Gravity Wave Heat Fluxes: A Lagrangian Approach

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(Manuscript received 8 October 1986, in final form 30 December 1987)

ABSTRACT

The effect of a vertically propagating, internal gravity wave on the vertical flux of potential temperature (heat) is considered by averaging the local heat flux vector over a potential temperature surface. This approach gives the wave heat flux a simple physical picture which is not readily apparent from the more common Eulerian formulation. This method also allows the eddy diffusion coefficient to be a function of the phase of the wave. Such a phase dependent eddy diffusion has been previously considered from an Eulerian viewpoint as a model of a convectively unstable gravity wave. Here, the Lagrangian method confirms and corrects the Eulerian results. Earlier work is extended by modeling a constant amplitude "breaking" wave, as well as by considering eddy diffusion coefficients that are asymmetric with respect to the wave breaking region. In all cases studied, localizing the eddy diffusion to the region of wavebreaking decreases the average heat flux.

1. Introduction

For some time now, turbulence has been recognized to be an important component of the dynamics of the middle atmosphere. The distributions of trace gases and aerosols, as well as the thermal structure of the middle atmosphere are determined to some extent by turbulent diffusion. Even more striking are the closed zonal wind jets in the middle atmosphere which are due, in part, to the momentum flux convergence associated with dissipating internal gravity waves. This dissipation is believed to be due to turbulence arising from instabilities within the wave field (Hodges 1967; Lindzen 1981). Thus, turbulent diffusion indirectly affects even the largest scales of the middle atmosphere circulation.

Both observational and theoretical studies have contributed to our understanding of turbulence in the middle atmosphere. Observations using radars, lidars, rockets, and balloons have provided evidence of two types of instabilities as sources of turbulence and wave dissipation. These are the dynamical (Kelvin-Helmholtz) instability and the convective instability. The former has been observed throughout the lower and middle atmosphere, and is believed to be associated preferentially with gravity wave motions with near-inertial frequencies. Convective instabilities, on the other hand, are believed to occur primarily in connection with relatively high-frequency (essentially two-dimensional) wave motions. The observational evidence of and theoretical basis for these instabilities and their effects were reviewed by Fritts and Rastogi (1985). Because energy and momentum fluxes are dominated by wave motions with high intrinsic frequencies and large vertical group velocities (Fritts 1984), the convective instability of the gravity wave field is believed to be the primary source of turbulence in the middle atmosphere.

While we are now beginning to understand the processes by which turbulence is produced in the middle atmosphere, our knowledge of the role and effects of such turbulence is meager at present. It is generally believed that the turbulence arising from saturating gravity waves acts both to limit wave amplitudes and to diffuse heat and constituents. As a first approximation, a number of studies have assumed that turbulence acts as a uniform eddy diffusion on the wave motions and background wind, temperature, and constituent profiles (Hodges 1969; Hines 1970; Lindzen 1981; Holton 1982; Matsuno 1982; Schoeberl et al. 1983). In contrast, observations (Barat and Bertin 1984; Fritts et al. 1986) suggest that turbulence may be highly localized within the wave field, and Strobel et al. (1985) have argued that an eddy diffusion of ~ 50 $m^2 s^{-1}$ is as large as can be reconciled with observed temperature profiles. However, this value is considerably less than the average turbulent diffusivity inferred from observations (see Hocking 1985). Thus, there are clear reasons to question the assumption of and the conclusions based upon a uniform turbulent diffusion.

Motivated by observations of turbulence localization and inferences of excessive heat fluxes (Schoeberl et al. 1983), the effects of a nonuniform turbulent diffusion were examined by Chao and Schoeberl (1984) and Fritts and Dunkerton (1985). The former study argued that the localization of turbulence would lead

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to increased dissipation in the velocity field and a reduction in the downward heat flux. Fritts and Dunkerton performed an Eulerian calculation of the heat flux due to both wave and turbulent motions and showed that the total flux could be reduced substantially as a result of turbulence localization. Another result of this study was the prediction of turbulent Prandtl numbers, for both wave motions and mean profiles, comparable with those inferred by Justus (1967), and able to reconcile observed and required values of turbulent diffusion. The central assumption in this treatment, and one consistent with observations, was that turbulence achieved a maximum intensity at that phase of the wave motion in which the local environment was statically unstable. This acts to reduce the total (wave plus turbulence) heat flux because most of the turbulence then acts in an environment with local gradients smaller than the mean state gradient.

Most of this paper is concerned with a greatly simplified, highly idealized model of a single, steady, two-dimensional internal gravity wave. It should be remembered that the atmosphere is much more complex. In particular, no studies have been done which justify the use of eddy diffusion to describe the convective overturning of a three-dimensional, nonsteady, internal gravity wave. The use of eddy diffusion to describe the damping of stable, nonbreaking waves has also not been investigated. Because of the many assumptions used in this paper, the work presented here should be regarded as an interpretation of a commonly used model of a breaking gravity wave, and not as a justification for using such a simple model.

The purpose of the present paper is to consider the heat flux due to internal gravity waves from a Lagrangian point of view. Both Schoeberl et al. (1983) and Fritts and Dunkerton (1985) used an Eulerian framework for their thermal diffusion calculations. This paper closely follows Fritts and Dunkerton (hereafter referred to as FD), repeating their calculation for the total eddy diffusion potential temperature flux in section 2. The main advantage of using a Lagrangian viewpoint to describe the work of FD is the simple physical picture that it provides of the gravity wave-turbulence interaction. A justification of the Lagrangian averaging procedure used in section 2 is presented in section 3. Extension of the theory to nonconstant mean diffusion values is given in section 4, and the Eulerian approach, with emphasis on a spectrum of noninteracting waves, is reviewed in section 5. Finally, section 6 provides a summary discussion.

2. Eddy diffusion across a potential temperature surface

This section considers the same problem as posed in FD: a two-dimensional, hydrostatic, internal gravity wave, propagating in an incompressible fluid. The potential temperature equation is taken to be

$$\theta_t + \mathbf{u}\cdot\nabla\theta = \nabla\cdot(\nu\nabla\theta), \qquad (1)$$

where θ is potential temperature, $\mathbf{u}$ is the vector velocity field, and ν is the eddy diffusion coefficient. Following FD, the wave solution is assumed to be sinusoidal, so that the potential temperature field is given by

$$\theta = \bar{\theta}_z\left(z - \frac{\alpha}{m}\sin\phi\right), \qquad (2)$$

with,

$$\phi = kx - mz, \qquad (3)$$

where k and m are the horizontal and vertical wavenumbers, and α is a wave amplitude parameter. As in FD, both the wave amplitude α and the mean potential temperature gradient $\bar{\theta}_z$ are taken to be slowly varying so that their derivatives can be neglected. The phase (3) is chosen so that k and m positive correspond to an upward propagating wave, and, for convenience, a frame of reference moving with the wave is used.

