Inertial Trajectories on a Rotating Earth

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ABSTRACT

The trajectories of inertial flows on a rotating earth are calculated, in an attempt to reconcile the differing heuristic suggestions in the literature on the subject. It is shown that westward propagating "nearly closed" orbits are possible away from the equator. For orbits crossing the equator, we find a stationary, "figure-eight-like" orbit, together with eastward and westward propagating modes. Near the pole, the convergence of longitudes causes the trajectories to be deflected cyclonically in contrast to the deflection of the Coriolis force, giving rise to a westward propagating mode that meanders about a central latitude.

1. Introduction

Inertia currents are characterized by the simplest possible dynamics: no forces whatsoever act on any element of fluid. Under the assumptions of zero pressure gradient, no friction or boundary effects, and that buoyancy forces suitably cancel, a two-dimensional inertial motion on a geopotential surface will occur. The length scale associated with such a flow is quite large, as no retarding forces obstruct the propagation of a fluid element, and is usually of the order of the earth's radius. Newton's first law ensures that in an inertial frame the trajectory of a fluid particle subject to some finite initial velocity will be constant velocity in a straight line. In a rotating frame such as the earth, the trajectory will appear to be more complicated. The neglect of all forces renders observations of inertial motion in the atmosphere or ocean as unobservable as the straight line counterpart in an inertial system. Nonetheless, the concept of inertial motion provides a useful paradigm for many studies of geophysical phenomena.

The simplest such trajectory is the familiar "inertial circle," which results when the variation of Coriolis parameter with latitude is neglected—the f-plane approximation. In such motions, a fluid element moves anticyclonically in a circle whose radius of curvature is equal to the particle speed divided by the Coriolis parameter. These oscillations are described in all textbooks on the subject (e.g., Haltiner and Martin 1957; Von Arx 1962; Holton 1979; Houghton 1986; Gill 1982).

In midlatitudes, the textbooks also agree. It is assumed that f varies linearly with poleward displacement (the β-plane approximation). Heuristic arguments suggest that the inertial circle will migrate slowly westward because of changes in the radius of curvature with f (Haltiner and Martin 1957; Von Arx 1962). However, differences arise between textbooks when motions near the equator are considered (the equatorial β plane). It is clear that variations in f are not small, so that the resulting motions are more complicated. Haltiner and Martin (1957, their Fig. 12-5) anticipate an eastward sinusoidal-like trajectory, and hence no circular motion, because the sign of the radius of curvature changes when the equator is crossed. This result is a little perplexing, since on either side of the equator—but away from it—the trajectory is presented as a westward-migrating circle. Von Arx (1962, his figure 4-14), on the other hand, anticipates a complicated westward oscillatory drift composed of a clockwise circle north of the equator and an anticlockwise circle south of the equator (i.e., both anticyclonic), combining to give a westward drifting figure-eight-like orbit.

Thus the oceanographer, reading Von Arx, and the meteorologist, reading Haltiner and Martin, form different ideas about inertial trajectories. This is partly due to the fact that the relevant velocities differ by two or more orders of magnitude. Yet there are differences of opinion on what must surely be the simplest possible

