Generalized Hydraulic Solutions Pertaining to Severe Downslope Winds

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ABSTRACT

Nonlinear steady-state solutions for stratified two-layer flow over a ridge are found. In parameter space, these solutions lie between the interfacial case of Long and the constant stratification case of Smith. The solutions predict the depth of the layer which will descend and accelerate. Qualitative agreement with Bora and Boulder observations is found.

1. Introduction

In a recent paper (Smith, 1985; hereafter S85) a mathematical model of the severe downslope wind phenomenon was proposed. The key element in this model is the decoupling of the low-level accelerated flow from the undisturbed flow aloft by a slowly moving neutral region of variable depth. The boundary condition imposed on the lower layer by the neutral region is similar to that in two-layer internal hydraulics with a deep upper layer (Long, 1954; hereafter L54); the qualitative nature of transitional flow (a monotonic drop) over a ridge is similar as well. Because of this it is natural to imagine that a close relationship must exist between the two types of model. The purpose of this paper is to illustrate such a relationship.

The most obvious difference between the L54 and S85 models is the vertical profile of density. The former has a neutral layer capped by a thin density interface while the latter has constant stratification. We proceed to fill the gap between these two models by considering the two-layer model shown in Fig. 1. This model can be made to approach either the interfacial or continuous model by appropriate choice of parameters, and also allows a closer approximation to certain atmospheric or laboratory flows of intermediate character.

Before discussing our results, we first digress to mention the deeper differences between the types of hydraulics described by Smith and Long. In Long's work, the two homogeneous layers are considered to be immiscible and therefore the identity of the layers and the significance of their interface are never in question. If the incoming flow is such that it does not allow a steady transitional (sub- to supercritical) flow over a ridge, an upstream wave of adjustment may be generated to correct the situation.

The model of Smith is different in that there is no predetermined layer definition. There is also no maximum horizontal wave speed so that the terms sub- and supercritical are problematic when applied to the undisturbed flow. In addition to upstream waves of adjustment, this situation allows other types of adjustment; for example:

1) The local selection of a particular streamline to be the upper boundary of the accelerating flow. A mixed layer forms above this streamline, decoupling it from the flow aloft.

2) The fluid may select *not to form* a neutral decoupling region, thus preferring a continuous field of vertically propagating gravity waves, instead of a layerlike hydraulic flow. This seems to occur when Nh/U is small in unidirectional flow and in backsheared flow when the wind-reversal is at the improper altitude.

Examples of these various possibilities are given by Clark and Peltier, 1984. In short, the flow of stratified fluid over a ridge has more degrees of freedom than the immiscible two-layer models suggest. The investigation of these adjustment processes cannot be done with the steady equations used by Smith and herein, but requires approaches such as numerical or laboratory simulation. The steady-state equations, however, provide a method for finding *possible* steady state solutions.

2. Generalized hydraulics

a. The two-layer model

Consider an incompressible stratified flow approaching a ridge with uniform speed. The stability profile consists of two layers of constant stability as shown in Fig. 1. We presume that the fluid selects a certain critical streamline in the upper layer to serve as the top of the disturbed flow. If the critical streamline lies in the lower fluid the analysis reduces to S85. If the critical streamline lies in the upper fluid, the method of S85 must be extended to include two active layers. Long's equation

$$\delta_{i_{zz}} + l^2\delta_i = 0 \quad \text{for} \quad \delta_i(x, z) \tag{1}$$

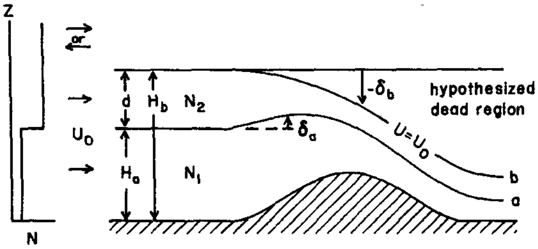

FIG. 1. Schematic of the hypothesized steady-state severe wind flow configuration.