It is useful to define a vector field,

$$\mathbf{F} \equiv -\nu\nabla\theta, \qquad (4)$$

which will be referred to as the "heat flux vector" since the divergence of $\mathbf{F}$ gives the rate of change of potential temperature. The heat flux vector has units of K/(s m^2) multiplied by a unit volume m^3. Using (2) and (3) then gives

$$F_h = \nu\bar{\theta}_z\frac{k}{m}\alpha\cos\phi \qquad (5)$$

$$F_v = -\nu\bar{\theta}_z(1 + \alpha\cos\phi) \qquad (6)$$

where F_h and F_v are the x and z components of $\mathbf{F}$. Now, instead of taking an Eulerian average of $\mathbf{F}$ as in FD, we consider an average of $\mathbf{F}$ along a potential temperature surface. Several reference vectors are sketched in Fig. 1 for two different wave amplitudes. The heat flux vectors are, by definition, normal to the θ-surface, and are well defined, even when a local upward component exists ($\alpha > 1$). However, in all cases the heat flux vectors always point from regions of high θ to regions of low θ.

Note that FD considered the possibility of different eddy diffusion coefficients in the horizontal and vertical flux components which would allow for $\mathbf{F}$ and $-\nabla\theta$ to be in different directions. Here we simplify by taking $\mathbf{F}$ and $-\nabla\theta$ always in the same direction, which implies that either the eddy diffusion coefficient is independent of direction, or that eddy diffusion is different along θ-surfaces (horizontal component) than across θ-surfaces (vertical component). Such behavior may not be an accurate description of a real breaking wave but considering the lack of more specific data at this time, the simplest assumption will be taken. Also, as shown by FD, for $k^2/m^2 \ll 1$ the horizontal component of the eddy heat flux is much smaller than the vertical component, so that only the vertical component needs to be considered for many internal gravity waves.

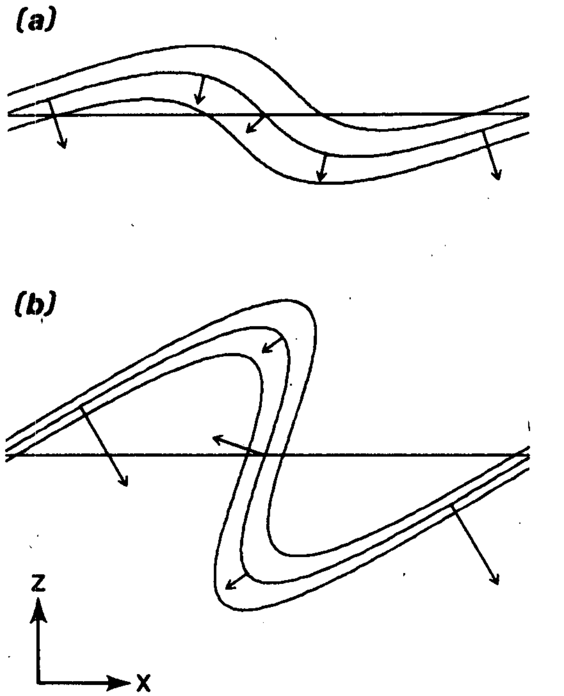


FIG. 1. Sketch of three θ-surfaces and heat flux vectors for (a) $\alpha = 0.5$, and (b) $\alpha = 1.5$. The vertical and horizontal wavelengths are equal, and the eddy diffusion coefficient is assumed to be constant.

Averaging the heat flux vector over the θ-surface gives the average heat flux across the θ-surface:

$$\bar{F}^{\theta} \equiv -\frac{k}{2\pi}\int |\mathbf{F}|\, dS, \qquad (7)$$

where the integral is over one wave cycle, and dS is an element of length along the θ-surface. Since the problem is two-dimensional, the average over the θ-"surface" is just the average over the θ-"curve" shown in Fig. 1. The integral is normalized by the horizontal wavelength. The notation $\bar{F}^{\theta}$ is used to distinguish the average in (7) from the Lagrangian mean average defined by Andrews and McIntyre (1978). The relation between the two averages is considered in section 3. It is not necessary to consider a vector element of length in this case because $\mathbf{F}$ is normal to the θ-surface.

From (2) and (3), the position of a surface of constant θ is given by,

$$z = \frac{\alpha}{m}\sin\phi \qquad (8)$$

$$x = \frac{1}{k}(\phi + \alpha\sin\phi) \qquad (9)$$

so the dS can be written in terms of $d\phi$,

$$dS = \left[\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2\right]^{1/2} d\phi \qquad (10)$$

or

$$dS = \left[\frac{1}{k^2}(1 + \alpha\cos\phi)^2 + \frac{\alpha^2}{m^2}\cos^2\phi\right]^{1/2} d\phi. \qquad (11)$$

Finally, (7) is rewritten using (5), (6) and (11) as

$$\bar{F}^{\theta} = -\bar{\theta}_z \int_0^{2\pi} \nu\left[(1 + \alpha\cos\phi)^2 + \alpha^2\frac{k^2}{m^2}\cos^2\phi\right]\frac{d\phi}{2\pi}. \qquad (12)$$

From (12) it is apparent that the integral can be thought of as a new eddy diffusion coefficient, which takes the gravity wave motion into account.

Equation (12) can also be written as

$$\bar{F}^{\theta} = -\bar{\theta}_z\bar{\nu}\left[1 - 2\beta\alpha + (1 + k^2/m^2)(1 + \gamma)\frac{\alpha^2}{2}\right], \qquad (13)$$

where

$$\bar{\nu} \equiv \int_0^{2\pi} \nu \frac{d\phi}{2\pi} \qquad (14)$$

$$\beta \equiv -\frac{1}{\bar{\nu}}\int_0^{2\pi} \nu\cos\phi\frac{d\phi}{2\pi} \qquad (15)$$

$$\gamma \equiv \frac{1}{\bar{\nu}}\int_0^{2\pi} \nu\cos 2\phi\frac{d\phi}{2\pi}. \qquad (16)$$

Note that the eddy diffusion coefficient has not yet been specified. It can be any periodic function of the wave phase, from the uniform diffusion case to an extremely localized diffusion case. The quantity β is defined as in FD so that (13) can be compared with FD (18). There are only two differences. First, FD allows for different values of ν in the horizontal and vertical directions, where here ν is assumed to be isotropic. Second, (13) contains the quantity γ which is not seen in FD. This difference occurs because four terms involving triple correlations were neglected in FD (12). Appendix A shows that retaining these triple correlation terms leads to the same term proportional to γ as in (13), revealing that both Eulerian and θ-surface averaging yield the same general result.

