Stochastic Forcing and Prediction of Low-Frequency Planetary Scale Flow

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ABSTRACT

A dynamical model incorporating observed field data is used to estimate the potential importance of linear and nonlinear vorticity advection to climate forecast models. Forecasts of 30-day averages benefit from inclusion of the linear advection term, but the nonlinear advection appears only marginally helpful. For intermediate averaging times (e.g., 10 days), both advection terms appear to be important. Analysis of the nonlinear terms suggests that they could be most adequately parameterized as a noise process that is "white" in wavenumber space and "red" in the time domain.

1. Introduction

In the course of developing a skillful, yet efficient climate forecasting model, one wonders what role nonlinear dynamics will play in the predictive skill of the model. Clearly a linear model would be simple to interpret—yet would it be as skillful as its nonlinear counterpart? Obtaining a partial answer to this question is the main thrust of this paper. A secondary goal is to suggest the general mathematical framework in which to parameterize the nonlinear terms assuming they are required in a model but are not computed explicitly.

Our primary question, predictability resulting from the nonlinear advection of vorticity, is apparently not discussed much in the literature. Cooley (1958), using 31 days of geopotential height data, computed the advection and then used it in a statistical model to see if the prediction of 12- and 24-hour values of 500 and 1000 mb heights were enhanced. He concluded that nonlinear terms contributed little or nothing to the forecast skill. Lorenz (1977), however, pointed out that Cooley had used an instantaneous value of the nonlinear term in his model. Allowing for the integrable nature of the effects of these terms in predicting future atmospheric conditions, Lorenz was able to show that the inclusion of the nonlinear term did significantly enhance the skill of a 24-hour prediction.

Recently Egger and Schilling (1983, 1984), investigating the nature of low frequency variability in the atmosphere, used observations of geopotential height to construct time series of the vorticity advection terms. Analyzing the statistical properties of these series, they conclude that the nonlinear terms can be characterized in the time domain as a red noise process. Hence, they conclude that a substantial fraction of the low frequency variability in the atmosphere can be characterized as a response to random forcing, although they explicitly did not address the problem from a "prediction" point of view. Accepting their results, and the results of Gambo (1982), suggests that a formulation of the climate prediction problem in terms of Langevin's equation (cf. Hasselmann, 1976) would be entirely appropriate, i.e., the climate system can be characterized as a system with memory forced by white noise.

The above ideas are further supported by the work of Kruse (1983) and, particularly, Bruns (1986), both of whom investigated the importance of the nonlinear advection terms to short term and monthly averaged forecasts. Their results suggest that nonlinear terms are of little help for the longer period forecast interval, although, as the authors point out, both analyses have some problems at these low frequencies. At shorter forecast lead times, they find that the nonlinear terms add significantly to the forecast skill.

These results suggest that the inclusion of the nonlinear advection terms will do little to improve the predictive skill of a model on climatic time scales. Indeed, if Egger and Schilling are correct, the variability introduced by these processes can be approximated by simply adding properly scaled red noise as a source term in the vorticity equation. This would give levels of variance in model results similar to those observed in the real atmosphere, but would do nothing for a model's predictive skill.

Parameterization of the nonlinear terms is often expressed as a diffusive approximation (e.g., White and Green, 1983). If the above results regarding the quasi-random nature (in time) of the nonlinear terms is correct, then such parameterizations are probably not wholly appropriate (cf. Egger and Schilling, 1984). As noted before, a simple noise model in the time domain would be adequate. However, none of the authors examined the structure of the nonlinear processes in wavenumber space and that information, presented in section 5, is important in deciding the form of the parameterization, if one is required.

The following sections outline the data and theoretical strategy of this paper. The next section investigates the skill associated with different components of the vorticity advection terms as they attempt to specify time-averaged properties of the vorticity field. A strict prediction framework is used to carry through this analysis, and so the method is distinctly different but complementary to that of Egger and Schilling (1984). Next the frequency–wavenumber structure of the nonlinear advection terms is investigated. The last section presents conclusions and shortcomings of this work.

