 1.1 \times 10^{-3} )</td>
<td>( 25 )</td>
<td>( 4.24 \times 10^{-2} )</td>
<td>( 4.00 \times 10^{-3} )</td>
<td>( 100 )</td>
<td>( 25 )</td>
</tr>
<tr>
<td>IIA</td>
<td>( 5.96 \times 10^{-1} )</td>
<td>( 5.73 \times 10^{-1} )</td>
<td>( 1.1 \times 10^{-1} )</td>
<td>( 0 )</td>
<td>( 3.98 \times 10^{-1} )</td>
<td>( 3.75 \times 10^{-1} )</td>
<td>( 1.1 \times 10^{-1} )</td>
<td>( 6 )</td>
<td>( 4.87 \times 10^{-1} )</td>
<td>( 6.51 \times 10^{-3} )</td>
<td>( 5000 )</td>
<td>( 8 )</td>
</tr>
<tr>
<td>IIB</td>
<td>( 2.39 \times 10^{-1} )</td>
<td>( 2.24 \times 10^{-1} )</td>
<td>( 1.1 \times 10^{-2} )</td>
<td>( 0 )</td>
<td>( 1.59 \times 10^{-1} )</td>
<td>( 1.45 \times 10^{-1} )</td>
<td>( 1.1 \times 10^{-2} )</td>
<td>( 8 )</td>
<td>( 2.67 \times 10^{-1} )</td>
<td>( 3.56 \times 10^{-3} )</td>
<td>( 5000 )</td>
<td>( 6 )</td>
</tr>
</tbody>
</table>
Table 3. Calculation of \( s_{1/2} \), critical equivalent depths, growth rates, and viscosity for the four jet profiles. See text for details.

<table>
<thead>
<tr>
<th>Profile</th>
<th>( h ) (m)</th>
<th>( \sigma_1 ) (0)</th>
<th>( \delta_1 ) (0)</th>
<th>( \sigma'_1 ) (0)</th>
<th>( s_{1/2} )</th>
<th>( h_{\text{crit}} ) (m)</th>
<th>( \sigma_1 ) (0)_{\text{crit}}</th>
<th>( \delta_1 ) (0)_{\text{crit}}</th>
<th>( \nu_{\text{crit}} ) (m² s⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA</td>
<td>0.001</td>
<td>.08256</td>
<td>.939</td>
<td>(-3.70 \times 10^{-5})</td>
<td>47</td>
<td>0.0134</td>
<td>.0673</td>
<td>.766</td>
<td>(1.29 \times 10^{-2})</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>.07074</td>
<td>.805</td>
<td>(-2.69 \times 10^{-5})</td>
<td>51</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>.03286</td>
<td>.374</td>
<td>(3.36 \times 10^{-4})</td>
<td>—</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IB</td>
<td>0.0001</td>
<td>.04413</td>
<td>.953</td>
<td>(-1.11 \times 10^{-5})</td>
<td>63</td>
<td>0.00138</td>
<td>.0376</td>
<td>.812</td>
<td>(2.04 \times 10^{-3})</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>.03927</td>
<td>.848</td>
<td>(-7.83 \times 10^{-6})</td>
<td>71</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>.02175</td>
<td>.470</td>
<td>(8.79 \times 10^{-5})</td>
<td>—</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIA</td>
<td>0.1</td>
<td>.57329</td>
<td>.962</td>
<td>(-2.35 \times 10^{-3})</td>
<td>16</td>
<td>1.91</td>
<td>.481</td>
<td>.807</td>
<td>(1.31 \times 10^{-1})</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>.52089</td>
<td>.874</td>
<td>(-1.78 \times 10^{-3})</td>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.0</td>
<td>.33044</td>
<td>.554</td>
<td>(7.82 \times 10^{-3})</td>
<td>—</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIB</td>
<td>0.01</td>
<td>.22445</td>
<td>.939</td>
<td>(-6.31 \times 10^{-4})</td>
<td>19</td>
<td>0.102</td>
<td>.189</td>
<td>.791</td>
<td>(2.76 \times 10^{-1})</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>.18945</td>
<td>.793</td>
<td>(-6.72 \times 10^{-3})</td>
<td>53</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>.10224</td>
<td>.428</td>
<td>(8.75 \times 10^{-3})</td>
<td>—</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using (4.1) and assuming the dissipation rate equals the growth rate at \( h_{\text{crit}} \) [i.e., \( \gamma = \sigma_1(0)_{\text{crit}} \)], we obtain the following expression for the dimensional eddy viscosity needed to neutralize the symmetric mode at \( h_{\text{crit}} \).

\[
\nu_{\text{crit}} = \sigma_1(0)_{\text{crit}}(2\Omega) \frac{g h_{\text{crit}}}{N^2} \tag{4.2}
\]

Assuming a typical tropospheric value for \( N^2 \) (10⁻⁴ s⁻²) in (4.2), Table 3 lists the value of \( \nu_{\text{crit}} \) for each jet profile. Since dissipation decreases with increasing vertical scale, as discussed above, it follows that these values of \( \nu_{\text{crit}} \) will stabilize the symmetric modes for \( h < h_{\text{crit}} \), while the asymmetric modes will still be unstable.

