NOTES AND CORRESPONDENCE

On the Transformations between Temporal and Spatial Growth Rates

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ABSTRACT

This note compares the error distributions for three transformation formulae between temporal growth rate and spatial growth rate with the linearized barotropic vorticity equation. The sech^2 and the tanh basic-state profiles are used for illustration. The transformation which uses the phase velocity gives a moderate error which does not have a strong dependence on the growth rate. The formulae derived by Gaster, and later by Nayfeh and Padhye, which employ the group velocity, have errors that are a function of the ratio of the spatial growth rate to the wavenumber. The errors from their formulae are small when the ratio is small, but the errors increase with the ratio so that all three transformation formulae give similar errors when the ratio is of order one. Nayfeh and Padhye's formula is rederived for the barotropic vorticity equation with a procedure which shows that ratio of growth rate to wavenumber must be small for accuracy.

1. Introduction

Basic currents in both the atmosphere and the ocean often have appreciable downstream variation. As a result, the traditional parallel flow stability theories, which employ temporal growth, cannot be directly applied to these flows. It has been shown by Tupaz et al. (1978), Williams et al. (1984) and Peng and Williams (1986) in studies of downstream, barotropic instability that spatial instability is more appropriate with downstream varying basic flows. The spatial growth exists when the basic flow does not support absolute instability. More relevant discussion concerning this is given in Peng and Williams (1987). They used spatial instability in the two-layer baroclinic model to help explain regions of enhanced cyclonic activity. Since spatial growth rates are likely to be useful in a variety of geophysical problems, it would be very desirable to have transformation formulae to obtain spatial growth rates from temporal growth rates or vice versa. In some studies it is easier to measure one type of growth rate than the other. Meanwhile, there are some cases when one type of instability has analytical solutions but not the other, so that a transformation formula would be very useful. In this note we evaluate the accuracy of some transformation formulae for geophysically relevant barotropic currents.

The temporally growing normal-mode solutions have complex frequencies or phase speeds and real wavenumbers, while the spatially growing solutions have complex wavenumbers and real frequencies. There is no general transformation formula which gives the exact relationship between the two types of growth rate. It was shown by Gaster (1965) that the simple transformation formula which depends on the phase speed (Schubauer and Skramstad, 1943) will introduce errors that are not negligible. Gaster (1962) demonstrated that for the same wavenumber, if the amplification rate is small, the frequencies for temporal and spatial growth are also equal with an error of the order of (maximum growth rate)^2. Under this condition, Gaster (1962) obtained a transformation from one type of growth rate to another in terms of the real group velocity. Only when the systems are nondispersive will the transformations using phase velocity and group velocity be the same. Recently, Nayfeh and Padhye (1979) reformulated this problem, and their results differ from Gaster's mainly in two aspects. First, the complex group velocity is used instead of the real group velocity as in Gaster's formula. Second, in order to obtain a proper comparison, a correction to the wavenumber (frequency) must be added when transforming the temporal (spatial) growth rate into the spatial (temporal) growth rate. The results they presented are very accurate, with almost all errors 2% or less.

In this note these approximations are compared for barotropic instability with special emphasis on the transformation by Nayfeh and Padhye (1979) whose limitations have not been previously discussed. Two different basic flows are studied. The first one is an easterly jet with a sech^2 profile, which approximates the jet that is observed south of the Tibetan Plateau at 200 mb during Northern Hemisphere summer. The second one is a tanh profile, which resembles the basic flow in the intertropical convergence zone.

In section 2, the three approximations will be compared for the two wind profiles. Some results from Nayfeh and Padhye will also be presented for comparison. In section 3, their formula will be rederived for the barotropic vorticity equation in order to better understand its range of validity.