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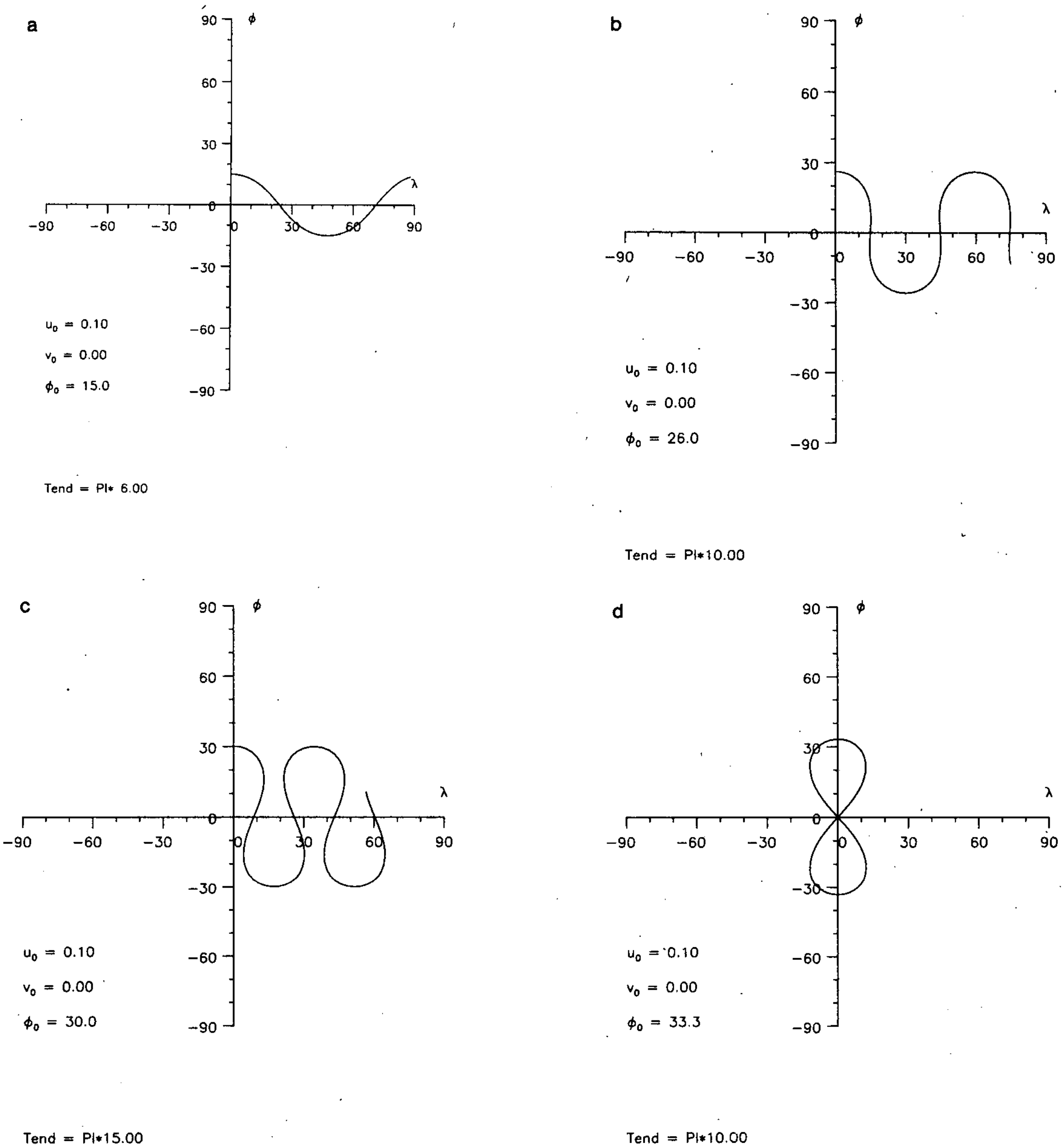
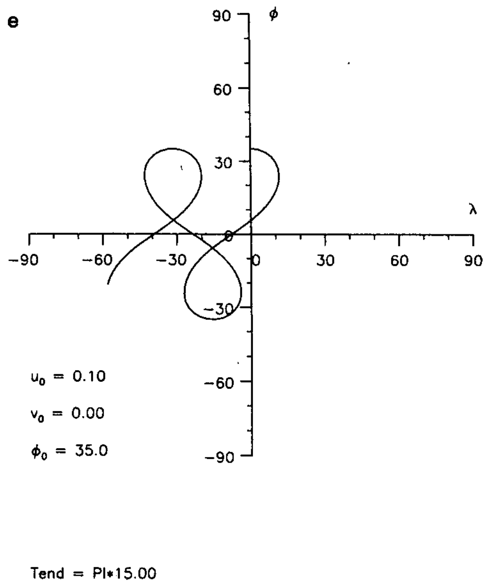