One advantage of using the potential temperature surface approach, instead of the Eulerian approach, is that it gives an intuitive picture of the heat flux in the presence of a wave. For example, consider the simplest case, where $\nu = \bar{\nu}$ is a constant, so that both β and γ are zero, and (13) becomes

$$\bar{F}^{\theta} = -\bar{\theta}_z\bar{\nu}\left[1 + (1 + k^2/m^2)\frac{\alpha^2}{2}\right]. \qquad (17)$$

The term proportional to α^2 is the Eulerian-averaged heat flux, which is often calculated separately from the rest of the eddy diffusion. Here, however, the increase in diffusion due to the presence of a wave can be thought of as coming from two separate (though not independent) effects:

1) The wave "stretches" the θ-surface, increasing the surface area, and hence, increasing the flux across the surface. This increase in surface area with wave amplitude is given by (10), the expression for dS.

2) The wave phase structure causes the θ-gradients to increase over most of the θ-surface, as sketched in Fig. 1, leading to an increase of the average θ-gradient along a θ-surface. This is expressed by the dependence of $\mathbf{F}$ on α, as seen in (5) and (6).

Therefore, the heat flux due to a gravity wave can be regarded as the increase of the turbulent eddy flux due to the distortions created in the θ-surface by the wave.

Another advantage of averaging over a θ-surface is that it is apparent from the setting up of the problem that the total heat flux (12) will always be downward. This is not as obvious from the Eulerian approach in which a number of separate terms are added together to give a result more in the form of (13). In fact, when the triple correlations (γ) were neglected in FD, it appeared that an upward total heat flux was possible in the limit of extremely localized turbulence. The role of γ in this limit is examined below.

It is interesting to note that, for the constant eddy diffusion coefficient case, the largest heat flux occurs when the air is moving downward through its equilibrium position, since this is where the θ-gradients are greatest. However, it may be somewhat unphysical to have so much diffusive mixing occurring where the atmosphere is most stable. An alternative hypothesis, for a wave propagating in a turbulent atmosphere, is to consider the eddy diffusion coefficient to be a function of the local θ-gradients: a small eddy diffusion coefficient where the θ-gradients are strongest, and a large eddy diffusion coefficient where the θ-gradients are weakest. Thus, localized eddy diffusion coefficients, if maximum where θ-gradients are weakest, can be regarded as a model of wave–turbulence interaction for a nonbreaking wave ($\alpha < 1$), as well as a model of the turbulence generating, breaking wave ($\alpha > 1$) considered by FD.

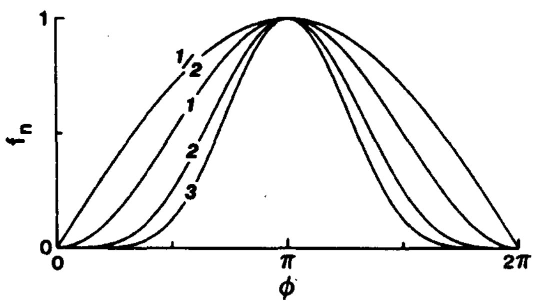


FIG. 2. The localization of the eddy diffusion function f_n as a function of wave phase for n = ½, 1, 2 and 3.

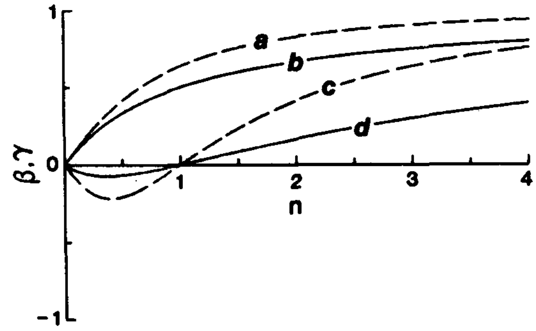


FIG. 3. β as a function of n for (a) the locally uniform function, and (b) the smooth function given by (19). γ as a function of n for (c) the locally uniform function, and (d) the smooth function given by (19).

a. The smoothly varying localization function of FD

The remainder of this section considers some specific examples of localized eddy diffusion. First, consider the function used by FD,

$$\nu = \nu_0 f_n(\cos\phi), \quad (18)$$

with

$$f_n(\cos\phi) = \left(\frac{1 - \cos\phi}{2}\right)^n. \quad (19)$$

The function f_n places the maximum value of ν at $\phi = \pi$, the region of wave overturning for $\alpha > 1$, and the region of weakest θ-gradients for $\alpha < 1$. The parameter n determines the degree of localization of the turbulent diffusion: that is, $n = 0$ corresponds to a constant value of ν, and the larger n to even more localized diffusion. Figure 2 displays f_n for a few values of n. If, as in FD, the mean value of ν is to be kept constant then ν_0, the maximum value of ν, will increase as n increases.

Using (18) and (19) for ν and integrating (14)–(16) (see appendix B for details) gives

$$\beta = \frac{n}{n + 1} \quad (20)$$

$$\gamma = \frac{n(n - 1)}{(n + 2)(n + 1)}. \quad (21)$$

Figure 3 shows β and γ as functions of n. Note that β is always positive, tending to decrease the magnitude of the total heat flux, while γ can be positive or negative, and therefore tends to increase or decrease the total heat flux. Both β and γ approach one in the limit of extremely localized diffusion ($n \rightarrow \infty$). This can be seen directly from (15) and (16) since $\nu/\bar{\nu}$ becomes a delta function centered on π in the limit $n \rightarrow \infty$.