with $i = 1, 2$ applies in each layer. In (1) $l^2 = N_0^2/U_0^2$. The horizontal speed is recovered from $U = U_0(1 - \delta_{i_z})$. Nonlinear boundary conditions apply at each of the three interfaces:

$$\left.\begin{aligned}\delta_2 &= \delta_b\\ \delta_{2_z} &= 0\end{aligned}\right\} \quad \text{on} \quad z = H_b + \delta_b \tag{2}$$

$$\left.\begin{aligned}\delta_1 &= \delta_a\\ \delta_2 &= \delta_a\\ \delta_{1_z} &= \delta_{2_z}\end{aligned}\right\} \quad \text{on} \quad z = H_a + \delta_a \tag{3}$$

$$\delta_1 = h \quad \text{on} \quad z = h. \tag{4}$$

For the present purpose we restrict Eqs. (1)–(4) somewhat by assuming a neutral lower layer $N_1 = 0$. With $l_1 = 0$, the solution to (1) is

$$\delta_1 = Az + B, \tag{5}$$

while in the upper layer $l_2 \neq 0$ and

$$\delta_2 = C \cos lz + D \sin lz. \tag{6}$$

b. Thin stable interface

As a limiting case, we consider the stable layer to be thin compared with the total depth of the system (i.e., $d = z_b - z_a \ll H_b$). Integrating (1) across this layer

$$\int_a^b [\delta_{2_{zz}} + l_2^2 \delta_2] dz = 0 \tag{7}$$

gives

$$-\delta_z + l_2^2 \int_a^b \delta_2 dz = 0. \tag{8}$$

In (8) and hereafter $\delta_z \equiv \delta_{1z} = \delta_{2z}$ at $z = z_a$. Although the layer thickness d is small, we cannot assume that δ_2 is constant across it. Instead, we use the conservation of volume flux

$$\int_a^b U dz \equiv \bar{U} d = U_0 d_0 \tag{9}$$

rewritten

$$\bar{U} \approx \frac{U_b + U_a}{2} = \frac{U_0 + U_0(1 - \delta_z)}{2} \tag{10}$$

to give an expression for d.

$$d = d_0 \Big/ \left(1 - \frac{\delta_z}{2}\right) \tag{11}$$

Now with (8), (11) and

$$\int_a^b \delta_2 dz \equiv \bar{\delta} d, \tag{12}$$

we obtain Bernoulli's equation in the form

$$\delta_z\left(1 - \frac{\delta_z}{2}\right) - \frac{g'}{U_0^2}\bar{\delta} = 0, \tag{13}$$

where $g' = N_0^2 d_0 = g(\Delta\rho/\rho)$. Using (4) and (5)

$$\left.\begin{aligned}h &= Ah + B\\ U &= U_0(1 - A)\\ \delta_a &= A(H_a + \delta_a) + B\end{aligned}\right\}$$

which together give the mass conservation for the lower layer

$$U_0 H_a = U(H_a + \delta_a - h). \tag{14}$$

A convenient form for Eqs. (13) and (14) is found by defining

$$H = H_0 + \bar{\delta} - h, \quad H_0 = H_a \sim H_b, \quad \bar{\delta} \sim \delta_a$$

so that

$$\frac{h}{H_0} = 1 - \frac{H}{H_0} + \frac{1}{2}\frac{U_0^2}{g' H_0}\left[1 - \left(\frac{H_0}{H}\right)^2\right]. \tag{15}$$

From (15) the critical mountain height for transition flow is (Long, 1954)

$$\frac{h^*}{H_0} = 1 + \frac{1}{2}F_0^2 - \frac{3}{2}F_0^{2/3}, \tag{16}$$

where $F_0 = U_0/\sqrt{g' H_0}$.

Overall, there is little advantage to the above derivation over the classical interface method except that it provides an estimate of the variation in the thickness [i.e., Eq. (11)] and Richardson number in the stable layer. The minimum Richardson number in the stable layer occurs on the uppermost streamline where

$$\mathrm{Ri} = (\bar{\delta} N_2/U_0)^{-2}, \tag{17}$$

so that for a fixed g' an interface with larger thickness (smaller N_2) will be less susceptible to Kelvin–Helmholtz instability.

c. Constant stratification

When the lower neutral layer becomes thin, the problem reduces to that in S85. We go beyond S85 only by presenting a closed form expression, analogous to (16), for the critical mountain height.

