Determination of the Heat-Transport Coefficient in Energy-Balance Climate Models by Extremization of Entropy Production

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ABSTRACT

Entropy production has been calculated as a function of the coefficient of meridional heat transfer for two seasonal energy-balance climate models. Both models display extrema in entropy production at values of the coefficient appropriate to the present climate. Inclusion of time dependence in the models is found to be the essential feature for successful application of entropy extremization.

1. Introduction

The principle of minimum entropy production posits that the steady state of a system in which irreversible processes occur is that state for which the rate of entropy production has the minimum value consistent with the constraints that prevent the system from reaching equilibrium (Prigogine, 1947; de Groot and Mazur, 1984). Since on a seasonally averaged basis the earth's climate system may be characterized as a steady state (Dutton, 1973), its rate of entropy production may be expected to be at a minimum. Prigogine's principle may therefore be used to constrain climate models.

Paltridge (1975, 1978) used this principle in a zonal-average equilibrium climate model in which meridional heat transport is expressed in terms of two variables, surface temperature and fractional cloud cover. He determined these two quantities by minimizing the net flow rate of entropy into the system from the external environment. However, Nicolis and Nicolis (1980) have argued that, in general, the time derivative of entropy production has no definite sign and assert that the steady state of the climate system does not necessarily correspond to a minimum of entropy production.

Golitsyn and Mokhov (1978) have used extremization of entropy production in the simpler and more conventional energy balance models of Budyko-Sellers type. They attempted to deduce the value of the meridional heat transfer coefficient in annually averaged models by minimizing the rate of entropy production. The variational calculations were not successful, however, in that the parameter values obtained were not in agreement with the empirical estimated values. Golitsyn and Mokhov concluded that real climate does not correspond to the existence of extremal properties in simple climate models.

In this report we examine the existence of an extremum in entropy production with respect to the heat transfer coefficient in two energy balance climate models, viz. the seasonal model of North and Coakley (1979) and the seasonal model of Ramanathan et al. (1979). A variational calculation is carried out with both models for the Northern Hemisphere. In the model of North and Coakley, the meridional heat transfer is represented in the standard Fourier form. An additional feature is the representation of the zonal average surface temperature by a Fourier-Legendre expansion, which reflects both latitude and time dependence. In the model of Ramanathan et al., the Budyko form of the horizontal heat flux is employed. We find that the entropy production is extremized for a value of the heat transfer coefficient which is near the empirical value. An important feature in both models is the shift in the extremum of entropy production to unrealistic values of the heat transfer coefficient if time dependence is suppressed. In addition, we note that the extrema in both models are characteristically different.

It may be noted here that the general problem of obtaining the climate state from a variational principle has also been investigated by other authors. In recent years Ghil (1976), North et al. (1979), Mobbs (1982) and Smith (1984) have obtained functionals, some of which may be related to entropy production, in variational formulations of the climate problem.

2. Entropy production in energy balance climate models

In vertically integrated models for an azimuthally symmetric spherical planet, the energy balance (heat

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diffusion) equation is, with the meridional and radial components made explicit,

$$\rho \frac{\partial u}{\partial t} + \nabla_R \cdot \mathbf{J} + \nabla_x \cdot \mathbf{J} = 0. \quad (1)$$

Here $\mathbf{J}$ is the heat flux, u the specific internal energy, ∇_R is the divergence operator in the radial (vertical) direction and ∇_x is the operator in the meridional direction using the sine of the latitude as the coordinate:

$$\nabla_x \equiv \frac{\partial}{\partial x}(1 - x^2)^{1/2}$$

where $x = \sin(\text{lat})$. Averaging over the azimuthal coordinate leaves Eq. (1) unchanged. Integration in the radial direction introduces the square of the planet's radius, but this factor is usually incorporated into the proportionality coefficients in each term. However, the term representing the vertical heat transport processes (primarily convection and radiation in the Earth's atmosphere) is typically parameterized as a scalar function of location and the surface air temperature, so

$$\frac{1}{R^2} \int \nabla_R \cdot \mathbf{J} r^2 dr = f(x, T). \quad (2)$$

The function f is often written in terms of the incident solar flux S and the outgoing infrared flux I,