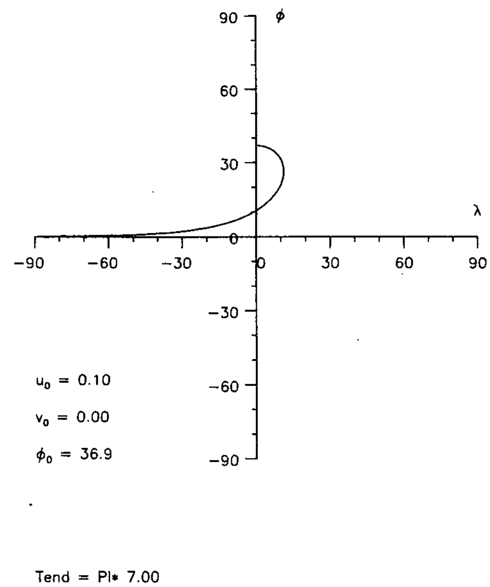
FIG. 1. Trajectories starting with an eastward velocity in increasing order of initial latitude. Initial values of u, v (v_0 is here zero), latitude ϕ, and time of integration T_{end} are shown. (a) Haltiner and Martin's (1957, Fig. 12-5). (d) Closed figure of eight orbit. (e) Von Arx's (1962, Fig. 4-14) westward migrating orbit. (f) The trajectory which approaches the equator asymptotically. (g) β plane westward migrating inertial circle contained entirely in the hemisphere of origin.

dynamics. Also, we can trivially consider a pure eastward or pure westward trajectory right on the equator (which is not subject to Coriolis forces since f vanishes there). How can the motions described above be converted smoothly to give a linear east–west motion?

At high latitudes (i.e., near the pole) both f and β plane physics are violated and the only orbit the textbooks suggest is a circle around the pole (Von Arx 1962, Fig. 4-14).

We were intrigued by these discrepancies; particularly because the textbook arguments all appear to be heuristic. Wiin-Nielsen's (1970) and Whipple's (1917) analytic treatments are of limited use because of their neglect of various terms in the equations of motion. Wiin-Nielsen neglected some metric terms, giving erroneous results at middle and high latitudes, and Whipple assumed motions to be near the equator. Yet, in Lagrangian coordinates, the problem on a spherical





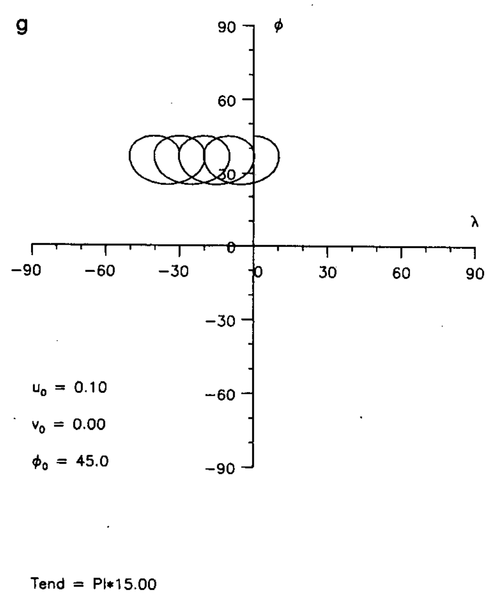


FIG. 1. (*Continued*)

earth is perfectly straightforward. In this paper, we display the range of possible inertial trajectories, including all previously postulated scenarios as special cases. Near the equator, we show the possibility of westward migrating inertial circles, in addition to eastward sinusoidlike trajectories. At high latitudes westward motion, oscillating around some given latitude, is also possible. Poleward of this latitude (whose value depends on the speed of the orbit), all trajectories are deflected equatorward because of the rapid convergence of longitude. Equatorward of this latitude, trajectories are deflected poleward by the Coriolis force. Thus, a trajectory starting westward at the right speed will simply encircle the pole in a closed orbit on the same latitude, as suggested by Von Arx.