From S85

$$\hat{h} = \hat{\delta} \cos(\hat{H}_0 + \hat{\delta} - \hat{h}). \tag{18}$$

Differentiating (18) and setting $\partial h/\partial \delta = 0$ at $h = h^*$, $\delta = \delta^*$ gives

$$\hat{\delta}^* = \frac{\cos(\hat{H}_0 + \hat{\delta}^* - \hat{h}^*)}{\sin(\hat{H}_0 + \hat{\delta}^* - \hat{h}^*)}. \qquad (19)$$

Using (18) and (19)

$$\hat{H}_0 = \hat{h}^* - \hat{\delta}^* + \arccos(\hat{h}^*/\hat{\delta}^*) \qquad (20)$$

where

$$\hat{\delta}^* = -\frac{1}{\sqrt{2}}[\hat{h}^{*2} + \hat{h}^*(\hat{h}^{*2} + 4)^{1/2}]^{1/2}.$$

In deriving (20), signs of radicals have been chosen to correspond to a positive mountain and accelerating flow. This Eq. (20) can be combined with the drag formula in S85 to give an a priori drag estimate.

d. The general case

In the more general case when the ratio of the upper to lower layer thickness

$$r = d/H_a \qquad (21)$$

takes on arbitrary values between $0 < r < \infty$, a numerical method must be used. From (1)–(4) with $N_2 = 0$,

$$\left.\begin{aligned} \hat{h} &= A\hat{h} + B \\ \hat{\delta}_a &= A(\hat{H}_a + \delta_a) + B \\ \hat{\delta}_a &= \hat{C}\cos(\hat{H}_a + \hat{\delta}_a) + \hat{D}\sin(\hat{H}_a + \hat{\delta}_a) \\ A &= -\hat{C}\sin(\hat{H}_a + \hat{\delta}_a) + \hat{D}\cos(\hat{H}_a + \hat{\delta}_a) \\ \hat{\delta}_b &= \hat{C}\cos(\hat{H}_b + \hat{\delta}_b) + \hat{D}\sin(\hat{H}_b + \hat{\delta}_b) \\ 0 &= -\hat{C}\sin(\hat{H}_b + \hat{\delta}_b) + \hat{D}\cos(\hat{H}_b + \hat{\delta}_b) \end{aligned}\right\} \qquad (22)$$

where $l = l_2$, $\hat{h} = lh$, $\hat{H}_a = lH_a$, $\hat{H}_b = lH_b$, $\hat{\delta}_a = l\delta_a$, $\hat{\delta}_b = l\delta_b$, $\hat{A} = lA$, $\hat{B} = lB$, and $\hat{C} = lC$. The set (22) can be reduced and rearranged to give

$$\hat{\delta}_a^2 + \left(\frac{\hat{\delta}_a - \hat{h}}{\hat{H}_a + \hat{\delta}_a - \hat{h}}\right)^2 = \hat{\delta}_b^2 \qquad (23)$$

$$\hat{\delta}_a = \hat{\delta}_b \cos(\hat{H}_b - \hat{H}_a + \hat{\delta}_b - \hat{\delta}_a). \qquad (24)$$

With $\hat{h}$, $\hat{H}_a$ and $\hat{H}_b$ specified, (23) and (24) can be solved numerically by finding $\hat{\delta}_a$ and then using $\hat{\delta}_a$ to find $\hat{\delta}_b$. Solutions which violate the conditions

$$\hat{h} < \hat{\delta}_a + \hat{H}_a \qquad (25)$$

$$\hat{\delta}_a < \hat{\delta}_b + \hat{d} \qquad (26)$$

are discarded. Inequality (25) prevents streamline "*a*" from entering the ground while (26) prevents streamlines "*a*" and "*b*" from crossing.

We can further restrict the solutions to (23) and (24) by requiring that the final state, with $\hat{h}$ returned to zero, satisfy

$$\hat{\delta}_a < 0 \qquad (27a)$$

$$\hat{\delta}_b < 0 \qquad (27b)$$

for

$$\hat{h}(x) \geq 0. \qquad (27c)$$

This ensures an unambiguous drop and acceleration of the flow over positive topography. This is the only type of transitional airflow for which we have solid evidence from the laboratory (Rottman and Smith, unpublished work), the field (Smith, 1987), and from numerical models (Clark and Peltier, 1984; Durran, 1986; and others). Solutions to (23) and (24) satisfying (25), (26) and (27) are found for

$$\cot(\hat{d}) < \hat{H}_a < \cot(\hat{d} + \pi) \qquad (28)$$

as shown in Fig. 2. In part of this domain

$$\hat{d} > \frac{\pi}{2} \qquad (29)$$

the "*a*" streamline rises slightly before it falls over the obstacle (e.g., Fig. 4).