2. Numerical comparisons

This study uses the linearized equation for conservation of vorticity in two-dimensional flow. The governing equation in nondimensional form is (§7.3, Pedlosky, 1979),

$$\left(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}+\lambda\right)\nabla^2\psi'+\left(\beta-\frac{\partial^2\bar{U}}{\partial y^2}\right)\frac{\partial\psi'}{\partial x}=0, \qquad (1)$$

where ψ' is the disturbance streamfunction, λ the frictional coefficient and β is the derivative of the Coriolis parameter (here λ and β are constants). The basic flow for the Bickley jet is described by

$$\bar{U}(y)=-a\,\mathrm{sech}^2\left(\frac{y}{b}\right) \qquad (2)$$

where a is the maximum speed and b is the half-width of the jet. The boundary conditions for the channel domain are

$$\psi'=0 \quad \text{at} \quad y=\pm D. \qquad (3)$$

For the ITCZ flow approximation, the basic flow is

$$\bar{U}(y)=0.5\tanh\left(\frac{y}{d}\right)+U_0 \qquad (4)$$

where U_0 is the constant basic state and the boundary conditions are that the streamfunction vanishes at infinity, i.e.,

$$\psi'=0 \quad \text{at} \quad y=\pm\infty. \qquad (5)$$

When the normal mode solution

$$\psi'=A(y)e^{i(\alpha x-\omega t)} \qquad (6)$$

is introduced, Eqs. (1) and (3) or (5) constitute an eigenvalue problem. If the frequency ω is a real number, the complex wavenumber gives a spatially growing wave. When the wavenumber α is a real number, complex frequency leads to a temporally growing wave. The parameters a, b in (2) and d in (4) will be altered to change the sharpness of the jet profile in order to obtain a large range of the instability level, which is measured by the amplification rate (growth rate). We will use this problem to study the errors that arise from transforming temporal to spatial growth rates using the different relations discussed in the Introduction.

In the following equations, subscripts r and i indicate the real and imaginary parts and ω_i is the temporal growth rate and α_i is the spatial growth rate. The first transformation formula uses the phase velocity, following Schubauer and Skramstad (1943),

$$\alpha_i=-\frac{\omega_i}{c_r} \qquad (7)$$

where c_r is the real phase velocity.

The second formula, which was proposed by Gaster (1962), uses the real group velocity,

$$\alpha_i(S)=-\frac{\omega_i(T)}{\partial\omega_r/\partial\alpha_r} \qquad (8)$$

where $\partial\omega_r/\partial\alpha_r$ is the real group velocity.

The third formula, which was derived by Nayfeh and Padhye (1979), is

$$\delta\alpha=-\frac{\omega_i}{\omega'} \qquad (9)$$

where $\omega'=\partial\omega/\partial\alpha$ is the complex group velocity. The real part of (9) is the transformation formula from temporal to spatial growth rate with the imaginary part of (9) being the required correction to shift the wavenumber for proper comparison. A more detailed derivation of (9) is given in section 3.

The computations of the transformed spatial growth rates for the sech^2 profile are summarized in Table 1, and compared with spatial growth rates from direct calculation. For the cases shown in Table 1, $\beta=-0.75$, $\omega=-0.75$ and $\lambda=-0.05$ nondimensionally in (1).

Column 1 in Table 1 is the real wavenumber and column 2 is the corresponding spatial growth rate. The positive sign for the imaginary part of the wavenumber in column 2 is due to the easterly jet specified in (2), which indicates spatial growth in the negative x direction. Columns 4, 6, and 8 are the spatial growth rates transformed from temporal growth rate using (7), (8) and (9), respectively. Columns 5, 7 and 9 give the relative errors for columns 4, 6 and 8 with respect to column 2. These errors are calculated, for instance, for G as

$$\text{error of } G=\frac{\left||\alpha_i|-|G|\right|}{|\alpha_i|}. \qquad (10)$$

The quantity $|\alpha_i/\alpha_r|$ which is listed in column 3 is the ratio between the amplification rate (α_i) and the local wavenumber (α_r). Although in the present case, α_i and $|\alpha_i/\alpha_r|$ are of the same order, variations of all the errors due to the different transformations are discussed as functions of $|\alpha_i/\alpha_r|$ instead of α_i. The reasons for this will be explained in section 3.

