Improving the Anelastic Approximation

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ABSTRACT

A new diagnostic equation is presented which exhibits many advantages over the conventional forms of the anelastic continuity equation. Scale analysis suggests that use of this "pseudo-incompressible equation" is justified if the Lagrangian time scale of the disturbance is large compared with the time scale for sound wave propagation and the perturbation pressure is small compared to the vertically varying mean-state pressure. No assumption about the magnitude of the perturbation potential temperature or the strength of the mean-state stratification is required.

In the various anelastic approximations, the influence of the perturbation density field on the mass balance is entirely neglected. In contrast, the mass-balance in the "pseudo-incompressible approximation" accounts for those density perturbations associated (through the equation of state) with perturbations in the temperature field. Density fluctuations associated with perturbations in the pressure field are neglected.

The pseudo-incompressible equation is identical to the anelastic continuity equation when the mean stratification is adiabatic. As the stability increases, the pseudo-incompressible approximation gives a more accurate result. The pseudo-incompressible equation, together with the unapproximated momentum and thermodynamic equations, forms a closed system of governing equations that filters sound waves. The pseudo-incompressible system conserves an energy form that is directly analogous to the total energy conserved by the complete compressible system.

The pseudo-incompressible approximation yields a system of equations suitable for use in nonhydrostatic numerical models. The pseudo-incompressible equation also permits the diagnostic calculation of the vertical velocity in adiabatic flow. The pseudo-incompressible equation might also be used to compute the net heating rate in a diabatic flow from extremely accurate observations of the three-dimensional velocity field and very coarse resolution (single sounding) thermodynamic data.

1. Introduction

Researchers studying small-scale atmospheric circulations frequently approximate the full equation of continuity with the anelastic continuity equation

$$\nabla \cdot (\bar{\rho}\mathbf{V}) = 0, \qquad (1)$$

where $\mathbf{V}$ is the three-dimensional velocity vector, physical height is the vertical coordinate, and $\bar{\rho}$ is defined in one of two different ways. These different definitions for $\bar{\rho}$ are not equivalent and result from different versions of the anelastic approximation.

Equation (1) was first discussed by Batchelor (1953), who defined $\bar{\rho}(z)$ to be the density in an adiabatically stratified, horizontally uniform reference state. The name "anelastic" was coined by Ogura and Phillips (1962) who derived (1), together with approximate momentum and thermodynamic equations, through a rigorous scale analysis. Their scaling analysis assumes: first, that all deviations of the potential temperature $\delta\theta$ from some constant mean value Θ are small, and second, that the time scale of the disturbance is similar to the time scale for gravity wave oscillations. As a consequence of the first assumption, their definition of $\bar{\rho}$ is identical to Batchelor's. Hereafter, the approximate equations described by Ogura and Phillips will be referred to as the "original anelastic approximation." An important attribute of the original anelastic system is that it does not support sound waves, allowing the governing equations to be numerically integrated using a much larger time step than that required to integrate the complete compressible system. The original anelastic system also conserves energy.

One advantage of the rigorous scaling arguments presented by Ogura and Phillips is that they allow one to estimate the validity of the approximation without actually solving the equations. The terms that are neglected in the original anelastic equations are an order $\epsilon = \delta\theta/\Theta$ smaller than those which are retained. Thus in the case of dry convection, where mixing will keep the environmental lapse rate close to adiabatic, ϵ will be small and the anelastic equations can be used to represent nonacoustic modes with complete confidence. If the phenomenon of interest is deep moist convection or gravity wave propagation, however, the mean-state stability can be sufficient to make ϵ rather

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large. As an example, the $\delta\theta$ across a 10 km deep isothermal layer is approximately 40% of the mean Θ. In such a case, the a priori error estimates obtained from the original anelastic scaling analysis are rather discouraging.

Since some of the largest errors in the original anelastic approximation are generated by deviations of the mean-state potential temperature from a constant reference value, several authors (Dutton and Fichtl 1969; Gough 1969; Wilhelmson and Ogura 1972; Lipps and Hemler 1982) have presented alternative "soundproof" equations in which the thermodynamic variables associated with the adiabatic reference state are replaced with their values in the actual mean state. Although these authors make different approximations in the momentum equations, and thus derive different "anelastic" systems, they all obtain a continuity equation of the form (1) in which $\bar{\rho}(z)$ is defined as the density in the actual (nonadiabatic) mean state. Hereafter, when $\bar{\rho}$ is defined with respect to the actual mean state, (1) will be referred to as the "modified anelastic continuity equation" in order to distinguish it from the original anelastic continuity equation proposed by Ogura and Phillips (1962). In this paper, the behavior of the modified anelastic continuity equation will be examined in connection with the particular set of approximate governing equations (hereafter, the "modified anelastic system") proposed by Wilhelmson and Ogura (1972). The Wilhelmson–Ogura system is not necessarily the "best" of the various approximate systems that include the modified anelastic continuity equation; for example, the system developed by Lipps and Hemler (1982) has better energy conservation properties. Wilhelmson and Ogura, however, write the momentum equations in a form that is convenient for comparison with the original anelastic and the pseudo-incompressible systems, and their equations have been widely used in numerical models.