As in S85, once a transitional solution has been found, increasing $\hat{d}$ by 2π yields another solution, albeit with some ascending streamlines in the middle of the stable layer. Such a flow is probably realizable, as shown by CP84 (with $\hat{H}_a = 0$), but has not yet been observed in the atmosphere.

The relaxation of (27a) allows, in a certain range of parameters, a class of transition in which the lower "*a*" interface moves up while the upper "*b*" interface moves down. We have no evidence that such a flow can exist in practice. Relaxation of both (27a) and (27b) allows a transition with the stable layer rising. With $r = d/H_a \ll 1$, this would be interpreted as a transition from supercritical to subcritical flow, a known impossibility. For the purposes of this paper we will not consider these possibly unphysical modes.

An example of a realizable two-layer transitional flow is given in Figs. 3 and 4 for the case $\hat{H}_b = 3\pi/2$, $\hat{H}_a = 3\pi/4 = \hat{d}$, and $r = 1$. With these parameters the critical mountain height is about $\hat{h}_c = 1.05$, only slightly different than the solution of S85 with stratification in the lower layer, as well. This nondimensional solution could correspond to a situation with $U = 20$ m s^{-1}, $N = 0.01$ sec^{-1}, $H_b = 9400$ m, $H_a = 4700$, and $h = 2100$ m as shown in Fig. 4. Note that the "*a*" streamline begins to rise before it falls. Velocity shear is confined to the stratified layer. The Richardson number falls below 1/4 beyond the point indicated.

3. The "effective depth" approximation

The dependence of the two-layer solutions on the two free parameters r and $\hat{H}_b$ can be approximated in a simple and surprisingly accurate fashion. We first consider the "effective altitude" of the stable layer to be

$$H_{\text{eff}} = H_a + \gamma d, \qquad (30)$$

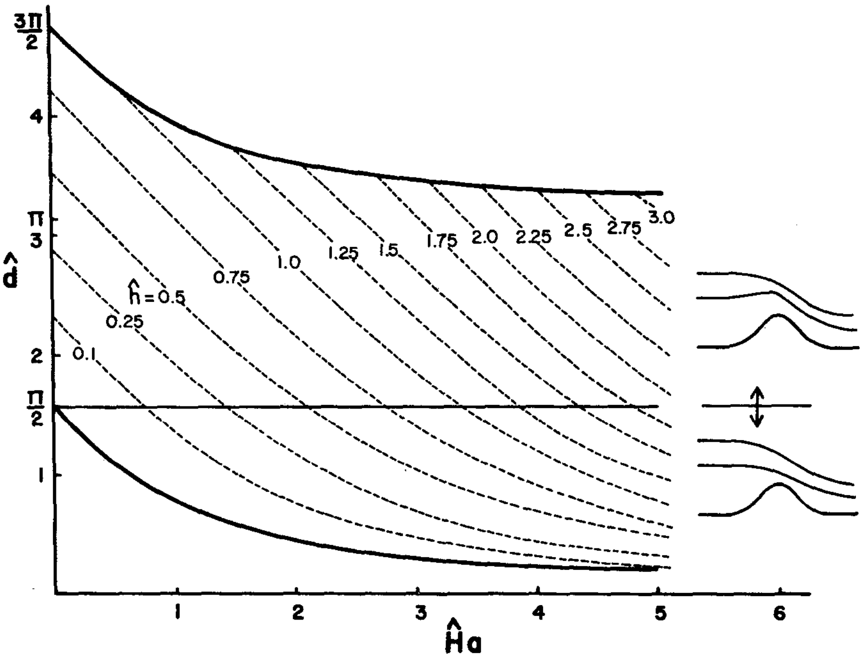

FIG. 2. Critical mountain height $\hat{h}$ given by Eqs. (23) and (24).

and its strength to be

$$g' = N_2^2 d = \frac{\Delta\rho}{\rho} g. \qquad (31)$$

Putting (30) and (31) into (16) and equating the resulting h^* with the exact one computed from (23) and (24) gives $\gamma(r, \hat{H}_b)$.