For waves having a horizontal wavelength which is much larger than its vertical wavelength, the term $k^2/$

m^2 can be neglected with respect to one in (13). Defining the quantity,

$$G \equiv 1 - 2\beta\alpha + (1+\gamma)\frac{\alpha^2}{2}, \tag{22}$$

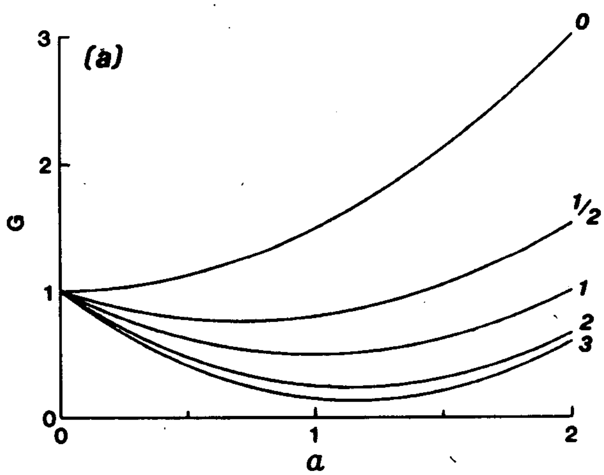
gives a convenient factor for expressing the change in the total heat flux due to the presence of a gravity wave. Figure 4a shows G as a function of α for several values of n. The effect of γ can be seen in Fig. 4b which shows G with γ set equal to zero for all n (Fig. 4b shows the same function as Fig. 3 in FD.) As expected from Fig. 3, γ becomes increasingly important as the turbulence becomes more localized ($n > 1$), and prevents the total heat flux from ever becoming upward.

b. *Uniform localized diffusion*

Next, consider a uniformly localized, eddy diffusion coefficient. The eddy diffusion coefficient is taken to be ν_0 for $\epsilon_n \leqslant \phi \leqslant 2\pi - \epsilon_n$ and zero elsewhere, thus centering the eddy diffusion about π, the region of wavebreaking. Defining $\epsilon_n = n\pi/(n+1)$ makes the dependence of localization on n similar to that used in the previous function, where $n = 0$ corresponds to no localization, and $n \to \infty$ corresponds to the delta function localization. For this locally constant function:

$$\beta = \frac{n+1}{\pi}\sin\left(\frac{n\pi}{n+1}\right) \tag{23}$$

$$\gamma = -\frac{n+1}{2\pi}\sin\left(\frac{2n\pi}{n+1}\right). \tag{24}$$



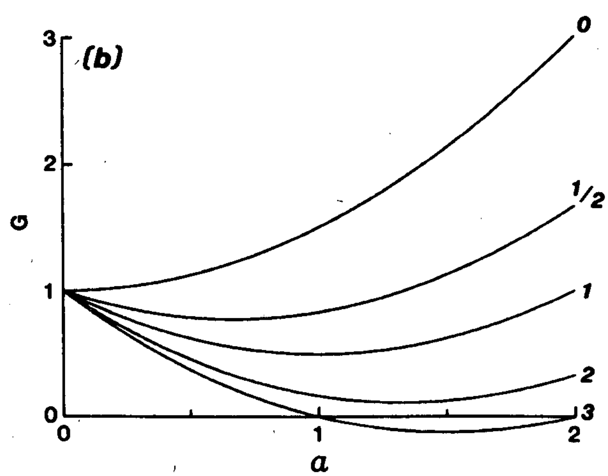


FIG. 4. G for the smooth function (19) as a function of α for n = 0, ½, 1, 2, 3 (a), and with γ = 0 (b).

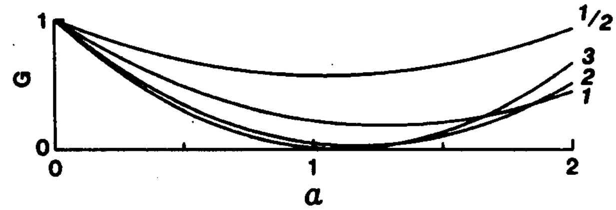


FIG. 5. G for the locally uniform function as a function of α for n = ½, 1, 2, 3.

Figure 3 shows β and γ for the uniformly localized function. While β is only somewhat greater than β of the smoothly varying function, γ has a much greater amplitude than γ for the smoothly varying function. This difference is seen in Fig. 5, which shows the heat flux reduction factor G for the uniformly localized function. Qualitative features remain the same, but the heat flux reduction near $\alpha = 1$ is greater for the uniformly localized function. Note that $n = 1$ implies $\gamma = 0$, so that the difference between Fig. 4 and Fig. 5 at $n = 1$ is due entirely to the difference between the two β terms.

c. *Other localization functions*

Up to this point, the eddy diffusion coefficients chosen have been symmetric and centered about π, the phase at which wavebreaking occurs. However, it is easy to evaluate β and γ for a symmetric ν centered about any phase of the wave. Moving the diffusion off center can be thought of as modeling downstream advection of localized turbulence. Assuming that ν is symmetric about π, and shifting ν to $\pi + \phi_0$ gives

$$\beta = \frac{1}{\bar{\nu}}\int_0^{2\pi} \nu(\phi+\phi_0)\cos\phi\,\frac{d\phi}{2\pi} \tag{25}$$

$$\gamma = -\frac{1}{\bar{\nu}}\int_0^{2\pi} \nu(\phi+\phi_0)\cos 2\phi\,\frac{d\phi}{2\pi}. \tag{26}$$

Changing variables to $\phi' = \phi + \phi_0$ and making use of the symmetry of ν results in

$$\beta = \frac{1}{\bar{\nu}}\cos\phi_0\int_0^{2\pi} \nu(\phi')\cos\phi'\,\frac{d\phi'}{2\pi} \tag{27}$$

$$\gamma = -\frac{1}{\bar{\nu}}\cos 2\phi_0\int_0^{2\pi} \nu(\phi')\cos 2\phi'\,\frac{d\phi'}{2\pi} \tag{28}$$

so that β becomes $\cos\phi_0$ times the previous β and γ becomes $\cos 2\phi_0$ times the previous γ. Thus, for $n = 1$ and $\alpha = 1$, moving the peak of the eddy diffusion coefficient away from the overturning region (where the θ-gradients are weakest) will tend to increase the magnitude of the total heat flux.

Alternatively, asymmetric eddy diffusion coefficients, peaking at π, can be constructed from the FD function (19) by specifying one value of n from 0 to π and another value of n (denoted by p) from π to 2π. Some of these functions can be seen in Fig. 2. The expressions for β and γ are then just the weighted average of the β and γ for the two chosen functions. That is,

$$\beta = \frac{\bar{\nu}_n}{\bar{\nu}}\beta_n + \frac{\bar{\nu}_p}{\bar{\nu}}\beta_p \tag{29}$$

$$\gamma = \frac{\bar{\nu}}{\bar{\nu}}\gamma_n + \frac{\bar{\nu}_p}{\bar{\nu}}\gamma_p, \tag{30}$$

where β_n and γ_n are given by (20) and (21) respectively, and $\bar{\nu}_n$ is given by (B2). As an example, consider the case $n = 0$ and $p = 1$. Then $\gamma = 0$ and $\beta = \beta_1/3$, so that the magnitude of the heat flux reduction is ⅓ that of the symmetric $n = 1$ case. In general, the results for this asymmetric function (29)–(30) lie between the two symmetric results and are weighted toward the symmetric result for the lower value of n.