It is difficult to construct a priori estimates of the error associated with the various soundproof systems involving the modified anelastic continuity equation, because most of these systems have not been derived through a consistent scale analysis. The system proposed by Lipps and Hemler (1982) is the exception, but in this case one scaling assumption is that $d\bar{\theta}/dz$ is small, a condition that is not well satisfied in very stable regions like the stratosphere. In this paper, we demonstrate that sound waves can be filtered from the governing equations with minimal approximation by replacing the compressible continuity equation with the "pseudo-incompressible equation"

$$\nabla \cdot (\bar{\rho}\bar{\theta}\mathbf{V}) = \frac{H}{c_p\bar{\pi}}, \qquad (2)$$

where $\bar{\rho}(z)$ and $\bar{\theta}(z)$ are the vertically varying mean-state density and potential temperature, $\bar{\pi}(z) = (\bar{p}(z)/p_0)^{R/c_p}$ is the mean-state Exner function, and H is the rate of heating per unit volume. No significant modifications are required in the other governing equations. The primary difference between (1) and (2) is the presence of the mean potential temperature inside the divergence operator in the pseudo-incompressible equations. In the limit $\epsilon \rightarrow 0$, when the original anelastic continuity equation must hold, (2) reduces to (1) because $d\bar{\theta}/dz$ and the heating become negligible.

The remainder of this paper is organized as follows. Section 2 discusses the derivation of the pseudo-incompressible equation (2) through scale analysis. The physical nature of this approximation is discussed in section 3. Section 4 provides a quantitative comparison of the pseudo-incompressible and anelastic continuity equations. The energy conservation properties of the pseudo-incompressible and anelastic systems are compared in section 5. Practical applications of the pseudo-incompressible approximation are discussed in section 6. Section 7 contains the conclusions.

2. Derivation of the pseudo-incompressible equation through scale analysis

The equation of state for dry air may be written

$$\pi = \left(\frac{R}{p_0}\rho\theta\right)^{R/c_v}. \qquad (3)$$

Taking the logarithm and total derivative of (3) yields

$$\frac{c_v}{R\pi}\frac{D\pi}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\theta}\frac{D\theta}{Dt}, \qquad (4)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Substitution from the continuity and thermodynamic equations into (4) gives

$$\frac{c_v}{R\pi}\frac{D\pi}{Dt} + \nabla \cdot \mathbf{V} = \frac{H}{c_p\rho\theta\pi}. \qquad (5)$$

Let the total thermodynamic fields be divided into a vertically varying mean state and a perturbation as follows:

$$\pi = \bar{\pi}(z) + \pi'(x, y, z, t);$$
$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t);$$
$$\theta = \bar{\theta}(z) + \theta'(x, y, z, t).$$

Substituting for $\rho\theta$ from the equation of state (3), and assuming that $\pi' \ll \bar{\pi}$, (5) may be approximated as

$$\frac{c_v}{R\bar{\pi}}\frac{D\pi'}{Dt} + \frac{c_v w}{R\bar{\pi}}\frac{d\bar{\pi}}{dz} + \nabla \cdot \mathbf{V} = \frac{RH}{c_p p_0 \bar{\pi}^{c_p/R}}. \qquad (6)$$

If the first term in (6) can be neglected, the equation of state for the mean fields may be used to write the remaining terms as

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$$\frac{w}{\bar{\rho}\bar{\theta}}\frac{d\bar{\rho}\bar{\theta}}{dz} + \nabla\cdot\mathbf{V} = \frac{H}{c_p\bar{\rho}\bar{\theta}\bar{\pi}}, \qquad (7)$$

which is equivalent to the pseudo-incompressible equation (2). Therefore, let us examine the circumstances under which the first term in (6) is negligible. Since the heating rate has no essential influence on this question we temporarily assume that the flow is adiabatic.