$$f(x, T) = I(x, T) - S(x).$$

Using the specific internal energy $u = cT$, where c is the specific heat, Eq. (1) takes the form

$$\rho c \frac{\partial T}{\partial t} + f(x, T) + \nabla_x \cdot \mathbf{j} = 0, \quad (3)$$

where j is the horizontal (i.e., polar or latitudinal) heat flux obtained from the vertical integration. For a system with constant volume, the Gibbs relation gives the change in specific entropy, ds,

$$ds = \frac{du}{T} = \frac{cdT}{T}. \quad (4)$$

Combining Eqs. (3) and (4) yields the entropy production equation

$$\rho \frac{\partial s}{\partial t} + \frac{f(x, T)}{T} + \frac{1}{T} \nabla_x \cdot \mathbf{j} = 0. \quad (5)$$

An integration of (5) over the remaining dimension will produce an expression for the total entropy production for this system. Since the gradients for irreversible thermodynamics must be simple (no sign change), the integration results in an expression that is suitable for one-half of a hemispherically symmetric planet. Integrating (5) over x, the sole coordinate, produces

$$\frac{\partial S}{\partial t} = \int_0^1 \rho \frac{\partial s}{\partial t} dx$$
$$= -\int_0^1 \frac{f(x, T)}{T} dx - \int_0^1 \frac{1}{T} \nabla_x \cdot \mathbf{j} dx. \quad (6)$$

The first term involving $f(x, T)$ gives the entropy produced by external flows,

$$\frac{d_e S}{dt} = -\int_0^1 \frac{f(x, T)}{T} dx. \quad (7)$$

The remaining term involving the heat flow within the system gives internal entropy production,

$$\frac{d_i S}{dt} = -\int_0^1 \frac{1}{T} \nabla_x \cdot \mathbf{j} dx. \quad (8)$$

If Eq. (8) is integrated by parts and the condition of heat flux symmetry ($j = 0$ for $x = 0$) imposed,

$$\frac{d_i S}{dt} = -\left[\frac{1}{T}(1 - x^2)^{1/2} j\right]_0^1 + \int_0^1 j(1 - x^2)^{1/2} \frac{\partial}{\partial x}\left(\frac{1}{T}\right) dx$$
$$= \int_0^1 \mathbf{j} \cdot \nabla_x \left(\frac{1}{T}\right) dx. \quad (9)$$

With the heat flux in the Fourier form, used by North (1975) and North and Coakley (1979),

$$\mathbf{j} = -D_0 \nabla_x T, \quad (10)$$

Eq. (9) becomes

$$\frac{d_i S}{dt} = D_0 \int_0^1 \frac{1}{T^2} (\nabla_x T)^2 dx \geqslant 0 \quad (11)$$

as is expected for internal entropy production in a dissipative system. We now examine the possibility that the model will evolve toward a steady state. Differentiating the internal entropy production (11) with respect to the heat flow coefficient D_0,

$$\frac{d_i^2 S}{dD_0 dt} = \frac{\partial}{\partial D_0} D_0 \int_0^1 \left(\frac{1}{T} \nabla_x T\right)^2 dx$$
$$= \int_0^1 \left(\frac{1}{T} \nabla_x T\right)^2 dx + 2D_0 \int_0^1 \frac{1}{T} \nabla_x T \frac{\partial}{\partial D_0} \left(\frac{1}{T} \nabla_x T\right) dx. \quad (12)$$

In this last expression, the sign of the second term can be established by considering the implications of a change in the heat flow in the earth's climate system. The parameter D_0 is a measure of the heat flow effectiveness. If D_0 is increased by a small amount, the heat flow improves. This takes heat from the low latitudes where the incoming radiant energy exceeds the outgoing and delivers it to the high latitudes where the outgoing radiation exceeds the insolation. This improvement will serve to decrease the temperatures slightly at low latitudes while increasing them at high latitudes. The logarithmic derivative will thus decrease in magnitude making the second term negative. This shows clearly that the right-hand side of Eq. (12) can be equal to zero, i.e., the rate of entropy production passes through an extremum when varied with respect to D_0.