3. Relation to GLM theory

In section 2, the procedure of averaging the heat flux vector over a θ-surface was not justified, except in an intuitive way. This section provides a more rigorous justification of that procedure by considering the relation of such an average to the generalized Lagrangian mean (GLM) theory of Andrews and McIntyre (1978). The material presented in this section can be found in a more general form in Andrews and McIntyre, but for the specific two-dimensional problem being considered here the general theory simplifies considerably.

The Lagrangian mean in Andrews and McIntyre is defined in terms of the parcel displacement vector $\boldsymbol{\xi} = (\xi', \delta')$ as

$$\bar{\psi}^L \equiv \overline{\psi^\xi} \tag{31}$$

with

$$\psi^\xi = \psi(\mathbf{x} + \boldsymbol{\xi}, t) \tag{32}$$

where the overbar in (31) denotes an Eulerian average, which will be taken to be an average over the periodic horizontal x-direction, and ψ is an arbitrary scalar function of $\mathbf{x}$. It is clear that the Lagrangian average of the heat flux vector is not equal to the θ-surface average, since the Lagrangian average is taken over x, while the θ-surface average is taken over the arc length of the surface. In fact, for a plane, slowly varying, internal gravity wave, the displacement vector lies along the lines of constant phase, so that

$$\mathbf{F}^\xi \approx \mathbf{F} \tag{33}$$

$$\bar{\mathbf{F}}^L \approx \bar{\mathbf{F}} \tag{34}$$

to lowest order.

The Lagrangian mean of the potential temperature equation (1) is

$$(\bar{\theta}^L)_t + \bar{W}^L(\bar{\theta}^L)_z = -\overline{\nabla\cdot\mathbf{F}}^L. \tag{35}$$

Note that, unlike the Eulerian mean potential temperature equation, no wave heat flux terms appear in (35). This means that all the effects of eddy diffusion are contained in the one term on the right-hand side of (35). The right-hand side of (35) can be rewritten by using the chain rule,

$$\begin{aligned}(\psi^\xi)_x &= (\psi_x)^\xi(1 + \xi'_x) + (\psi_z)^\xi\delta'_x \\ (\psi^\xi)_z &= (\psi_x)^\xi\xi'_x + (\psi_z)^\xi(1 + \delta'_z)\end{aligned} \tag{36}$$

and its inverse,

$$\begin{aligned}(\psi_x)^\xi &= \frac{1}{J}\{(\psi^\xi)_x(1 + \delta'_z) + (\psi^\xi)_z(-\delta'_x)\} \\ (\psi_z)^\xi &= \frac{1}{J}\{(\psi^\xi)_x(-\xi'_z) + (\psi^\xi)_z(1 + \xi'_x)\}\end{aligned} \tag{37}$$

where

$$J = 1 + (\xi'_x + \delta'_z) + (\xi'_x\delta'_z - \xi'_z\delta'_x). \tag{38}$$

Equation (37) can be rewritten as

$$\begin{aligned}(\psi_x)^\xi &= \frac{1}{J}\{[\psi^\xi(1 + \delta'_z)]_x + [\psi^\xi(-\delta'_x)]_z\} \\ (\psi_z)^\xi &= \frac{1}{J}\{[\psi^\xi(-\xi'_z)]_x + [\psi^\xi(1 + \xi'_x)]_z\}.\end{aligned} \tag{39}$$

Using (39) then gives

$$\overline{\nabla\cdot\mathbf{F}}^L = \overline{\frac{1}{J}\{F_h{}^\xi(1 + \delta'_z) + F_v{}^\xi(-\xi'_z)\}_x} + \overline{\frac{1}{J}\{F_h{}^\xi(-\delta'_x) + F_v{}^\xi(1 + \xi'_x)\}_z}. \tag{40}$$

Up to this point no approximations have been made.

Assuming incompressibility, so that $\xi'_x + \delta'_z = 0$, and a slowly varying transverse wave, so that $\xi'_x\delta'_z - \xi'_z\delta'_x = 0$, gives $J = 1$ so that

$$\overline{\nabla\cdot\mathbf{F}}^L \approx \{\overline{F_h{}^\xi(-\delta'_x) + F_v{}^\xi(1 + \xi'_x)}\}_z \tag{41}$$

$$\approx \left\{\frac{k}{2\pi}\int_0^{2\pi}\mathbf{F}^\xi\cdot\mathbf{dS}\right\}_z \tag{42}$$

where $\mathbf{dS} = (-d\delta', dx + d\xi')$. The vector $\mathbf{dS}$ is just the area element vector for the material surface described by $x + \xi$, so that the integral in (42) is the average of $\mathbf{F}$ over the material surface. If, as assumed in section 2, the material surface coincides with a potential temperature surface, then

$$\mathbf{dS} = \frac{\nabla\theta}{|\nabla\theta|} dS. \tag{43}$$

That is, $\mathbf{dS}$ will be perpendicular to the θ-surface with a magnitude dS given by (11). For a slowly varying, transverse wave, $\mathbf{F}^{\xi} \approx \mathbf{F}$, and (42) becomes

$$\overline{\nabla \cdot \mathbf{F}}^{L} \approx (\bar{F}^{\theta})_z. \tag{44}$$

Neglecting mean vertical advection, and approximating $\bar{\theta}^{L}$ as $\bar{\theta}$ then gives

$$\bar{\theta}_t \approx -(\bar{F}^{\theta})_z. \tag{45}$$

Thus, for this simple problem consisting of a transverse, slowly varying wave, the intuitive idea of averaging the heat flux vector over a θ-surface is seen to be the correct approximation to the GLM theory.

4. Variations of mean diffusion with localization

In section 2 the mean (or average) value of the eddy diffusion coefficient was kept constant as the total heat flux was evaluated for different degrees of localization. This section considers a simple model of a saturated gravity wave, where the diffusive wave damping (generated by wavebreaking) removes just enough energy from the wave to keep the wave amplitude constant in the presence of an exponentially decreasing mean density. For this simple model, it is shown that the mean value of the turbulent diffusion needs to be increased as turbulence becomes more localized within the wave field.