Define the following scales and nondimensional variables: $(x, y) = L(x^*, y^*)$, $z = Dz^*$, $(u, v) = U(u^*, v^*)$, $w = Ww^*$, $\bar{\theta} = \Theta\bar{\theta}^*$, $\bar{\pi} = 1\cdot\bar{\pi}^*$, $\pi' = P\pi^*$, $t = Tt^*$. In the preceding, T is a Lagrangian time scale and unity is the scale for $\bar{\pi}$. Suppose $O(\partial u/\partial x) = O(\partial w/\partial z)$, so that $U/L = W/D$, then the nondimensional adiabatic form of (6) can be written

$$\frac{c_p\Theta P}{c_s^2 T}\bar{\theta}^*\frac{D\pi^*}{Dt^*} + \frac{c_v U}{RL}\frac{w^*}{\bar{\pi}^*}\frac{d\bar{\pi}^*}{dz^*} + \frac{U}{L}\nabla^*\cdot\mathbf{V}^* = 0, \qquad (8)$$

where ∇^* is the divergence in nondimensional coordinates and c_s is the speed of sound in the basic-state. The first term can be neglected with respect to the last term when

$$\frac{c_p\Theta P}{c_s^2 T} \ll \frac{U}{L}. \qquad (9)$$

The nondimensional x-momentum equation can be written

$$\frac{U}{T}\frac{Du^*}{Dt^*} + \frac{c_p\Theta P}{L}\bar{\theta}^*\frac{\partial\pi^*}{\partial x^*} = 0. \qquad (10)$$

In order to have a nontrivial balance in (10), the scale for P may be chosen such that $P = UL/(c_p\Theta T)$. Substituting for P in (9), one obtains the criteria

$$T \gg \frac{L}{c_s}. \qquad (11)$$

Therefore, *the use of the pseudo-incompressible equation is justified when the Lagrangian time scale associated with the disturbance is much greater than the time scale associated with sound wave propagation.* The only additional assumption which must be made is that $\pi' \ll \bar{\pi}$.

3. The pseudo-incompressible approximation

It has just been demonstrated that the pseudo-incompressible equation is a good approximation when the Lagrangian time scale of the motion is much greater than the time scale for sound wave propagation and $\pi' \ll \bar{\pi}$. The following provides an alternative physical interpretation of the approximation required to obtain the pseudo-incompressible equation. Let us define a "pseudo-incompressible density" ρ^* satisfying the equation of state:

$$\bar{\pi} = \left(\frac{R}{p_0}\rho^*\theta\right)^{R/c_v}. \qquad (12)$$

Equation (12) is identical to the full equation of state (3) except that the total pressure π has been replaced by the mean-state pressure $\bar{\pi}$. Unlike the true density, ρ^* is unaffected by changes in the perturbation pressure field. The term "pseudo-incompressible" has been chosen because the response of ρ^* to compression or expansion is limited to that forced by changes in the mean-state pressure.

Consider the following set of equations for an inviscid flow:

$$\frac{Du}{Dt} - fv + c_p\theta\frac{\partial\pi'}{\partial x} = 0, \qquad (13)$$

$$\frac{Dv}{Dt} + fu + c_p\theta\frac{\partial\pi'}{\partial y} = 0, \qquad (14)$$

$$\frac{Dw}{Dt} + c_p\theta\frac{\partial\pi'}{\partial z} = g\frac{\theta'}{\bar{\theta}}, \qquad (15)$$

$$\frac{D\theta}{Dt} = \frac{H}{c_p\rho^*\bar{\pi}}, \qquad (16)$$

$$\frac{D\rho^*}{Dt} + \rho^*\nabla\cdot\mathbf{V} = 0. \qquad (17)$$

Equations (13)–(15) are identical to the momentum equations in the complete compressible system (note that although the hydrostatically balanced mean-state pressure has been removed, the pressure-gradient term has not been linearized). Equation (16) is identical to the complete thermodynamic equation except that the effects of the perturbation pressure field are ignored in the coefficient of the heating term[1]. The essential difference between these equations and the complete compressible system is found in (17), where the pseudo-incompressible density has been substituted for the true density in the mass balance.

The preceding system of equations can be simplified by eliminating ρ^*. Note that

$$\rho^*\theta = \bar{\rho}\bar{\theta} \qquad (18)$$

because the mean fields satisfy the equation of state. This relationship can be used to replace ρ^* by $\bar{\rho}\bar{\theta}/\theta$ in the heating term in the thermodynamic equation. As a slightly less trivial exercise, one can take the convec-

[1] This is consistent with an approximation commonly used in numerical models in which the influence of π' on H is neglected by omitting the dependence of the saturation mixing ratio on the perturbation pressure (Wilhelmson and Ogura 1972).