The errors introduced by using phase velocity for conversion (listed in column 5), are rather uniform irrespective of the size of $|\alpha_i/\alpha_r|$. As expected, the errors introduced by using Gaster's formula (column 7) increase rapidly as $|\alpha_i/\alpha_r|$ increases. However, the errors introduced by using Nayfeh and Padhye's formula (column 9) also increase rapidly as $|\alpha_i/\alpha_r|$ increases. The values in column 6 and column 8 are very close. For clearness, these relations are graphically shown in Fig. 1, in which the error distributions for these three transformation are plotted against the ratio $|\alpha_i/\alpha_r|$. The solid line is the error for the phase velocity formula, the dashed line is the error for Gaster's formula and

TABLE 1. Comparison of the growth rates and errors from three different approximations.*

1 Wavenumber α_r	2 Spatial growth rate α_i	3 Ratio $\lvert\alpha_i/\alpha_r\rvert$	4 From Eq. (5) (SS)	5 Error of SS	6 From Eq. (6) G	7 Error of G	8 From Eq. (7) (NP)	9 Error of NP
−1.51100	0.01353	0.0089	−0.01075	0.2055	−0.01316	0.0273	−0.01350	0.0022
−1.50556	0.04607	0.0306	−0.03636	0.2108	−0.04476	0.0284	−0.04547	0.0130
−1.49997	0.07982	0.0532	−0.06298	0.2109	−0.07720	0.0328	−0.07817	0.0207
−1.49361	0.11530	0.0772	−0.09093	0.2113	−0.11074	0.0395	−0.11188	0.0296
−1.47679	0.19164	0.1298	−0.15109	0.2116	−0.18094	0.0558	−0.18224	0.0490
−1.45168	0.27468	0.1892	−0.21610	0.2132	−0.25352	0.0770	−0.25509	0.0713
−1.39246	0.40581	0.2914	−0.31587	0.2216	−0.35848	0.1166	−0.36115	0.1091
−1.30670	0.52804	0.4041	−0.40186	0.2393	−0.44460	0.1580	−0.44982	0.1481
−1.24311	0.59223	0.4764	−0.44196	0.2537	−0.48502	0.1810	−0.49085	0.1712
−1.15617	0.65644	0.5677	−0.47594	0.2750	−0.52070	0.2068	−0.52558	0.1993

* SS: Schubauer and Skramstad (1943); NP: Nayfeh and Padhye (1979).

the dotted line is the error for Nayfeh and Padhye's formula. When the parameters ω, β and λ are varied over a reasonable range, the error distributions are similar to those in Fig. 1.

For the basic flow with tanh profile [Eq. (4)], the numerical calculations are made by taking the lateral boundaries at a large distance ($y = \pm 10$) instead of at infinity. The results obtained are very close to the calculations by Michalke (1965) for the same basic-state profile, indicating that it is sufficient for the present study. An example is given in Fig. 2, where $\omega = 0.3$, $\beta = 0.1$, and $\lambda = 0.01$. The magnitude of the growth rate is varied by changing d in (4). In this case, the error using the phase velocity (solid line) decreases with the ratio $|\alpha_i/\alpha_r|$ and then it increases again. Gaster's formula (dotted line) and Nayfeh and Padhye's formula (dashed line) have the same error distribution, with Nayfeh and Padhye's formula being slightly better.

Overall, when the ratio $|\alpha_i/\alpha_r|$ is small, the transformation which uses the phase velocity has large errors around 20–30%, and both Gaster's formula and Nayfeh and Padhye's formula give small errors. Errors from the latter formulae increase rapidly as $|\alpha_i/\alpha_r|$ increases and they are about equal. When the ratio is large, which is the more relevant situation for atmospheric instability, all three formulae give about the same error range.

As further illustration, we compute the transformed growth rate for the two-layer baroclinic model in a channel. The description of the model can be found in Pedlosky (1979). In this model, the transformed growth rate using either phase velocity or real part of the group velocity (Gaster's formula) give very satisfactory results with errors around 0.2–2%. This is because the system is only slightly dispersive so that the phase velocity is very close to the group velocity. When the system is nondispersive or weakly dispersive, both formulae using phase speed and real part of the group velocity are very accurate and it is not necessary to consider Nayfeh and Padhye's formula.