2. Data

The data set used in this study consists of seven years (1969–75) of the symmetric spherical harmonics for the Northern Hemisphere 500 mb height field. An additional year of information was not used because of the nonhomogeneous nature of these data in the smaller scales, an apparent result of a change in analysis technique. The height fields (G) were available at half-day intervals. This data set was provided by G. Hannoschoech of the Max-Planck Institut für Meteorologie, although it originated from Speth and Kirk (1981).

The heights were converted to streamfunctions using the linear balance relationship

$$\nabla \cdot f \nabla \psi = \nabla^2 g G, \tag{1}$$

where g is the gravitational acceleration (9.8 m s^{-2}) and G is the height of the 500 mb surface. A description of the recursion relationships needed for this calculation is given by Lorenz (1977). The height field was triangularly truncated at $N = 15$, $M = 15$. The derived streamfunction field was then truncated at $N = 16$, $M = 15$ with only odd harmonics. Explicitly

$$\psi(y, \lambda) = \sum_{m=0}^{M} \sum_{n=m+1}^{N} (\psi c_n{}^m \cos m\lambda + \psi s_n{}^m \sin m\lambda) P_n{}^m \tag{2}$$

where $\psi c_n{}^m$ and $\psi s_n{}^m$ are the cosine and sine coefficients. The $P_n{}^m$ are normalized such that the areal average of the variance is simply the sum of the squares of the individual ψ's.

3. Theoretical approach

The basic idea of this work is to use the data set described in section 2 to compute time series of the various streamfunction advection terms. These series will then be integrated to see how well they can specify subsequent changes in the vorticity field. The use of the prediction formalism to evaluate the role of the advection terms in climate modeling is conceptually different than the recent work described above; but it is similar in principle to the strategy used by Lorenz (1977) for the short-term forecast problem. In addition, forecasts of time averages by this method will be examined. Time averaging helps to increase predictability times, in part, by removing some of the random high frequency noise.

The parameterization of the nonlinear Jacobian will then be investigated by estimating the normal modes of the nonlinear advection in the wavenumber/time domain. This will be done via a complex empirical orthogonal function (CEOF) analysis described in section 3c.

a. Model

It is assumed that the inviscid quasi-geostrophic nondivergent barotropic vorticity equation represents the internal atmospheric dynamics of interest. Given this reference we are interested in the balance

$$\frac{\delta}{\delta t} \nabla^2 \psi = -J(\psi, \nabla^2 \psi + f), \tag{3}$$

where J is the Jacobian operator. In terms of streamfunction

$$\frac{\delta}{\delta t} \psi = -\nabla^{-2} J(\psi, \nabla^2 \psi + f).$$

Specifically, we are interested in the ability of integrals of J to explain changes in ψ, i.e.,

$$\int_0^{T_1} \frac{\delta \psi}{\delta t} dt = -\int_0^{T_1} \nabla^{-2} J(\psi, \nabla^2 \psi + f) dt,$$

which can be expressed in integral form as

$$\psi(T_1) = \psi(0) - \int_0^{T_1} \nabla^{-2} J(\psi, \nabla^2 \psi + f) dt. \tag{4}$$

When a forecast of a time average is required, we have

$$\frac{1}{T_2} \int_0^{T_2} \psi(T_1) dT_1$$

$$= \psi(0) - \frac{1}{T_2} \int_0^{T_2} \int_0^{T_1} \nabla^{-2} J(\psi, \nabla^2 \psi + f) dt dT_1. \tag{5}$$

If finite differences are used to express the time derivatives and integrals, then a forward difference approximation gives

$$\psi(\Delta t) = \psi(0) - \Delta t \nabla^{-2} J(0)$$

$$\psi(2\Delta t) = \psi(\Delta t) - \Delta t \nabla^{-2} J(\Delta t)$$

$$= \psi(0) - \Delta t \nabla^{-2} [J(0) + J(\Delta t)]$$

$$\psi(q\Delta t) = \psi(0) - \Delta t \nabla^{-2} \sum_{r=0}^{q-1} J(r\Delta t),$$

where Δt is the time between observations. (Note that the quantities in parentheses for the Jacobians refers here to the time at which the evaluation of the Jacobian is done.) Similarly the forecast of the time average is

$$\frac{1}{Q}\sum_{q=1}^{Q}\psi(q\Delta t)=\psi(0)-\frac{\Delta t\nabla^{-2}}{Q}\sum_{q=0}^{Q-1}\sum_{r=0}^{q}J(r\Delta t). \quad (6)$$