Our calculations indicate that

$$\gamma = 0.5 \pm 0.05 \qquad (32)$$

over most of parameter space except when $r < 0.5$, in which case the deviation from $\gamma = 0.5$ does not matter very much. This result (32) agrees with the intuitive idea that a thick stable layer might be approximated by a thin density interface of equal strength located at its mean position. It follows from (30)–(32) that

$$F_0^2 = \frac{U_0^2}{N_2^2 d H_{\rm eff}} = \frac{(1+r)^2}{r(1+r/2)\hat{H}_b^2} \qquad (33)$$

for use in (16). In the worst case, $H_a = 0$, $H_{\rm eff} = \frac{1}{2} H_b$, $r = \infty$, we obtain from (16) and (33)

$$\hat{h}^* = \frac{\hat{H}_b}{2}\left[1 + \frac{1}{\hat{H}_b^2} - \frac{3}{2}\left(\frac{\sqrt{2}}{\hat{H}_b}\right)^{2/3}\right] \qquad (34)$$

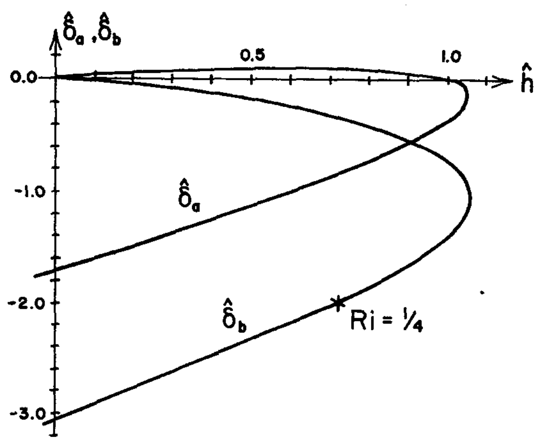

FIG. 3. A particular solution curve to Eqs. (23) and (24) for $\hat{H}_b = 3\pi/2$, $\hat{H}_a = 3\pi/4$.

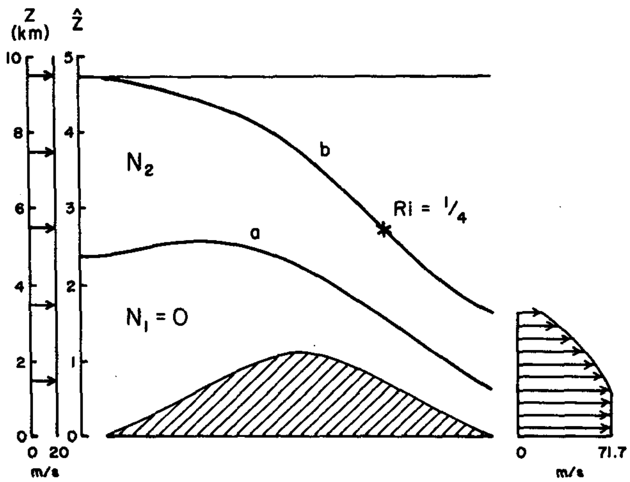

FIG. 4. The flow field corresponding to the solution curves in Fig. 3. Kelvin–Helmholtz instability is likely downstream of the point *.

which still compares within 10% with the exact result (20) as shown in Fig. 5.

4. Bora and Boulder downslope winds

Observations of the height of the split streamline in Bora and Boulder windstorms were compared with the predictions of hydraulic theory in Smith, 1987. Only the limiting cases given by 16 and 20 were available at that time. The present theory allows a more meaningful comparison because of the slightly improved representation of the upstream stability profile. To accomplish this, each upstream potential temperature profile was fit by eye with a lower neutral layer and an upper layer of constant stability. The parameters describing this fit are given in Table 1. A prediction for the height of the split streamline (d_p) is taken from Fig. 2 or, if r is small, from Eqs. (16) and (32). In the last column of Table 1, some idea is given of the sensitivity of d to h by presenting the value of h which would be associated with the observed d.

The following conclusions may be drawn from Table 1. The general agreement between d and d_p gives support for the idea that the steady state hydraulic equations can be used to predict the split streamline altitude when severe wind conditions exist. The agreement, however, is not much better than that reported in Smith (1987) using the large and small r limits. It appears that, by chance, the six cases in Table 1 fall into large r cases (M7, Boulder) and small r cases (M22, M25, A15). Only M6 has an $r \approx 1$, for which the present theory is needed, and this case may have been modified somewhat by latent heat effects.

The data provides further evidence for the similarity of the Bora and Boulder windstorms in that the agreement between the theory and the Boulder storm is comparable with that for the Bora cases.