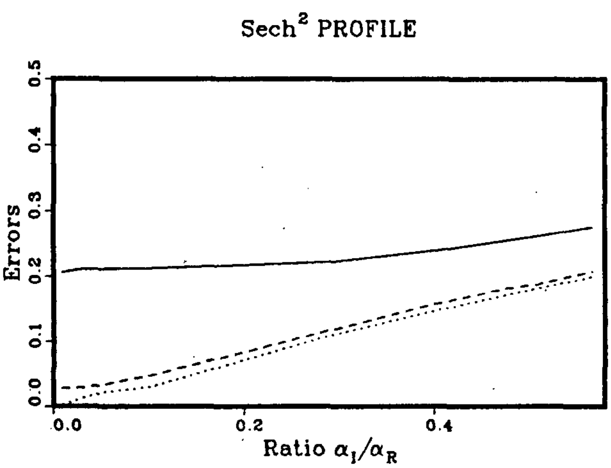

FIG. 1. Error distributions for a case with sech profile using different transformation formulae: Schubauer and Skramstad (solid line), Gaster (dashed line), and Nayfeh and Padhye (dotted line).

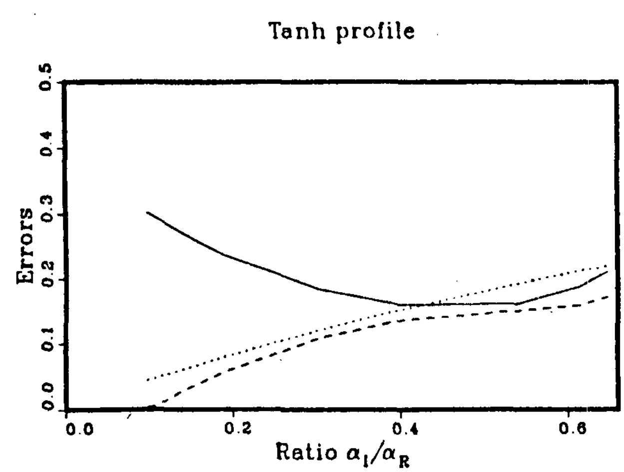

FIG. 2. As in Fig. 1 except for a case with tanh profile.

For comparison with our results, we extract data from Tables 1 and 2 of Nayfeh and Padhye to construct our Table 2, which contains their errors as a function of $|\alpha_i/\alpha_r|$ for each case. All the errors in Table 2 are small, ranging from 0.0052 to 0.0438, while the $|\alpha_i/\alpha_r|$ are also small, ranging from 0.00136 to 0.02325. This table shows that Nayfeh and Padhye's good results were obtained when $|\alpha_i/\alpha_r|$ was small.

As mentioned earlier, during the transformation from one type of growth rate to another, the frequency and the wavenumber cannot be kept the same. When transforming from temporal to spatial growth rate, results obtained in keeping the frequency the same are slightly better than keeping the wavenumber the same, i.e., a 2–4% improvement. The correction to the wavenumber indicated by Nayfeh and Padhye (1979) is very close to the adjustment of the wavenumber in order to obtain constant frequency.

3. Analysis of source of error

In this section we will discuss Nayfeh and Padhye's analysis in order to show that the error in their relation should depend on $|\alpha_i/\alpha_r|$. First we will rederive their formula for barotropic vorticity equation. The following analysis was used by Pedlosky (1979) to demonstrate the physical meaning of group velocity.

Consider the motion governed by (1) and (3). The total wave field viewed as a wave packet has two separate scales of oscillation. In the normal time scale, the local oscillation determines the period over which the wave amplitude appears to be constant. In the "slow" time scale, the amplitude of the wave field changes gradually. Due to the bounded channel domain in the y-direction, the amplitude has slow variation in the x-direction only. Two different slow scales can thus be introduced:

$$X=\epsilon x$$
$$T=\epsilon t \qquad (11)$$

where ϵ is a small parameter which is a measure of the slowness of the temporal and spatial variations of the field. The derivatives in (1) are replaced by