The total Jacobian is separated into linear and nonlinear components as follows: The streamfunction is separated first into a zonal mean component and a wave component by

$$\psi=\bar{\psi}+\psi'$$

where $\bar{\psi}$ is the zonal mean and ψ' is the deviation from the zonal mean. Substituting this expression into the Jacobian gives

$$J(\psi,\nabla^2\psi+f)=\{J(\bar{\psi},\nabla^2\psi')+J(\psi',\nabla^2\bar{\psi}+f)\} + \{J(\psi',\nabla^2\psi')\}=J_L+J_{NL}, \quad (7)$$

where J_L and J_{NL} refer to the linear and nonlinear Jacobians respectively.

The Jacobians were estimated for each spectral component via spectral transform techniques. An excellent description of this calculation can be found in Machenhauer (1979). Previously coded versions of the transform method provided by Egger, Tribbia and Washington were also used to check the new calculations made for this study. The final results of these operations were time series of J_L and J_{NL} every 12 hours for seven years. These series will be used in (6) to estimate the relative importance of the Js to future values of ψ.

b. Skill measure

The skill measure used to determine the relative importance of the Jacobians is the complex correlation function defined by

$$C=\frac{\langle\psi_0\hat{\psi}^*\rangle}{\langle\psi_0\psi_0^*\rangle^{1/2}\langle\hat{\psi}\hat{\psi}^*\rangle^{1/2}}, \quad (8)$$

where $\hat{\psi}$ is the streamfunction forecast of the time averages from the right-hand side of (6), ψ_0 the observed time averaged streamfunction from the left-hand side of (6) and asterisks denote complex conjugation. In this notation, the ψ_0 and $\hat{\psi}$ (i.e., the Js) are composed of the cosine/sine coefficients of their spectral expansion (cf. section 2) and hence expressed as complex quantities. The wavenumber indices (m, n) have been suppressed for simplicity. The squared modulus of C is

$$R=CC^* \quad (9)$$

and represents the fraction of variance in ψ that can be explained by the Jacobians. The distribution of R is evaluated in wavenumber space for different forecast averaging times and provides the answer to one of the fundamental questions addressed in this paper.

c. Structure of the J_{NL} in wavenumber space

Effective parameterization of J_{NL} requires knowledge of its dependence on wavenumber (m, n). A simple way to investigate this dependence is through a CEOF decomposition of the spectral distribution of J_{NL}. The notation can be simplified by dropping the subscript NL and compressing the sub/superscript double (m, n) into a single subscript "i". The eigenmodes of the data set were found from the Hermitian covariance function

$$CV_{ij}=\langle J_i^* J_j\rangle, \quad (10)$$

which has real eigenvalues λ_k and complex eigenvectors B_k. The spectrum of the λ_k can be used to characterize various modes of CV as either "noise" or "nonnoise" (Preisendorfer et al., 1981) and to investigate the statistical degeneracy between adjacent eigenmodes (e.g., see Horel, 1984). For our present purposes a "state vector" representation will be useful for discussing the degree of coherence between the J_{NL} in the wavenumber domain. The state vector is conveniently expressed in polar form as

$$V_k=S_k e^{i\phi_k} \quad (11)$$

where

$$S_k=[B_kB_k^*]^{1/2}$$

$$\phi_k=\tan^{-1}\frac{\mathrm{Im}B_k}{\mathrm{Re}B_k}$$

where it is understood that the B_k carry an implicit wavenumber dependence. This information, together with the uniqueness of the λ_k and principal components of the J-set provide much of the data needed to empirically parameterize the nonlinear Jacobians.