2. Formulation

We nondimensionalize with the natural scalings: lengths on R, the radius of the earth; time on $1/(2\Omega)$, where Ω is the frequency of the earth's rotation; and velocities on $2\Omega R$. This makes the velocity scale around 1 km s^{-1}, which is clearly unphysically large. Much of the interesting phenomena occur with nondimensional velocities of order 0.1, which is not unreasonable for

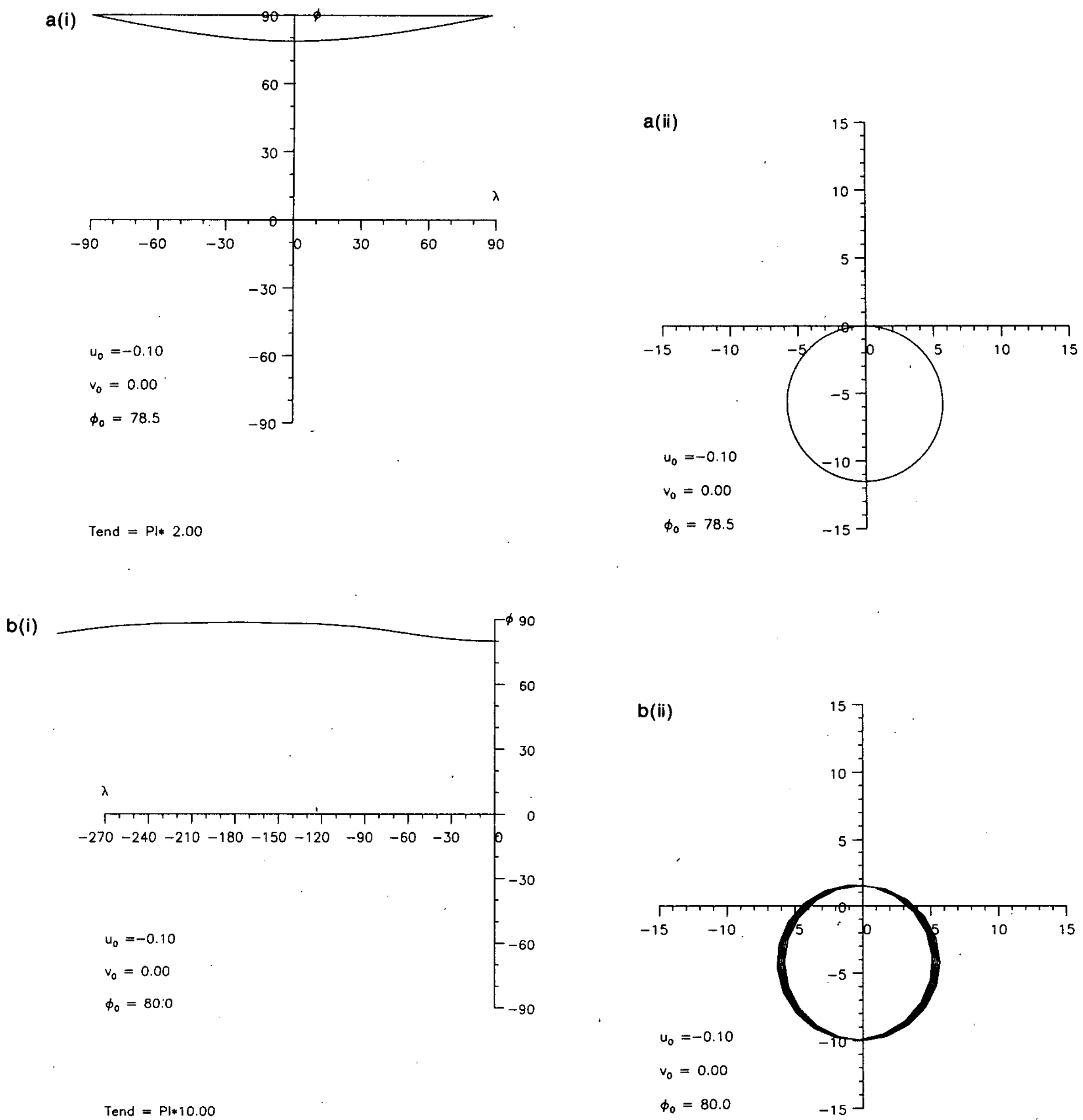


FIG. 2. Trajectories starting with a westward velocity. Initial values of u, v (v_0 is here zero), latitude ϕ and time of integration T_{end} are given. (a) An orbit which passes very close to the pole: (i) (λ, ϕ) coordinates; (ii) polar view, radius = colatitude, azimuth = longitude. (b) Westward-migrating sinusoidlike orbit induced by rapid longitude convergence near the pole. At the part of the orbit nearest the pole, there is cyclonic deflection because of the convergence of longitude: (i) and (ii), as in (a). (c) Steadily propagating orbit around the pole (Von Arx 1962, Fig. 4-14): (i) and (ii) as in (a). (d) Rapid westward flow oscillating around the equator.