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T}$$
$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial X}, \qquad (12)$$

and (1) becomes

$$\left(\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T}+\bar{U}\frac{\partial}{\partial x}+\epsilon\bar{U}\frac{\partial}{\partial X}+\lambda\right)\times\left(\frac{\partial^2}{\partial x^2}+2\epsilon\frac{\partial^2}{\partial X\partial x}+\epsilon^2\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial y^2}\right)\psi'+\left(\beta-\frac{\partial^2\bar{U}}{\partial y^2}\right)\left(\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial X}\right)\psi'=0. \qquad (13)$$

Since ϵ is a small parameter, ψ' can be expressed in terms of ϵ:

$$\psi'=\psi_0(x,X,y,t,T)+\epsilon\psi_1(x,X,y,t,T)+\cdots. \qquad (14)$$

Now substitute (14) into (13) and collect terms of the same order of ϵ. The first-order terms yield

$$\left(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}+\lambda\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi_0+\left(\beta-\frac{\partial^2\bar{U}}{\partial y^2}\right)\frac{\partial\psi_0}{\partial x}=0, \qquad (15)$$

which is an equation in the normal variables only. This equation can admit the following solution:

$$\psi_0=R_eA(X,T)\zeta(y)e^{i(\alpha x-\omega t)}. \qquad (16)$$

Substitution of (16) into (15) yields the familiar relation

$$(-i\omega+i\bar{U}\alpha+\lambda)\left(-\alpha^2\zeta+\frac{\partial^2\zeta}{\partial y^2}\right)+\left(\beta-\frac{\partial^2 U}{\partial y^2}\right)(i\alpha)\zeta=0. \qquad (17)$$

The balanced equation at order (ϵ) is

$$\left(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}+\lambda\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi_1+\left(\beta-\frac{\partial^2\bar{U}}{\partial y^2}\right)\frac{\partial\psi_1}{\partial x}$$
$$=-2\left(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial X}+\lambda\right)\frac{\partial^2\psi_0}{\partial X\partial x}-\left(\frac{\partial}{\partial T}+\bar{U}\frac{\partial^2\psi_0}{\partial y^2}\right)\times\left(\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial y^2}\right)-\left(\beta-\frac{\partial^2 U}{\partial y^2}\right)\frac{\partial\psi_0}{\partial X}. \qquad (18)$$

The right-hand side of (18) can be evaluated with the introduction of the order-one solution (16) which becomes

$$\mathrm{rhs}=\left[-2(-i\omega+\bar{U}i\alpha+\lambda)\frac{\partial A}{\partial X}(i\alpha)\zeta-\left(\frac{\partial}{\partial T}+\bar{U}\frac{\partial}{\partial X}\right)\left(-\alpha^2\zeta+\frac{\partial^2\zeta}{\partial y^2}\right)A-\left(\beta-\frac{\partial^2\bar{U}}{\partial y^2}\right)\frac{\partial A}{\partial X}\zeta\right]e^{i(\alpha x-\omega t)}. \qquad (19)$$

Since the left-hand side of (18) has the same structure

TABLE 2. Errors extracted from Tables 1 and 2 of Nayfeh and Padhye (1979).

$\lvert\alpha_i/\alpha_r\rvert$	Errors
0.00155	0.0260
0.01500	0.0072
0.01990	0.0059
0.02286	0.0052
0.02325	0.0072
0.01717	0.0157
0.00791	0.0281
0.00136	0.0438

as the first-order equation (15), the same homogeneous solution as (16) also applies to the order (ϵ) solution. The right-hand side of (19), which is the forcing part, oscillates with the same frequency as the homogeneous part. Therefore, the solution would contain a secular growth with time. For time ($1/\epsilon$), the solution $\epsilon\psi$ will become as large as the first-order solution and violate the expansion (14). Therefore, terms on the right-hand side of (19) which are proportional to $\exp[i(\alpha x - \omega t)]$ have to be suppressed. With the help of (17), this leads to

$$\frac{\partial A}{\partial T}+\left[\frac{2i\alpha\zeta(-i\omega+i\alpha\bar{U}+\lambda)}{(-\alpha^2\zeta+\partial^2\zeta/\partial y^2)}-\frac{(-i\omega+\lambda)}{i\alpha}\right]\frac{\partial A}{\partial X}=0. \quad (20)$$