4. Predictability due to Jacobians

The goal of this section is to estimate the relative importance of the Jacobians to the hindcast skill of a climate forecast model. The procedure is straightforward. We first estimate the relative skill of making an N-day average forecast of the spectral components of ψ from knowledge of the value of the component at an initial time (ψ_0). This is equivalent to the quasi-persistence model discussed by Roads and Barnett (1984) since the complex correlation is being used here to estimate the skill.

The next three steps involve the use of (6) and (8) with the addition of the J_L, J_{NL}, or $J_L + J_{NL}$, respectively. The results show the skill of a model that included "perfect" knowledge of linear vorticity advection, nonlinear advection and finally the total advection (the use of perfect ignores the errors associated with observed data and the use of spectrally truncated terms in the nonlinear calculations). These results essentially represent a "specification" as opposed to a true "prediction" and in a sense are an unfair comparison to the persistence model, which is a true prediction.

However, if perfect knowledge of a particular physical process does not help substantially the specification of future time averages of ψ, then one may well decide to omit that process from a prediction model.

a. Results

The essence of the results is illustrated in Figs. 1–3, which show the specification skill (9) of the four models for three different averaging times. The principal results are as follows.

The major skill is given by persistence at a forecast lead time of 1 day (Fig. 1a). The addition of either component of the Jacobian only has a modest effect for wavenumbers below $m \sim 4$, $n \sim 9$. However, increased skill is apparent for large n (>15) for all m. This point will be addressed later. The J-terms actually reduce the skill at the highest M-wavenumbers by a few percent, e.g., reduction from 20% to 10% near $m = 12$, $n = 14$. The inclusion of the full set of Jacobians (Fig. 1d) does little, if any, better than either the linear or nonlinear terms by themselves over most of the wavenumber range. However, in the energy containing modes ($m \sim 1$, $n \sim 6$) there is a modest increase in skill, a result in agreement with Lorenz (1977).

A large fraction of the skill of a 10-day average forecast still comes from persistence (Fig. 2a). The overall skill obtained by adding both advection terms (Fig. 2d) is appreciably better, however. The addition of J_L increases skill for the ultralong waves (e.g., $m = 1$, $n = 2$ (Fig. 2b)) but the inclusion of the J_{NL} tends to obscure the skill in that region associated with J_L. However, J_{NL} is seen to contribute appreciable skill at all higher wavenumbers.

Forecasts of 30-day-averaged streamfunctions obtain modest skill from persistence alone, a result well documented elsewhere (e.g., Roads and Barnett, 1984). Addition of the linear advection term in the integral of (5) produces a substantial increase in skill in wavenumber regions noted above (Fig. 3b). Again a minimum skill occurs in (m, n) space in the vicinity of $m \sim 4$, $n \sim 11$. This feature is discussed later. The variable J_{NL} adds skill over that expected from persistence (Fig. 3c), particularly in the general region of wavenumber space where the skill due to J_L tended to be small. But the skill obtained using the total Jacobian (Fig. 3d) is actually less than that with persistence and J_L alone. Hence the effect of J_{NL} on skill is dichotomous.

b. Discussion

Specification of time averages obtain a large fraction of the skill from the linear advection terms over and above that expected from persistence alone.[1] The skill contribution due to J_{NL} is modest, at best, in most of the wavenumber domain. However, there are limited regions where the inclusion of the J_{NL} can be appreciable. These regions generally occur on the high wavenumber side of the most energetic regions of the ψ-spectrum. In contrast, the J_{NL} can also detract from skill in other regions of wavenumber space. These facts suggest that models designed for specific regions of wavenumber space may need to include or exclude the J_{NL} depending on the wavenumbers to be resolved. Alternatively, one can envision a statistical–dynamical model that adequately weights the J_{NL} where they are useful and ignores them otherwise.

Part of the skill of the linear Jacobian, however, may be artificial. Note that a large increase in skill is present at the truncation limits. This increase in skill was investigated further and found to be due to the term $J(\bar{\psi}, \nabla^2\psi')$. If the data set was truncated even further to include only terms from the first m, $n = 13$ then the skill was not increased at the new truncation limits. Thus we conclude that the Jacobian calculations (using finite truncations) do not cause this increase in skill at the largest values of n and the cause of the large skill increase must be in the data at the truncation limits. This different truncation does not specifically affect the other conclusions noted above.