We now infer the relationship between vertical scale of the unstable perturbation and zonal wavenumber of greatest instability. Figure 10 shows how zonal wavenumber of preferred instability (\( s_m \)) is related to equivalent depth. The ordinate is \( \tilde{s} = s_m(\theta_N - \theta_0) / \cos \theta_m \), where \( \theta_m = (\theta_S + \theta_N) / 2 \) is the latitude of the center of the inertially unstable region. With this normalization of wavenumber, \( \tilde{s} = 1 \) indicates a perturbation zonal length scale (wavelength/2\( \pi \)) equal to the geometric width of the unstable region [\( a(\theta_N - \theta_0) \)]. The abscissa is \( h/h_{\text{crit}} \) on a logarithmic scale. By definition, for \( h < h_{\text{crit}} \) the symmetric (\( s = 0 \)) mode is preferred. We see that as \( h \) exceeds \( h_{\text{crit}} \) by a factor of 10–1000, \( \tilde{s} \) becomes \( O(1) \) and the preferred zonal scales approximate the cross-jet scale of the unstable region. To the extent that

![Fig. 9. Normalized growth rate (\( \delta_1 \)) of maximum instability versus \( \log(h/h_{\text{crit}}) \) for jets IA, IB, IIA and IIB. See text for definition of \( \delta_1 \) and \( h_{\text{crit}} \). Dashed curve shows the dependence of linear dissipation on vertical scale for a fixed eddy viscosity such that \( \gamma = \sigma_1(0) \) at \( h = h_{\text{crit}} \).](image1)

![Fig. 10. Normalized zonal wavenumber (\( \tilde{s} \)) of maximum instability versus \( \log(h/h_{\text{crit}}) \) for jets IA, IB, IIA and IIB. See text for definitions of \( \tilde{s} \) and \( h_{\text{crit}} \).](image2)
these modes are realized in nature, the instability is highly asymmetric.

5. Summary and conclusions

In this paper we have investigated the preferred instabilities for horizontally sheared flow away from the equator; i.e., for cases in which the region of inertial instability (\( \tilde{f} \tilde{\eta} < 0 \)) does not include the equator. For these jet profiles, Rayleigh's necessary condition for barotropic instability is always satisfied. However, we have found that the inertial instability has a significantly greater growth rate. These internal modes would therefore dominate the barotropic processes when the inertial instability condition is satisfied.

In the absence of dissipation, symmetric perturbations at an infinitesimal vertical scale approach the theoretical maximum possible growth rate, which is given by the largest geometric mean of the modified Coriolis parameter and the negative of the absolute vorticity. The symmetric perturbations have minimum cross-stream scale and structure.

At finite but often rather small vertical scale, nonsymmetric (\( s \neq 0 \)) instabilities are preferred over the symmetric instability. These instabilities could be readily excited in nature when dissipation stabilizes the symmetric modes of smallest vertical scale. This result corroborates and provides a likely explanation for the development of asymmetries in an initially symmetric model hurricane, as hypothesized by Anthes (1972); his model contained relatively coarse vertical resolution, which disallowed the growth of symmetric perturbations.

For symmetric modes, the meridional pressure gradient, which develops as the vertical scale increases, acts to oppose the instability. Hence the growth rate decreases rapidly as the equivalent depth increases. For nonsymmetric instabilities, the pressure gradients that develop cause acceleration and wave propagation in the zonal direction. In comparison nonsymmetric pressure gradients are less effective in reducing the instability. Therefore at finite vertical scale, the nonsymmetric modes grow faster than the symmetric.

Even for small vertical scales, the growth rate curves are quite flat as a function of \( s \). For arbitrary initial conditions, we therefore do not expect symmetric overturning to dominate the circulation at either infinitesimal or finite amplitude. Consequently, we recommend that inertial instability, at least of horizontally sheared flows, not be considered or referred to as symmetric instability.

With the horizontally sheared basic flow and meridionally limited region of instability, the unstable perturbations tend to be confined in all three space dimensions: both horizontal scales approximate the finite meridional scale of the unstable region and the vertical scale is minimized until dissipation counteracts the growth. Consequently, we suspect that our results may be applied at least qualitatively to more general flows—e.g., to situations in which the mean flow varies slowly in the vertical and jet axis directions.

In the case of a basic state with horizontal and/or vertical shear [\( \tilde{u}(y, p) \)], the basic state potential vorticity \( \tilde{f} \tilde{P} \) replaces the absolute vorticity in determining the instability criterion; cf. Stevens (1983). A necessary condition for inertial instability is then \( f\tilde{P} < 0 \) somewhere in the fluid where \( f\tilde{P} \) is defined as follows for geostrophic flow

\[
f\tilde{P} = \tilde{\Theta} \left( \frac{\partial u}{\partial p} \right)^2 \left( \frac{\partial \tilde{u}}{\partial p} \right)^2 \tilde{\eta} \left\{ \frac{\eta}{f} \right\} \left( \frac{\eta}{f} \right) - 1.
\]

where

\[
Ri = \frac{N^2}{(\rho g \partial u / \partial p)^2}
\]

is the Richardson number, and \( \tilde{\Theta} \) and \( \tilde{\eta} \) are the basic state potential temperature and density, respectively. The results of our analyses which considered only horizontally sheared flow, contrast significantly with those of Stone (1966, 1970), who found that symmetric disturbances are preferred when the inertial instability is derived solely from the vertical shear (\( \partial \tilde{u} / \partial p \)) in the mean flow. To understand the connection between this present work and that of Stone's, we are in the process of extending our analysis scheme to accommodate both horizontal and vertical shear. However, in light of this present work we suggest that inertial circulations due to horizontal shear, as in the outflow regions of hurricanes (Ooyama, 1969, and Holland and Merrill, 1984), could be strongly affected by asymmetric instabilities.

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REFERENCES