Differentiation of the dispersion relation (17) with respect to α will show that the bracket in (20) is the complex group velocity in the x direction, i.e.,

$$\frac{\partial A}{\partial T}+\omega_\alpha\frac{\partial A}{\partial X}=0 \quad (21)$$

where

$$\omega_\alpha=\frac{\partial\omega}{\partial\alpha}=\frac{2i\alpha\zeta(-i\omega+i\alpha\bar{U}+\lambda)}{[-\alpha^2\zeta+(\partial^2\zeta/\partial y^2)]}-\frac{(-i\omega+\lambda)}{i\alpha}. \quad (22)$$

Nayfeh and Padhye treated nonparallel flow with slow variation which allowed the separation of the problem into different orders of the small parameter ϵ, which measures the nonparallelism of the basic flow. With the procedure similar to the above analysis, the solvability conditions at next order lead to, in two-dimensional form,

$$g_1\frac{\partial A}{\partial t_1}+g_2\frac{\partial A}{\partial x_1}=h_1A, \quad (23)$$

where x_1, t_1 are slow variables scaled by ϵ and expressions for g_n and h_n are given in appendix A in Nayfeh and Padhye. For quasi-parallel flow, the nonlinear term on the right-hand side of (23) is dropped and the slow scales x_1, t_1 become x and t so that (23) becomes

$$\frac{\partial A}{\partial t}+\omega_\alpha\frac{\partial A}{\partial x}=0, \quad (24)$$

where $\omega_\alpha = g_2/g_1$ is the complex group velocity in x direction.

In the case of temporal stability, Nayfeh and Padhye further used the following expression

$$A=a\exp(-\omega_i t) \quad (25)$$

so that (24) becomes

$$\frac{\partial a}{\partial t}+\omega_\alpha\frac{\partial a}{\partial x}=\omega_i a \quad (26)$$

and the solution in their paper becomes

$$U=a(x,t)\zeta(y)\exp[i(\alpha x-\omega_r t)]. \quad (27)$$

To convert from the temporal to spatial stability, Nayfeh and Padhye constrain a to be independent of t in (26) and solve for a:

$$a=a_0\exp(\omega_i x/\omega_\alpha). \quad (28)$$

Expressing ω_α in polar form as $C_g\exp(i\theta)$, and substituting into (27), it becomes

$$U=a_0\zeta(y)\exp\{i[(\alpha+\delta\alpha)x-\omega_r t]+\sigma_s x\} \quad (29)$$

where

$$\sigma_s=\frac{\omega_i}{C_g}\cos\theta, \quad (30)$$

$$\delta\alpha=-\frac{\omega_i}{C_g}\sin\theta. \quad (31)$$

Equation (30) is the transformation formula with the corresponding shifting $\delta\alpha$ in the wavenumber given by (31). Equation (9) stated in section 2 is the complex form of (30) and (31).

Equation (24) is of the same form as (21) derived above. Both formulations are based upon the following assumption

$$\frac{1}{A}\frac{\partial A}{\partial x}\ll\alpha,\quad \frac{1}{A}\frac{\partial A}{\partial t}\ll\omega \quad (32)$$

which means there is a requirement that the amplitude have slow variation (measured by the spatial growth rate) compared to the local modulation (represented by the local wavenumber). The later neglect of the nonparallel effects by Nayfeh and Padhye does not relax the slow variation requirement. Our analysis gives the condition $|\alpha_i/\alpha_r| \ll 1$ directly because we did not include the slow variation of the basic flow initially. Since the slow variation of amplitude requirement is equivalent to the assumption that $|\alpha_i/\alpha_r|$ be small, it is clear that this analysis agrees with the comparisons in section 2.

In summary, Nayfeh and Padhye's (1979) formula is valid when the spatial amplification rate is small compared with the local wavenumber. The applicability of Gaster's formula and Nayfeh and Padhye's formula are the same. When the amplification rate is relatively large, the relations which use group velocity or phase velocity give results with approximately the same errors. More research is required to determine more general relations between the spatial and temporal growth rates.

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