5. Structures of J_{NL}

Results of the previous section suggest knowledge of the nonlinear Jacobian will improve forecasts of short term, multiday (e.g., 10 day) averages. Further, even though this term is not particularly helpful in forecasting longer term time averages it may still be desirable to include in a model its contribution to the streamfunction variance. If it is included, the computed fields will then have variance comparable to the real atmosphere. These two ideas motivate us to search for a simple way to parameterize J_{NL}.

a. Structure in wavenumber space

The procedures of section 3c were applied to the time series of the 72 spectra components of the J_{NL}. Each series contained 2555 daily values. The results of the complex EOF analysis of the correlation matrix of this data were as follows.

The leading eigenvalue of the covariance matrix (10) accounted for 3.2% of the variance in the data set while the second eigenvalue captured 3.1% and so on. The eigenvalues are clearly not distinct from each other (cf. Horel, 1984, for the test criteria). Further, application of several of the filtering rules discussed in Preisendorfer et al. (1981) showed the eigenvalue spectrum to be indistinguishable from that expected from a white noise field, i.e., 72 different time series, each of which is uncorrelated with the others.

As might be expected from the above result, the distribution of the components of the lowest order state

[1] Persistence was deliberately left in (6) to reproduce, as closely as possible, the actual integration of a model started from a given initial condition.

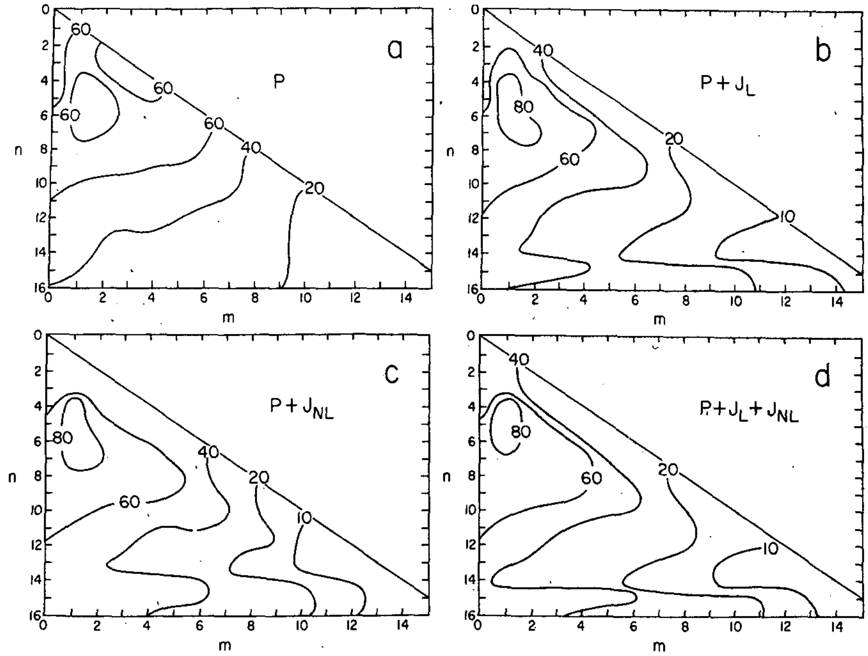

FIG. 1. Specification (% variance) for one-day averages of spectral components of the streamfunction due to (a) persistence, (b) persistence and the linear part of the Jacobian, (c) persistence and nonlinear part of the Jacobian, and (d) persistence and full Jacobian.

vector (11) showed no coherent structure among the (m, n) components in wavenumber space. This is illustrated in Fig. 4 where one can see that the magnitude of the state vectors is essentially independent of (m, n). Similarly, the relative phase, given by the direction of the vectors, appears random. These results clearly show the spectral distribution of the J_{NL} to be essentially white.