Before proceeding with the variational calculations we note that in a steady state the internal production of entropy balances the exchange of entropy with the environment, i.e.,

$$\int_{\text{year}} \frac{d_eS}{dt}\,dt = -\int_{\text{year}} \frac{d_iS}{dt}\,dt,$$

or

$$\int_{\text{year}} \int_0^1 \frac{f(x, T)}{T}\,dtdx = D_0 \int_{\text{year}} \int_0^1 \frac{1}{T^2} (\nabla_x T)^2 dtdx. \tag{13}$$

Thus the variation of entropy production with respect to D_0 may be examined by using either of the two equivalent expressions in (13).

3. Variational determination of the heat transport coefficient

a. Model of North and Coakley (1979)

In this model the seasonal energy balance is given as,

$$c(x, \phi)\frac{\partial T(x, \theta, t)}{\partial t} - D_0 \nabla^2 T(x, \phi, t) + A + BT(x, \phi, t) = QS(x, t)a(x, t), \tag{14}$$

where

x	the sine of latitude and ϕ is longitude,
$c(x, \phi)$	the effective thermal inertia per unit area
D_0	the coefficient of heat transport,
$A + BT(x, \phi, t)$	the familiar Budyko-Sellers parameterization of outgoing longwave flux, $I(T, x, t)$,
Q	the solar constant,
$S(x, t)$	the incoming solar radiation per unit area,
$a(x, t)$	the co-albedo.

Dependence on longitude ϕ is included in this equation primarily to take account of the contrast between the effective thermal inertia $c(x, \phi)$ over land and ocean. After integration around a latitude belt and introducing a parameter ν that accounts for land–sea interaction, North and Coakley obtained

$$C_{L,W}\frac{dT_{L,W}}{dt} - D_0 \frac{\partial}{\partial x}(1 - x^2)\frac{\partial}{\partial x} T_{L,W} + \frac{\nu}{f_{L,W}}(T_{L,W} - T_{W,L}) + A + BT_{L,W} = QS(x, t)a_{L,W}(x, t), \tag{15}$$

where subscripts L and W represent land and ocean values respectively. The zonally averaged temperature is obtained by appropriate weighting of the land and ocean temperatures:

$$T(x, t) = f_L T_L + f_W T_W, \tag{16}$$

where f_L and f_W are the land and ocean fractions. Temperature, co-albedo and incident solar radiation are each represented by a series of Legendre polynomials while time dependence is represented by a Fourier series of sines and cosines. For a hemisphere on a symmetric globe, the expression for internal entropy production, using Eqs. (8) and (10), is

$$\frac{1}{B}\frac{d_iS}{dt} = \int_0^1 \int_0^1 \frac{\frac{\partial}{\partial x}(1 - x^2)D\frac{\partial}{\partial x}\sum_{n=0} I_n P_n(x)}{T_0 + T_1(t)P_1(x) + T_2P_2(x)}\,dxdt, \tag{17}$$

where the notation of North and Coakley has been followed in that $D = D_0/B$ is the dimensionless heat transport parameter, the zonal surface temperature $T(x, t) = T_0 + T_1(t)P_1(x) + T_2P_2(x)$ and the outgoing infrared flux $I(x, t) = \sum_{n=0} I_n P_n$.

In the calculation based on a symmetric Northern Hemisphere, the following values of the parameters have been used in accordance with North and Coakley:

$$f_L = 0.6 \qquad f_W = 0.4$$
$$C_L/B = 0.16 \text{ years} \qquad C_W/B = 4.7 \text{ years.}$$
$$Q = 340 \text{ W m}^{-2}$$
$$A = 204.6 \text{ W m}^{-2}$$
$$\nu/B = 0.387$$

Also, we used North and Coakley's (1979) values for the insolation terms on the right-hand side of Eq. (14) in which $QS(x, t)a(x, t)$ is written as

$$\sum_{i=0}^{2} QH_i,$$

where:

$$QH_0 = 236, \quad QH_1(t) = -193(\omega t - 1.5 \text{ dys}) \quad \text{and} \quad QH_2 = -143.$$

The results of the variational calculations are shown in Fig. 1 where the negative of the internal entropy production, Eq. (17), is plotted against D. Thus, we see that the internal entropy production exhibits a maximum, which is equivalent to a minimum in external entropy exchange. We note that the extremum is found to occur at $D = 0.218$. This is reasonably close to $D = 0.238$ used by North and Coakley (1979). We find that the temperature distributions calculated with these two D values are in close agreement with each other.