Assuming that the incompressible Eq. (1) gives a first approximation to the constant amplitude compressible problem, the steady, horizontal mean, perturbation energy equation can be written as,

$$\frac{1}{\bar{\rho}}(C_g\bar{\rho}E)_z = \overline{u'\nabla\cdot(\mathrm{Pr}\nu\nabla u)} + \frac{N^2}{\bar{\theta}_z^{\,2}}\overline{\theta'\nabla\cdot(\nu\nabla\theta)}, \tag{46}$$

where $\bar{\rho}$ is the mean density, C_g is the vertical group velocity, $E \equiv \overline{u'^2}$ is the mean wave energy density, N is the buoyancy frequency, and Pr is the local Prandtl number, which gives the ratio of the horizontal momentum eddy diffusion coefficient to the potential temperature eddy diffusion coefficient. Equation (46) expresses the balance between the vertical wave energy flux convergence (on the left-hand side) and the wave dissipation from the momentum equation (the first term on the right-hand side) and the potential temperature equation (the second term on the right-hand side).

The Eulerian integrals in (46) can be written in terms of β and γ as in appendix A, giving

$$\frac{1}{\bar{\theta}_z^{\,2}}\overline{\theta'\nabla\cdot(\nu\nabla\theta)} \approx -\bar{\nu}\left[-\beta\alpha + (1 + k^2/m^2)(1+\gamma)\frac{\alpha^2}{2}\right], \tag{47}$$

and (using $u_z' = -\alpha N \sin\phi$),

$$\overline{u'\nabla\cdot(\nu\nabla u)} \approx -\bar{\nu}N^2\,\mathrm{Pr}(1 + k^2/m^2)(1-\gamma)\frac{\alpha^2}{2} \tag{48}$$

so that

$$\frac{1}{\bar{\rho}}(\bar{\rho}C_gE)_z \approx -\bar{\nu}N^2\alpha^2\left[\frac{1}{2}(1 + k^2/m^2) \times \{(1+\mathrm{Pr}) + (1-\mathrm{Pr})\gamma\} - \beta/\alpha\right]. \tag{49}$$

Note that terms involving γ (the triple correlations) tend to cancel since θ_z' and u_z' are 90 degrees out of phase, and, as (28) shows, γ goes as $\cos 2\phi_0$. For the remainder of the section the Prandtl number will be assumed to be one, and k^2/m^2 will be neglected.

For the uniform diffusion case, $\beta = 0$, and (49) reduces to the familiar, amplitude independent, formula for ν given by Hodges (1969), Lindzen (1981), and others:

$$\bar{\nu}_0 = -\frac{1}{\bar{\rho}}(\bar{\rho}C_gE)_z\frac{1}{N^2\alpha^2} \tag{50}$$

or

$$\bar{\nu}_0 = \frac{1}{2}C_g\frac{(C-\bar{U})^2}{N^2H}, \tag{51}$$

where C is the horizontal phase speed and H is the atmospheric scale height. In obtaining (51) from (50) it has been assumed the C_gE is independent of height, and that $E = (C-U)^2\alpha^2/2$. The zero subscript denotes that this is the $\bar{\nu}$ needed for the uniform diffusion case ($n = 0$). While a breaking wave amplitude ($\alpha = 1$) is often assumed in deriving (51), such an assumption is overly restrictive since any wave amplitude (for a given wave) can be kept constant by the amount of diffusion given by (51).

In terms of $\bar{\nu}_0$, (49) can be rewritten as

$$\bar{\nu} = \frac{\bar{\nu}_0}{1-\beta/\alpha}. \tag{52}$$

Since, for given wave parameters, $\bar{\nu}_0$ is constant, (52) shows that the mean value of eddy diffusion depends on both the degree of turbulence localization β and the wave amplitude α. If β is positive, so that the turbulence is localized where the θ-gradients are weakest, then a value of $\bar{\nu}$ greater than $\bar{\nu}_0$ is needed to keep a given wave at constant amplitude. More damping is needed because it is being applied in a region where the damping is not very effective.

Figure 6 shows the factor $G(1 - \beta/\alpha)$ as a function of α for several values of n, where the diffusion is localized by (18) and (19), the localization used by FD. The main result is the same as in FD. That is, the localization of the eddy diffusion to regions of weak θ-gradients reduces the magnitude of the downward heat flux. Here, however, the reduction is not as great as in FD since the mean eddy diffusion is increased to offset the tendency for wave growth with height. For example, for $n = 1$, $\alpha = 1$ the heat flux is reduced to ⅔ of its $n = 0$, $\alpha = 1$ value (Fig. 6), compared to a reduction of ⅓, which occurs for the same parameters for the constant ν case (Fig. 4).

5. The Eulerian approach

While the Lagrangian approach is useful for a single wave and gives a simple physical picture of the origin of the wave heat flux, its application to more than one wave is awkward because the θ-surface at any given time is generally quite complex. For an atmosphere consisting of a spectrum of noninteracting waves the Eulerian formulation may be easier to apply. In this section the Eulerian formulation of FD will be reviewed as it is extended to the case of a spectrum of constant amplitude noninteracting waves.

For the two-dimensional problem the Eulerian average of the potential temperature equation (1) is

$$\bar{\theta}_t + \bar{w}\bar{\theta}_z = (\overline{\nu\theta_z} - \overline{w'\theta'})_z. \qquad (53)$$

Following FD, the perturbation θ-equation is multiplied by θ' and averaged:

$$\frac{1}{2}(\overline{\theta'^2})_t + \overline{w'\theta'}\,\bar{\theta}_z + \overline{u'\theta'\theta'_x} + \overline{w'\theta'\theta'_z} = -\overline{\nu\theta'_z\theta_z} - \overline{\nu\theta'^2_x} + (\overline{\theta'\nu\theta_z})_z. \qquad (54)$$

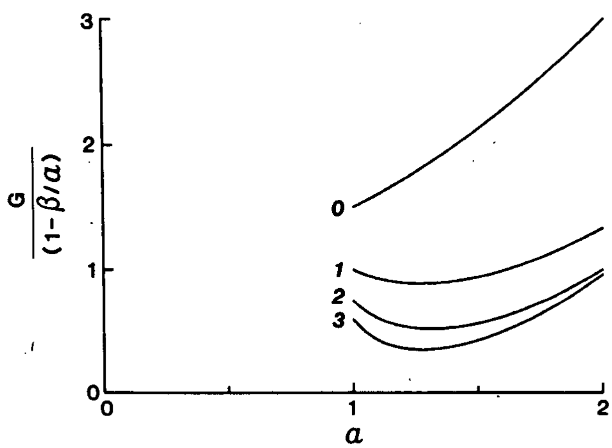


FIG. 6. $G/(1 - \beta/\alpha)$ as a function of α for $n = 0, 1, 2, 3$. β and γ are given by (20) and (21), respectively.