While the behavior of the J_{NL} may essentially be independent of (m, n) one should not expect a similar

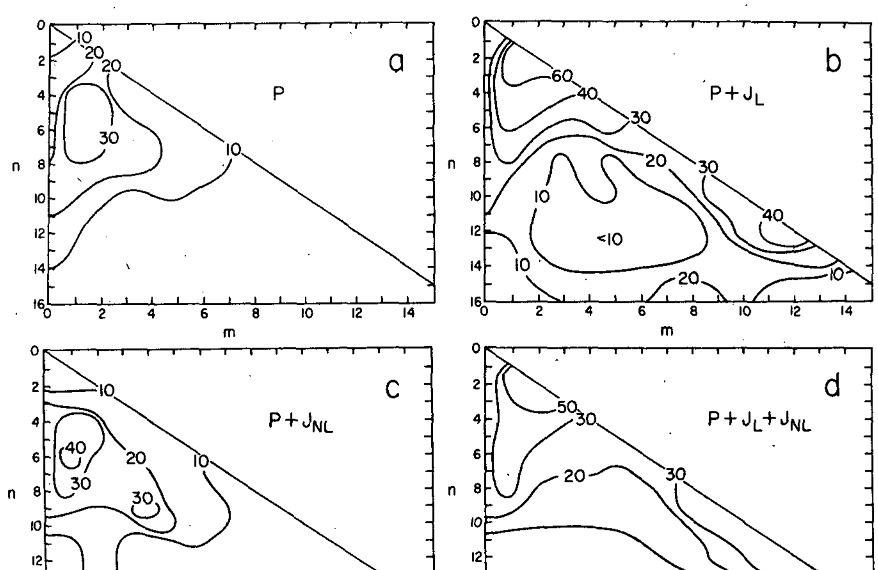

FIG. 2. As in Fig. 1 but for 10-day averages.

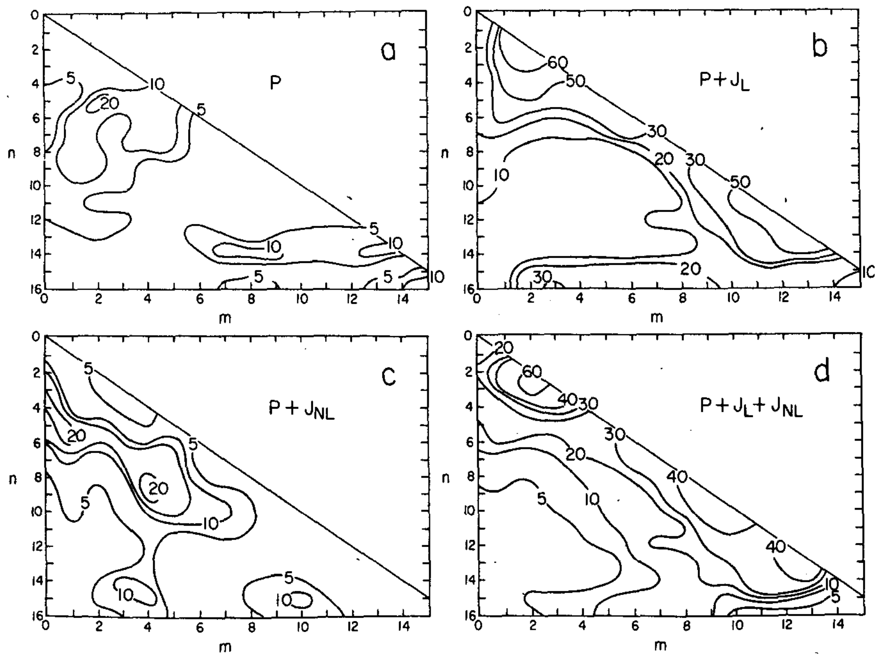

FIG. 3. As in Fig. 1 but for 30-day averages.

behavior for their energy (represented here by $J_{NL}J^*_{NL}$). Repeating the preceding analysis on this quantity showed that the first eigenmode was associated with 19% of the variance in the energy field and was statistically distinct from the higher modes. The associated (real) eigenvector (Fig. 5) shows a highly coherent and uniform structure in (m, n) of the J_{NL} energy field. Variations in this quantity for the wavenumbers near the diagonal $m = n$ are out of phase with those for $n \gg m$, i.e., the zonal Jacobians. Thus, scaling factors associated with a parameterization of the J_{NL} could include some (m, n) dependence to take account of the total energetics of the J_{NL}-field. However, the relatively low variance of the first eigenmode suggests this would be a second-order correction to any parameterization of J_{NL}.