the atmosphere (100 m s^{-1}). For oceanic applications, these arguments can serve only as mechanistic propositions; all realistic values of nondimensional velocity must be very small. The energetic inertial peak observed in oceanic velocity spectra has been attributed to various forcing mechanisms, such as fronts (Paldor 1983), winds (Stern 1977), etc. These forcing mechanisms excite the oscillations at the natural (inertial) frequency, but will interfere with the purely inertial motion discussed here, which involves no external forces.

Using spherical polar coordinates (λ, ϕ) to measure longitude and latitude, respectively, the nondimensional Coriolis parameter is $\sin\phi$. Using u, v to denote eastward and northward velocities, the Lagrangian equations of motion become simply

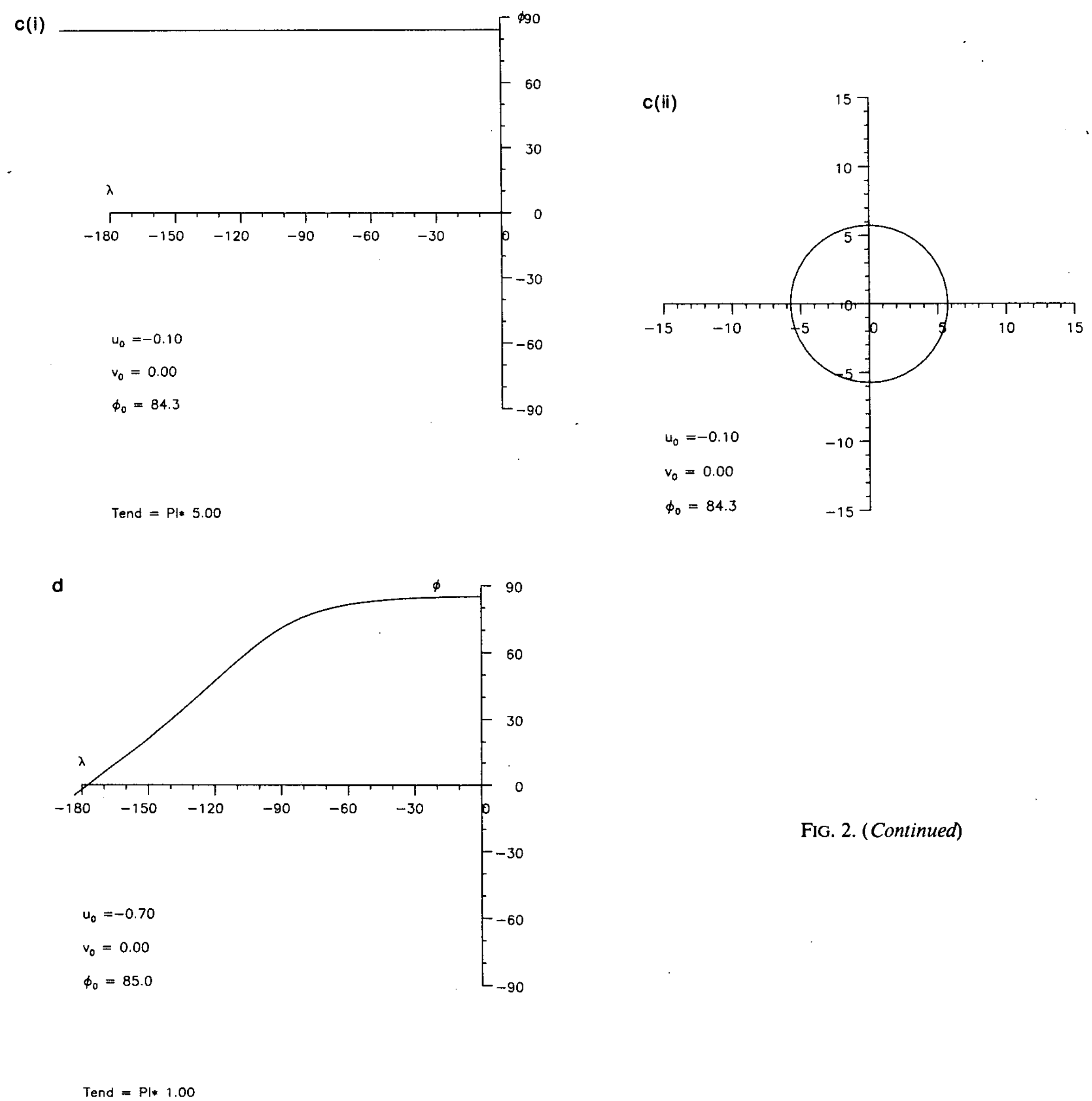


FIG. 2. (*Continued*)

$$du/dt = v \sin\phi(1 + u/\cos\phi) \tag{2.1}$$

$$dv/dt = -u \sin\phi(1 + u/\cos\phi) \tag{2.2}$$

$$d\phi/dt = v \tag{2.3}$$

$$d\lambda/dt = u/\cos\phi. \tag{2.4}$$

Here the first two equations are force balances in the east and north directions respectively, including the effect of longitude convergence since the motion encompasses a large fraction of the earth's surface. This term becomes dominant near the pole where $\cos\phi$ vanishes. The latter pair define v and u as the rate of change of latitude and longitude, respectively. In contrast to the f-plane or β-plane equations, these equations do not contain any free parameters, so that trajectories depend solely on initial conditions. The f plane is obtained by replacing ϕ in (2.1) and (2.2) with its initial value (e.g., ϕ_0), and the β plane is obtained by replacing $\sin\phi$ by $\sin\phi_0 + (\phi - \phi_0)\cos\phi_0$. Since λ does not appear explicitly, we can assume, without loss of generality, that its initial value is 0.

Since the set (2.1)–(2.4) is invariant under the transformation $\phi \rightarrow -\phi$; $v \rightarrow -v$, we can assume that the motion starts off in one hemisphere, say, the Northern Hemisphere. Motions starting in the Southern Hemisphere can be simply obtained by looking at the mirror image of the Northern Hemisphere trajectories.