If the overbar is considered to be a time average over the wave motion, as well as a horizontal average, then the two terms consisting of triple correlations on the left-hand side of (54) are zero, provided the waves are noninteracting (nonresonant). The term involving the time derivative is also zero since the wave amplitudes are assumed to be steady. The last term on the right-hand side of (54) is small for constant amplitude waves, assuming once again that the overbar represents a time as well as a space average so that any interference patterns created by the waves average out to zero. Of course, all of the terms singled out above will be small (or zero) for a single wave. Using these approximations and neglecting the mean vertical advection, (53) becomes,

$$\bar{\theta}_t = \left[\bar{\nu}\bar{\theta}_z + 2\overline{\nu\theta'_z} + \frac{\overline{\nu\theta'^2_z}}{\bar{\theta}_z} + \frac{\overline{\nu\theta'^2_x}}{\bar{\theta}_z}\right]_z. \qquad (55)$$

This equation is the same as that in FD, except for the triple correlations, which have not been neglected in (55). Factoring (55) as

$$\bar{\theta}_t = \left[\bar{\theta}_z\overline{\nu\left(1 + \frac{\theta'_z}{\bar{\theta}_z}\right)^2} + \overline{\nu\frac{\theta'^2_x}{\bar{\theta}_z}}\right]_z \qquad (56)$$

shows that the quantity in brackets can be written as a vector dot product:

$$\bar{\theta}_t = \left[\frac{\overline{\nu(\nabla\theta)^2}}{\bar{\theta}_z}\right]_z \qquad (57)$$

where $(\nabla\theta)^2 = \nabla\theta \cdot \nabla\theta$.

For the case of a single wave, (57) is just the average of the heat flux vector taken over a θ-surface, but (57) also applies to a spectrum of waves satisfying the above restrictions. The simple geometric picture is lost in the case of a spectrum of waves, but (57) shows that in such a case the wave heat flux can still be combined with the mean diffusion as a single term. Equation (57) can also be used to define an effective Prandtl number for the total diffusive flux. As pointed out by FD, the diffusion coefficient in the mean momentum equation should remain unchanged by the localization of diffusion because the velocity and potential temperature fields are in approximate quadrature (assuming once again that the diffusion is localized about the wave-breaking region), so that (57) becomes

$$\bar{\theta}_t = [\bar{\nu}\,\mathrm{Pr}_{\mathrm{eff}}^{-1}\bar{\theta}_z]_z \qquad (58)$$

where

$$\mathrm{Pr}_{\mathrm{eff}}^{-1} \equiv \overline{\frac{\nu}{\bar{\nu}}\left(\frac{\nabla\theta}{\bar{\theta}_z}\right)^2}. \qquad (59)$$

For the single wave case, this inverse effective Prandtl number is equal to the factor G (neglecting k^2m^{-2}), which is plotted in Fig. 4a for the localization function used by FD, and in Fig. 5 for the uniformly localized function. Considering the smoothly varying localization function shows that effective Prandtl numbers of 2 to 3 are possible for relatively moderate localizations ($n = 1$ or 2). Higher effective Prandtl numbers are possible for more extreme localizations. These results support the conclusions of Strobel et al. (1985) which show that the heat diffusion must be reduced relative to the momentum diffusion to explain the observed thermal structure of the middle atmosphere.

6. Discussion

The main result of this study supports the result of FD—that localizing the eddy diffusion to the region of weak potential temperature gradients, such as occur in the overturning region of a wave near the breaking amplitude, reduces the magnitude of the total (wave plus turbulent) heat flux. There is a small difference between the results presented here and those of FD, due to the inclusion of the triple correlation terms (which appear here as the factor γ). For diffusion which is not extremely localized ($n < 1$), the presence of γ tends to increase the magnitude of the heat flux reduction relative to that in FD, while for more localized diffusion ($n > 1$), the presence of γ tends to decrease the magnitude of the heat flux reduction relative to that in FD.

This study also extends the work of FD by examining other localization functions. The use of the uniformly localized eddy diffusion function shows that the general result is independent of the particular function chosen for the localization of the diffusion as long as the localization occurs at the same wave phase. As the region of turbulence localization is moved away from the region of weakest potential temperature gradients, or asymmetrically skewed away from the region of weakest potential temperature gradients, the magnitude of the heat flux increases. All these results can be easily understood in terms of the heat diffused across the wave-distorted potential temperature surface, as discussed in sections 2 and 3.

Another result of this study is that the gravity wave heat fluxes combine with the mean diffusion term in a natural way and that the total can be thought of in terms of a new "effective" Prandtl number as described in section 5. This effective, or wave-averaged, Prandtl number may be a useful way for large scale models to include the effects of gravity waves instead of separately parameterizing the gravity wave heat fluxes; especially since, for localized diffusion, all the gravity wave effects on potential temperature diffusion are not contained in the heat flux term.

While not changing the effective Prandtl number, section 4 investigates how the mean value of the diffusion would have to be adjusted for a height independent solution in a density stratification. When the diffusion is localized in a region of weak potential temperature gradients, the wave damping is less effective, necessitating a larger mean diffusion. Still, the results show that localized diffusion can reduce the magnitude of the total heat flux significantly.

One drawback to the simple model presented here is the assumption of the separation of scales between the eddy diffusion and the wave. A real breaking gravity wave may be more accurately described by a continuous range of scales. More work is needed, but regardless of future results pertaining to breaking waves, the wave heat flux picture presented here should apply to nonbreaking gravity waves damped by molecular diffusion, where the separation of scales between the wave and diffusive motions is an accurate assumption.

Finally, more work needs to be done concerning the case of a realistic spectrum of gravity waves with localized diffusion. This problem is too complicated to handle analytically because the turbulence created by one wave breaking may interact with other waves. A numerical approach is probably necessary. It is hoped, however, that simple models, such as presented here, will aid in the design and understanding of more complex models.

Acknowledgments. This research effort was supported by the Division of Atmospheric Sciences of the National Science Foundation under Grant ATM8404017.