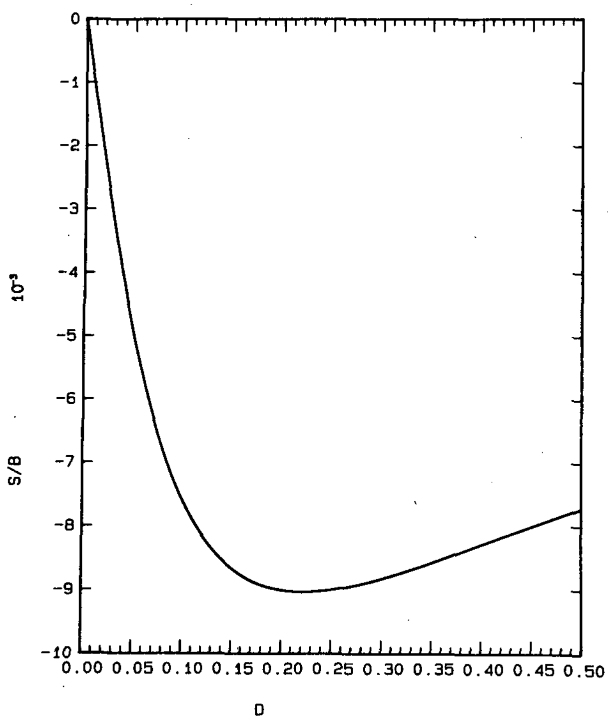

FIG. 1. Rate of entropy production as a function of heat transfer coefficient D in the model of North and Coakly for the northern hemisphere.

When time dependence is suppressed in the model, we recover the result of Golitsyn and Mokhov (1978) who found the extremum near $D = 0.16$. Sensitivity calculations with changes in the values of the solar constant Q, the land–sea interaction coefficient and the albedo α, did not produce significant changes in the location of the extremum for both the time dependent and time independent models. Among the parameters we examined, the location of the extremum showed some sensitivity to the phase of the infrared flux I.

b. *Model of Ramanathan et al. (1979)*

Here, we present a variational calculation of the entropy production as a function of the heat transport coefficient in the seasonal energy balance climate model of Ramanathan et al. (1979). An interesting difference from the previous model is that the Budyko form of horizontal heat flux parameterization is used. The model computes the zonal surface temperature for the Northern Hemisphere by solving the equation

$$C(x)\left\{\frac{\partial T(x,t)}{\partial t} + \omega[T(x,t) - \bar{T}(x)]\right\} = S(x,t)[1-\alpha(x,t)] - \lambda[T(x,t) - \langle T\rangle] - I(x,t), \quad (18)$$

where

- $C(x)$ the effective thermal inertia of the earth–atmosphere system (in J m^{-2} K^{-1});
- $\bar{T}(x)$ the annually averaged zonal surface air temperature;
- $\langle T\rangle$ the hemispherically averaged surface air temperature;
- ω $(=2\pi/\tau)$ for $\tau = 12$ months;
- $S(x, t)$ the incoming solar radiation (in W m^{-2}),
- $\alpha(x, t)$ the zonal albedo;
- λ the heat transport coefficient; and
- $I(x, t)$ the outgoing longwave flux.

The term $\omega(T - \bar{T})$ in Eq. (18) represents seasonal heat storage, primarily in the oceans, which causes lags between insolation and the temperature in each zone.

Ramanathan et al. (1979) found that the model reproduces the observed temperature distribution in the Northern Hemisphere for $\lambda = 3.4$ W m^{-2} K^{-1}.

The albedo α in Eq. (18) is a function of location, the cosine of the solar zenith angle and the local cloud cover and incorporates the ice-albedo feedback (Lian, 1978). We have solved Eq. (18) using a fifth and sixth order Runge-Kutta scheme.