In summary, the behavior of the J_{NL} appear independent of (m, n). Any patterns that might be discerned in Fig. 4 are highly unstable due to the degeneracy of the eigenvalues. The same results were obtained by analyzing the correlation matrix of the J_{NL}. However, while the wavenumber behavior of the J_{NL} are unrelated, their energy is not. There is a limited measure of correlation in the (m, n) distribution of the magnitudes of the J_{NL}, but this effect is of second-order importance to a parameterization of the J_{NL}.

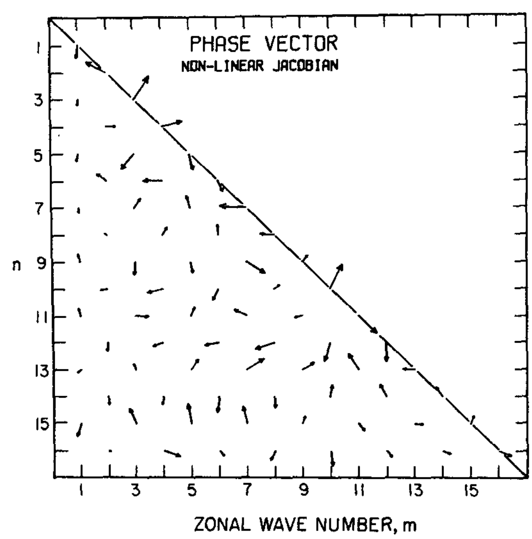

FIG. 4. Spectral distribution of state phase vector [Eq. (11)] for complex EOF decomposition of nonlinear Jacobian.

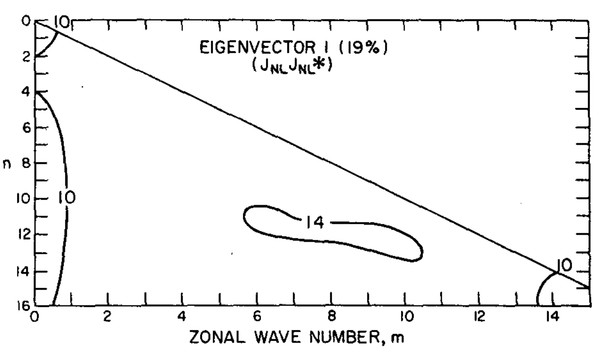

FIG. 5. Spectral distribution of mode 1 eigenvector from EOF decomposition of nonlinear Jacobian variance field.

b. Temporal structure

The temporal structure of the total Jacobian has been studied by Egger and Schilling (1983, 1984) for a limited region of (m, n)-space defined by rhomboidal truncation of order 5. They found the resulting advective terms were well modeled by a first-order Markov process that was independent of the planetary scale flow. A similar result was found by Gambo (1982).

For present purposes we have concentrated on the temporal behavior of J_{NL}. The autocorrelations of the detrended, daily time series were computed for the sine and cosine components of the J_{NL} separately. The lag 1 values for the cosine elements are displayed in Fig. 6. The values are quite comparable to those found by Egger and Schilling (1983, Fig. 1) for the total Jacobian. Following their conclusions, the correlation analysis was used to construct a first-order Markov model of the cosine elements of covariance function of the J_{NL} (cf. Jenkins and Watts, 1968, for the method). This model was subtracted from the actual data time series to give a "residual." The residuals were, in most cases, indistinguishable from white noise according to the criterion of Jenkins and Watts (1968, pg. 187).

In summary, the temporal behavior of the J_{NL} is rather well modeled by a simple first-order Markov process. This result agrees with that of Egger and Schilling for the full Jacobian.

c. Discussion

These results suggest that modeling the variance associated with the J_{NL} is rather straightforward. The temporal behavior corresponds to a first-order autoregressive (AR) process with appropriately scaled variance, i.e., red noise. Since the J_{NL} for each wavenumber are independent to first order, a separate (uncorrelated) AR process can be constructed separately for each (m, n). At a slightly higher level of sophistication, the scale factor (variance) of such representations could include consideration of the spectral distribution of the energy in the J_{NL} field. This procedure will effectively account for the rms variability in ψ caused by the J_{NL} forcing. It clearly will not do much to help predictive skill. In a more positive vein, these results suggest the practical relevance to climate dynamics of North's (1984) theoretical finding regarding the coincidence of EOFs and dynamical modes in simple systems driven by spatially white noise.