3. Inertial trajectories

It is straightforward to obtain two first integrals of the inertial motion. First, by multiplying Eqs. (2.1) and (2.2) by u and v respectively, adding, and integrating w.r.t. time, we find the standard energy constraint

$$u^2 + v^2 = C^2 \quad (3.1)$$

for some constant speed C. Thus, the speed remains constant at all times. In addition to this energy conservation integral, we can obtain another integral by dividing (2.1) by (2.3) (v cancels out) to get the first-order equation

$$du/d\{\cos\phi\} = -1 - u/\cos\phi. \quad (3.2)$$

Solving this simple equation yields

$$\cos\phi(\cos\phi + 2u) = \cos\phi_0(\cos\phi_0 + 2u_0), \quad (3.3)$$

which states the conservation of angular momentum (Cushman-Roisin 1982). The integral (3.3) gives a useful relation between the latitude of the fluid particle, ϕ, and the eastward component of the velocity vector, u, in terms of their initial values, which have subscript zero.

a. Eastward initial velocities

In this section we discuss trajectories emanating initially eastward, so that the initial condition $v_0 = 0$, $u_0 > 0$ will be assumed throughout. We assume that the initial velocity, u_0, is fixed (and positive), and vary the initial latitude, ϕ_0, only.

If the initial latitude is $\phi_0 = 0$, then the trivial solution is $u(t) = u_0$, $\lambda(t) = u_0 t$ so that the fluid element will remain at the equator moving eastward indefinitely. This is the analogue of the straight line trajectory of the inertial system, and this trajectory results because the Coriolis deflection vanishes there.

When the initial latitude is north of the equator, i.e., $\phi_0 > 0$, several types of trajectories are possible. If u_0 and ϕ_0 satisfy

$$\cos\phi_0(2u_0 + \cos\phi_0) > 1 \quad (3.4)$$

(i.e., the trajectory starts not too far from the equator) then the integral (3.3) implies that $\min\{u(t)\}$ is positive and therefore the trajectory will be eastward propagating sinusoidlike orbits (Fig. 1a) as suggested by Haltiner and Martin (1957). If equality holds in (3.4), then $\min(u) = 0$ when $\cos\phi = 1$, and the trajectory of Fig. 1b is encountered.