The internal entropy production is plotted as a function of λ in Fig. 2. We find a minimum at $\lambda = 3.3$ W m^{-2} K^{-1}, which is close to the empirical value of 3.4. It is notable that this minimum has a very local character and is one of the several extrema in the range of λ which is shown. Nevertheless we conclude that for the present climate state the model is consistent with

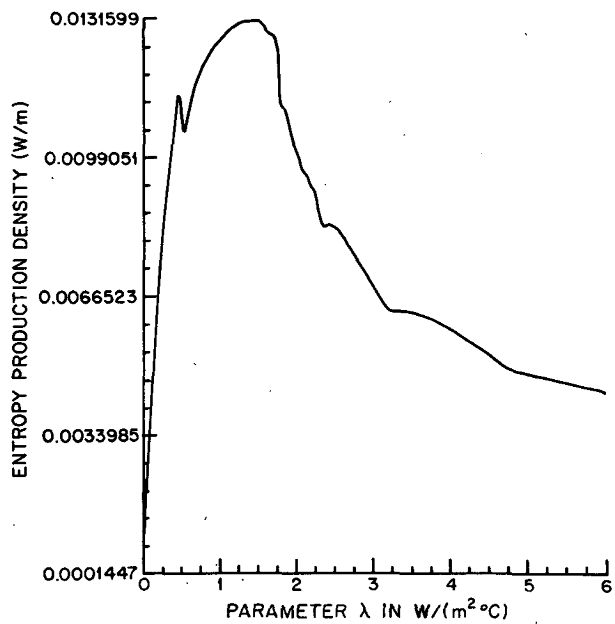

FIG. 2. Rate of entropy production as a function of the Budyko heat transfer coefficient λ in the model of Ramanathan et al. for the northern hemisphere.

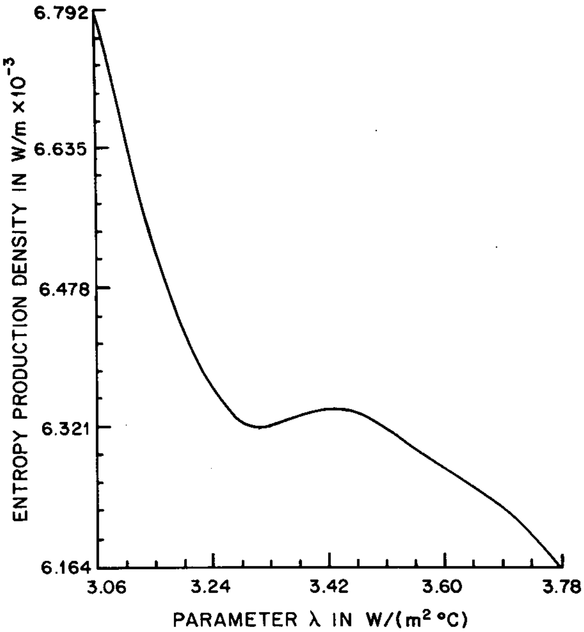

FIG. 3. Rate of entropy production versus λ in the model of Ramanathan et al. for the Northern Hemisphere (expanded scale).

Prigogine's theorem. It may be recalled that the theorem does not specify either the size of a minimum in entropy production or the number of minima possible for a system. The other minima in Fig. 2 signify different steady states in the model state space. In view of the highly parameterized representation of physical processes in this model these minima, away from the current climate state, are unlikely to have physical significance. In Fig. 3 we display the local minimum near the empirical value of λ on an expanded scale. Repetition of the variational calculation with a time-independent version of the model of Ramanathan et al. did not show a minimum in the range of λ values displayed in Fig. 3.

4. Conclusions

From the analysis presented here, it appears that entropy extremization can be imposed as an additional constraint on simple climate models. Our calculations with the models of North and Coakley (1979) and Ramanathan et al. (1979) indicate that the inclusion of time dependence is the essential model feature if entropy production is to be used as a variational quantity. We do not have an explanation of why seasonality brings the calculated meridional heat flux into agreement with observations; this is an important question that needs investigation.

It is known that the theorem of minimum entropy production for steady states can be proved only for linear systems. Meridional heat flux is parameterized as a linear process in the models of North and Coakley and Ramanathan et al., although ice-albedo feedback is nonlinear in the latter model. Many processes in the climate system are, of course, highly nonlinear. The scope of the applicability of Prigogine theorem to the modeling of climate interactions needs to be established. Our results show that it is feasible to use entropy extremization as a practical technique of closure for linearly modeled processes.

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