The structure of the J_{NL} in wavenumber space argues against the parameterization of the effect of these terms as a diffusion process. The behavior of the J_{NL}, on average, does not depend on the J_{NL} at neighboring wavenumbers. The structure of $\nabla^2\psi$ (not shown), which is often used in a diffusion parameterization, is quite different and does show coherent variations in (m, n). Thus, diffusion approximations that invoke such a dependence are not physically compatible with the nature of the process they seek to represent. All in all, the simplest representation of the nonlinear Jacobians in wavenumber space appears to be a noise model.

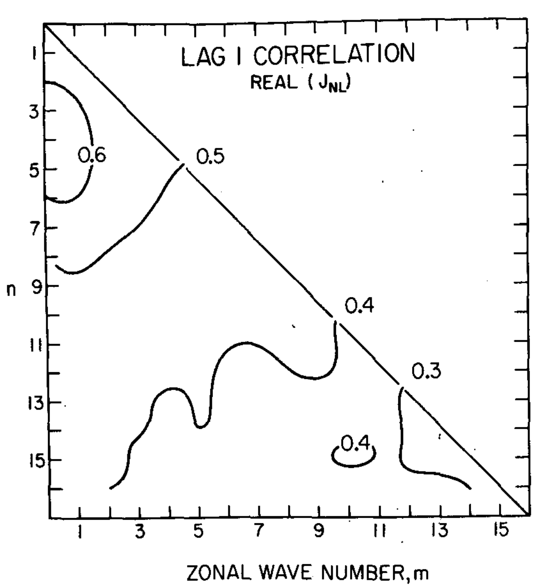

FIG. 6. Spectral distribution of autocorrelation for real part of nonlinear Jacobian.

6. Conclusions

The relative predictive importance of the linear and nonlinear advection terms has been investigated in the context of a spectral quasi-geostrophic barotropic vorticity equation. The advection terms have been estimated explicitly from actual data and then used in the dynamical framework to specify subsequent multiday average streamfunction data. Additionally, the problem of parameterizing the nonlinear advection term was investigated empirically. The results of these studies led to the following conclusions.

1) Forecasts of 30-day-averaged streamfunctions probably require that both persistence and linear vorticity advection be included in the prediction model. The nonlinear advection terms appear of modest importance for monthly average forecasts. These results suggest, but do not prove, the relevance of linear models to the climate forecast problem.

Several cautions are needed with this conclusion. It has been suggested by M. Cane (personal communication, 1985) that the role of the J_{NL} in governing atmospheric dynamics may be highly episodic (although limited time series plots of the J_{NL} used in this work do not suggest this). If this were the case, then our analysis methods would rather effectively obscure this fact. It is also important to remember that the analysis has really addressed the "specification" problem. It is possible, but unlikely, that the nonlinear terms could appear relatively unimportant to this problem but be relevant to the "forecast" problem (which we have not explicitly considered). Finally, the nonlinear terms examined here in the context of the spectrally truncated quasi-geostrophic, barotropic vorticity equation, may prove to be more important if analyzed by a more comprehensive model.

2) Both advection terms will likely add useful skill to the prediction of shorter term averages. In this regard we are in agreement with Lorenz (1977). This also indicates that appropriate filters, which weight most heavily the early days of a forecast, may ultimately help to increase the skill due to the Jacobians in the forecast of 30-day averages (see Roads, 1986).

3) The statistical character of the nonlinear advection terms can be characterized as white in wavenumber space and red in the time domain; the latter result in agreement with Egger and Schilling (1983, 1984). To model this behavior one can resort to standard procedures of time series analysis. However, such a parameterization will do little to enhance predictive skill aside from the persistence nature of the temporal variability.

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