APPENDIX A

Evaluation of the Triple Correlation Terms

In FD four triple correlation terms ($\overline{\nu'\theta'\theta'_{zz}}$, $\overline{\nu'_z\theta'\theta'_z}$, $\overline{\nu'_h\theta'\theta'_{xx}}$, $\overline{\nu'_{hx}\theta'\theta'_x}$) were neglected in deriving the expression for the heat flux. Retaining these terms (and setting $\nu'_h = \nu'$) results in an exact agreement between FD's Eulerian derivation of the heat flux and the present potential temperature surface derivation.

First, consider the two terms involving vertical derivatives. Starting with the form in which these terms enter the expression for the total heat flux and rewriting gives

$$-\frac{1}{\bar{\theta}_z}[\overline{\nu'\theta'\theta'_{zz}} + \overline{\nu'_z\theta'\theta'_z}] = -\frac{1}{\bar{\theta}_z}[(\overline{\nu'\theta'\theta'_z})_z - \overline{\nu'\theta'^2_z}] \quad \text{(A1)}$$

$$\approx \frac{1}{\bar{\theta}_z}\overline{\nu'\theta'^2_z}, \quad \text{(A2)}$$

since the derivative of the averaged quantity can be neglected in the slowly varying approximation. From (2) it follows that

$$\theta'^2_z = \bar{\theta}_z{}^2\alpha^2\cos^2\phi \quad \text{(A3)}$$

$$= \bar{\theta}_z{}^2\frac{\alpha^2}{2}(1 + \cos 2\phi). \quad \text{(A4)}$$

Then, (A2) and (A4) give

$$-\frac{1}{\bar{\theta}_z}[\overline{\nu'\theta'\theta'_z} + \overline{\nu'_z}] = \bar{\theta}_z \frac{\alpha^2}{2}\overline{\nu' \cos 2\phi} \quad \text{(A5)}$$

$$= \bar{\theta}_z \frac{\alpha^2}{2}\overline{\nu \cos 2\phi}$$

$$= \bar{\theta}_z \bar{\nu}\frac{\alpha^2}{2}\gamma. \quad \text{(A6)}$$

Similarly, since

$$\theta_x'^2 = \frac{k^2}{m^2}\bar{\theta}_z{}^2\frac{\alpha^2}{2}\cos^2\phi, \quad \text{(A7)}$$

the two horizontal terms become

$$-\frac{1}{\bar{\theta}_z}[\overline{\nu'\theta'\theta'_{xx}} + \overline{\nu'_x\theta'\theta'_x}] = \frac{k^2}{m^2}\bar{\theta}_z\bar{\nu}\frac{\alpha^2}{2}\gamma.$$

APPENDIX B

Evaluation of the Integrals in (14) and (15)

Given ν as specified by (18) and (19), (14) becomes

$$\bar{\nu} = \nu_0 \int_0^{2\pi}\left(\frac{1-\cos\phi}{2}\right)^n \frac{d\phi}{2\pi} \quad \text{(B1)}$$

$$= \nu_0 \bar{f}_n. \quad \text{(B2)}$$

This integral is symmetric about π, so only the interval from 0 to π need be considered. Changing variables to

$$x = \frac{1-\cos\phi}{2} \quad \text{(B3)}$$

so that

$$d\phi = x^{-1/2}(1-x)^{-1/2}dx, \quad \text{(B4)}$$

gives

$$\bar{f}_n = \frac{1}{\pi}\int_0^1 x^{n-1/2}(1-x)^{-1/2}dx \quad \text{(B5)}$$

or

$$\bar{f}_n = \frac{1}{\pi}B(n+\tfrac{1}{2}, \tfrac{1}{2}), \quad \text{(B6)}$$

where B is the beta function:

$$B(r, s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx \quad \text{(B7)}$$

$$= \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}. \quad \text{(B8)}$$

It follows that

$$\bar{f}_n = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}, \quad \text{(B9)}$$

since $\Gamma(1/2) = \sqrt{\pi}$. Some values of $\bar{f}_n$ are: $\bar{f}_0 = 1$, $\bar{f}_1 = 1/2$, $\bar{f}_2 = 3/8$, $\bar{f}_3 = 5/16$.

From (15) the integral,

$$-\frac{\bar{\nu}}{\nu_0}\beta = \int_0^{2\pi}\left(\frac{1-\cos\phi}{2}\right)^n \cos\phi \frac{d\phi}{2\pi}, \quad \text{(B10)}$$

is needed. Changing variables as above and noting that $\cos\phi = 1 - 2x$ gives

$$-\frac{\bar{\nu}}{\nu_0}\beta = \bar{f}_n - \frac{2}{\pi}\int_0^1 x^{n+1/2}(1-x)^{-1/2}dx \quad \text{(B11)}$$

$$= \bar{f}_n - \frac{2}{\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+2)} \quad \text{(B12)}$$

$$= \bar{f}_n\left[1 - \frac{2(n+1/2)}{n+1}\right] \quad \text{(B13)}$$

$$= -\bar{f}_n\frac{n}{n+1} \quad \text{(B14)}$$

or,

$$\beta = \frac{n}{n+1}, \quad \text{(B15)}$$

where the relation $\Gamma(x) = (x-1)\Gamma(x-1)$ has been used to write (B12) in terms of $\bar{f}_n$.

Finally, (16) gives,

$$\frac{\bar{\nu}}{\nu_0}\gamma = \int_0^{2\pi}\left(\frac{1-\cos\phi}{2}\right)^n \cos 2\phi \frac{d\phi}{2\pi}. \quad \text{(B16)}$$

Changing variables as before gives $\cos 2\phi = 1 - 8x + 8x^2$, and therefore,

$$\frac{\bar{\nu}}{\nu_0}\gamma = \bar{f}_n\left[1 - \frac{8\left(n+\frac{1}{2}\right)}{n+1}\right] + \frac{8}{\pi}\int_0^1 x^{n+3/2}(1-x)^{-1/2}dx \quad \text{(B17)}$$

$$= \bar{f}_n\left[-\frac{7n+3}{n+1}\right] + \frac{8}{\sqrt{\pi}}\frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma(n+3)} \quad \text{(B18)}$$

$$= \bar{f}_n\left[-\frac{(7n+3)}{n+1} + \frac{8\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}{(n+2)(n+1)}\right] \quad \text{(B19)}$$

$$= \bar{f}_n\left[\frac{n(n-1)}{(n+2)(n+1)}\right] \quad \text{(B20)}$$

or

$$\gamma = \frac{n(n-1)}{(n+2)(n+1)}. \quad \text{(B21)}$$

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