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tive derivative of (18) and substitute, from (16) and (17), to obtain

$$\frac{H}{c_p\bar{\pi}} - \rho^*\theta\nabla\cdot\mathbf{V} = w\frac{d\bar{\rho}\bar{\theta}}{dz}. \tag{19}$$

If (18) is used again, to replace $\rho^*\theta$ by $\bar{\rho}\bar{\theta}$ in (19), one obtains the pseudo-incompressible equation (2).

Thus, the equations (2) and (13)–(16) (with ρ^* replaced by $\bar{\rho}\bar{\theta}/\theta$ in the heating term in the thermodynamic equation) form a closed system of five equations in the unknowns u, v, w, θ and π', and this system is exactly equivalent to the set of equations (12)–(17). Comparison of (12)–(17) with the complete equations governing compressible fluid flow demonstrates that *the physical assumption, associated with the replacement of the exact continuity equation with the pseudo-incompressible form* (2), *is that the influence of perturbation pressure on perturbation density can be neglected in the mass balance.* Note that the pseudo-incompressible equation allows one to filter sound waves and close the system of governing equations without making any approximations in the momentum equations.

4. A comparison of the anelastic and pseudo-incompressible equations

The complete equation of continuity may be written

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\mathbf{V} = 0. \tag{20}$$

Equation (20) reduces to the modified anelastic continuity equation when $\rho^{-1}D\rho/Dt$ is approximated as $(w/\bar{\rho})d\bar{\rho}/dz$. Thus, *the physical approximation associated with the use of the modified anelastic continuity equation is that the perturbation density field has no influence on the mass balance.* Similarly, the original anelastic continuity equation may be obtained by constructing a mass balance which neglects all perturbations about the density profile in the adiabatic reference state. At least at a superficial level, these appear to be stronger physical constraints than that associated with the pseudo-incompressible approximation.

Further comparison of the anelastic continuity equations with the pseudo-incompressible equation is easiest when the flow is adiabatic. Therefore, the remainder of this section will focus on the case of adiabatic flow. Diabatic effects will be specifically considered in section 6. The relations

$$\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz} = -\left(\frac{N^2}{g} + \frac{g}{c_s^2}\right) \equiv -\alpha_1,$$

$$\frac{1}{\bar{\rho}\bar{\theta}}\frac{d\bar{\rho}\bar{\theta}}{dz} = -\frac{g}{c_s^2} \equiv -\alpha_2 \tag{21}$$

can be used to express the original anelastic continuity equation as

$$\nabla\cdot\mathbf{V} - w\frac{g}{c_{s_a}^2} = 0, \tag{22}$$

the modified anelastic continuity equation as

$$\nabla\cdot\mathbf{V} - w\left(\frac{N^2}{g} + \frac{g}{c_s^2}\right) = 0, \tag{23}$$

and the new pseudo-incompressible equation (assuming adiabatic flow) as

$$\nabla\cdot\mathbf{V} - w\frac{g}{c_s^2} = 0, \tag{24}$$

where $c_s = (\gamma R\bar{T})^{1/2}$ is the speed of sound in the mean state and c_{s_a} is the speed of sound in the adiabatic reference state. The difference between the pseudo-incompressible and original anelastic forms lies in the temperature dependence of the speed of sound. At the top of a 10 km deep isothermal layer, $g/c_{s_a}^2$ can be 40% greater than g/c_s^2. The difference between the modified anelastic continuity equation and the pseudo-incompressible equation is contained in the term N^2w/g; it is surprising that the pseudo-incompressible form is actually independent of the vertical gradient of potential temperature (i.e., the Brunt–Väisälä frequency), but the modified anelastic form is not. In an isothermal atmosphere N^2/g is roughly 40% of g/c_s^2. Evidently the difference between either (22) or (23) and the pseudo-incompressible form (24) can be significant when the mean stratification is very stable.

The anelastic continuity and pseudo-incompressible equations can easily be compared against each other by performing diagnostic calculations. Suppose that the atmosphere is isothermal and the horizontal divergence is constant with height (and the motion is adiabatic). Equations (22)–(24) may be vertically integrated, subject to the condition $w = 0$ at the lower boundary. The vertical velocity obtained from the original anelastic continuity equation is

$$w_{oa} = -\frac{\nabla_h\cdot\mathbf{V}_h}{(1 - z/H_s)^{c_v/R}}\frac{RH_s}{c_p}[1 - (1 - z/H_s)^{c_p/R}], \tag{25}$$

where $H_s = c_p\Theta/g$ is the scale height in the reference state and $\nabla_h\cdot\mathbf{V}_h$ is the horizontal divergence. The modified anelastic continuity equation yields the relation

$$w_{ma} = -\frac{\nabla_h\cdot\mathbf{V}_h}{\alpha_1 e^{-\alpha_1 z}}(1 - e^{-\alpha_1 z}) \tag{26}$$

and the pseudo-incompressible equation gives

$$w_{pi} = -\frac{\nabla_h\cdot\mathbf{V}_h}{\alpha_2 e^{-\alpha_2 z}}(1 - e^{-\alpha_2 z}), \tag{27}$$