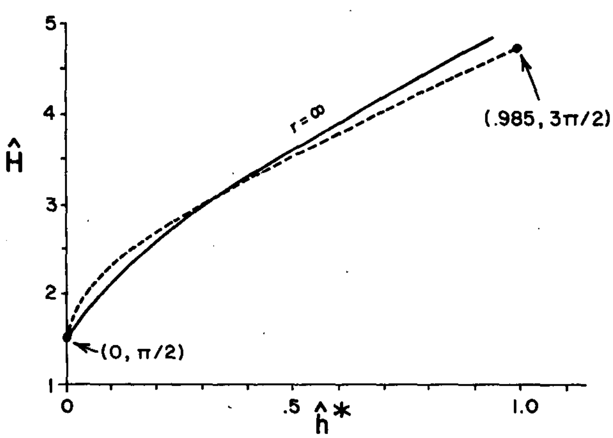

FIG. 5. The relationship between the upstream layer depth $\hat{H}_b$ and critical mountain height $\hat{h}^*$ for the constant stratification case: (a) exact solution for $\hat{H}_0$ from Eq. (20) (dashed curve); (b) approximate solution for $\hat{H}_b$ [Eq. (34)] based on the idea of an effective depth (solid line).

The systematic error evident in Table 1 is an approximate 25% overestimate of d_p, especially in the large r cases. This may arise from 3-D effects, the failure to take into account wind shear in the environment, or the influence of upstream blocking. The last column indicates that if low-level blocking were equivalent to a 30% or 40% decrease in the effective mountain height, closer agreement could be obtained. The hydraulic theory seems to predict idealized flows in numerical simulations (CP84, see S85) and laboratory experiments (Rottman and Smith, 1987) more accurately than it predicts atmospheric phenomena.

5. Summary

We have demonstrated the nature of hydraulic flows which lie between the constant stratification and interfacial stratification limits. The monotonic behavior between the two limits and the surprising accuracy of the "effective depth" approximation indicate that there is no essential physical difference between the two limits.

The application of the new results to Bora and Boulder windstorm cases given only a marginal improvement over the limiting forms used in S87. Apparently by chance, the six cases considered fell close to one limit or the other and thus did not provide a wholly satisfactory test of the present theoretical extension. Further field observations, numerical and laboratory simulations may be needed to test the theory more conclusively.

The tendency of the theory to overestimate the depth of the accelerating layer (by 25% in some cases) we take to be an effect of wind shear, blocking, or three-dimension effects in the real atmosphere. In idealized numerical simulations (CP84, see S85) and in laboratory simulations (Rottman and Smith, 1987) the hydraulic theory tends to do better.

The larger question of how the fluid approaches one of the possible steady-state solutions remains unanswered. Further time-dependent numerical and laboratory modeling is required. This question is best-addressed using the constant stratification model as there is in that case no preexisting atmospheric structure to confuse the issue of depth selection. Ultimately, however, the case with a deep stratified layer over a neutral lower layer (or some other type of structure) should be considered as it provides a more general context for the study of depth selection, decoupling and hydraulic acceleration.

The method described here can be extended, in principle, to any number of layers. Two limitations may arise, however, in practice. First, with an increasing number of layers comes an increasing number of spurious solutions which must be characterized and discarded. Second, any attempt to represent wind shear as slight velocity discontinuities between layers would lead to instability in the upstream state. A method for

TABLE 1. Observations of fit of the split streamline in Bora and Boulder windstorms.

	Observed					Predicted			
Case	H (m)	N_2 (s^{-1})	H_a[@] (m)	U (m s^{-1})	θ crit °K	d[@] (m)	$r = \frac{d}{H_a}$	d_p (m)[†]	h_p (m)[††]
M6, Bora	800	0.016	1300	16	292	2000	1.5	3000	400
M7, Bora	800	0.017	500	10	288	1700	3.4	2350*	350
M22, Bora	800	0.026	2900	13	290	500	0.17	500**	800
M25, Bora	800	0.023	1600	4	288	500	0.31	350**	1000
A15, Bora	800	0.019	1400	7	290	700	0.5	1100**	600
Boulder	1600	0.011	1800	40	328	8000	4.4	10000	900

* Not in allowed region–choose boundary point.
** Small r makes use of Fig. 4 inaccurate—use Eqs. 16 and 32.
† d_p comes from Fig. 4 using h, N_2, U, H_a to compute $\hat{h}, \hat{H}_a$.
†† h_p comes from Fig. 4 using N_2, H_a, U, d to compute $\hat{d}$ and $\hat{H}_a$.
@ d and H_a are uncertain by at least ±200 m.

treating a finite amplitude disturbance on a smoothly sheared layer would be more useful.

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