When the initial conditions satisfy

$$1/\cos\phi_0 > \cos\phi_0 + 2u_0 > 1$$

(i.e., slightly farther northward) then $u(t)$ becomes negative but remains always larger than $-u_0$ so that the trajectory shown in Fig. 1c is encountered. Moving the initial latitude, ϕ_0, farther northward (i.e., decreasing $\cos\phi_0$), we encounter the equatorial figure-eight trajectory shown in Fig. 1d. The condition for obtaining this orbit can be derived from a complicated integral, which is not discussed here. Moving the initial latitude further northward yields the equatorial westward migrating coupled circles (Fig. 1e), as suggested by Von Arx (1962). Moving the initial latitude slightly northward such that

$$\cos\phi_0 = 1 - 2u_0,$$

the meridional velocity has to vanish when $\cos\phi = 1$, and the trajectory will therefore approach the equator asymptotically, as in Fig. 1f.

Eventually, the initial latitude is large enough for the trajectory not to reach the equator, so that it is contained within the hemisphere of origin (Fig. 1f). This happens, according to (3.3), when

$$\cos\phi_0 < 1 - 2u_0.$$

At this initial latitude we obtain the midlatitude, westward migrating, β-plane inertial circle suggested by both Haltiner and Martin and Von Arx.

b. Westward initial velocities

We now assume that $u_0 \leqslant 0$, $v_0 = 0$ and examine the trajectories encountered when ϕ_0 is varied between 0 and $\pi/2$ and u_0 assumes different values. When $\phi_0 = 0$, the trivial solution is a trajectory which remains on the equator indefinitely, as in section 3a. Away from the equator we first encounter the westward migrating β-plane inertial circles, as in Fig. 1g. These will be obtained whenever $\max\{u(t)\}$ reaches $|u_0|$. According to (3.3), this occurs whenever $\cos\phi_0 > 2|u_0|$. As the initial latitude increases, the trajectory approaches the pole closer and closer (e.g., Fig. 2a). Although the pole cannot lie precisely on any trajectory, an orbit can be constructed that passes arbitrarily close to the pole.

Increasing ϕ_0 further, this condition is violated and $\max\{u(t)\}$ is smaller than $|u_0|$. Equation (3.3) shows that at this initial latitude $u(t)$ can never vanish, so a westward motion, oscillating in latitude, is produced, as in Fig. 2b. When the trajectory starts such that $\cos\phi_0 = -u_0$, dv/dt and du/dt both vanish according to (2.1) and (2.2). The only solution is then $\phi(t) = \phi_0$; $u(t) = u_0$, $d\lambda/dt = -1$, and the trajectory remains on the same latitude at all times, as in Fig. 2c.

It is interesting to note that when the trajectory starts at a latitude larger than $\cos^{-1}u_0$, dv/dt is negative, so that v is directed equatorward, as in the higher latitude part of the orbit in Fig. 2b. The reason for this is that the convergence of the longitudes dominates over the Coriolis term.

One other, rather unphysical case can occur, if u_0 is sufficiently large and negative so that there is only one latitude for which (3.3) yields a solution with $u = u_0$ (rather than the two values evident in Fig. 2b). From (3.3), this involves

$$u_0 < -\frac{1}{2} - \frac{1}{2}\cos\phi_0.$$

In this case, the orbit must meander between $\phi = \pm\phi_0$, moving rapidly westwards (e.g., Fig. 2d).

c. *Other initial velocities*

It is trivial to show that trajectories which do not begin with $v = 0$ must become one of those shown above, so that all cases have been discussed.

To show this, one only has to replace the initial conditions u_0, v_0, ϕ_0 by the initial conditions

$$U_0 = \pm(u_0{}^2 + v_0{}^2)^{1/2}; \quad V_0 = 0$$

and choose a new initial latitude Φ_0 such that

$$\cos\Phi_0(\cos\Phi_0 + 2U_0) = \cos\phi_0(\cos\phi_0 + 2u_0)$$

(which can always be achieved) so that one of the trajectories in section 3a or 3b is encountered.

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