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where α_1 and α_2 are defined in (21). At a height of 10 km in a 250 K atmosphere, the original anelastic vertical velocity w_{oa} exceeds the pseudo-incompressible result w_{pi} by 23.6%, and the modified anelastic vertical velocity w_{ma} exceeds w_{pi} by 26%. Which velocity is correct? The true continuity equation is not a diagnostic equation, so it does not allow any of these results to be compared with the "true" solution. The actual vertical velocity could perfectly satisfy either anelastic continuity equation, or it could perfectly satisfy the pseudo-incompressible equation, or it could be something completely different—in each case the imbalance is accommodated by the density tendency $D\rho'/Dt$.

Thus, one cannot determine the relative superiority of (1) and (2) without reference to a complete solution of the governing equations. Exact solutions to the compressible equations are in short supply, but analytic solutions are available describing the behavior of small amplitude waves in an isothermal atmosphere. Let us, therefore, examine the effects of each approximate equation on the propagation of linear waves in an isothermal atmosphere.

The governing equations for two-dimensional (x, z) perturbations about a basic state at rest may be written

$$\frac{\partial u'}{\partial t} + c_p\bar{\theta}\frac{\partial \pi'}{\partial x} = 0, \tag{28}$$

$$\frac{\partial w'}{\partial t} + c_p\bar{\theta}\frac{\partial \pi'}{\partial z} = g\frac{\theta'}{\bar{\theta}}, \tag{29}$$

$$\frac{\partial \theta'}{\partial t} + \frac{\bar{\theta}}{g}N^2 w' = 0 \tag{30}$$

$$\delta_1\frac{c_v}{R\bar{\pi}}\frac{\partial \pi'}{\partial t} + \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} - w'\left(\frac{g}{c_s^2} + \delta_2\frac{N^2}{g}\right) = 0. \tag{31}$$

In the preceding, the flow is assumed to be adiabatic and inviscid, and the Coriolis acceleration is neglected. The basic-state variables $\bar{\rho}$, $\bar{\theta}$, and $\bar{\pi}$ are functions of z; perturbations are denoted by primes. The complete linearized continuity equation is obtained by setting $\delta_1 = 1$, $\delta_2 = 0$ in (31); $\delta_1 = \delta_2 = 0$ gives the linearized form of the pseudo-incompressible equation (2); and $\delta_1 = 0$, $\delta_2 = 1$ gives the linearized modified anelastic equation (1). Discussion of the linear solutions to the original anelastic equations will be deferred until the end of this section.

It is useful to remove the effect of the decrease in density with height by defining new variables

$$\tilde{u} = \left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}u', \qquad \tilde{w} = \left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}w',$$

$$\tilde{\pi} = \left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}c_p\bar{\theta}\pi', \quad \tilde{\theta} = \left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}\frac{g}{\bar{\theta}}\theta', \tag{32}$$

where ρ_0 is a constant reference density. In terms of these new variables, the governing equations become

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{\pi}}{\partial x} = 0, \tag{33}$$

$$\frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{\pi}}{\partial z} + \Gamma\tilde{\pi} = \tilde{\theta}, \tag{34}$$

$$\frac{\partial \tilde{\theta}}{\partial t} + N^2\tilde{w} = 0, \tag{35}$$

$$\frac{\delta_1}{c_s^2}\frac{\partial \tilde{\pi}}{\partial t} + \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{w}}{\partial z} - \left(\Gamma + \delta_2\frac{N^2}{g}\right)\tilde{w} = 0. \tag{36}$$

Here

$$\Gamma = \frac{1}{2\bar{\rho}}\frac{d\bar{\rho}}{dz} + \frac{g}{c_s^2} = -\frac{1}{2\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{1}{\bar{\theta}}\frac{d\bar{\theta}}{dz}. \tag{37}$$

In the case of an isothermal atmosphere, N^2, c_s^2 and Γ are constant, and solutions may be obtained of the form

$$(\tilde{u}, \tilde{w}, \tilde{\theta}, \tilde{\pi}) = (u_0, w_0, \theta_0, \pi_0)e^{i(kx+mz-\omega t)}. \tag{38}$$

Substituting (38) into the governing equations (33)–(36) and eliminating u_0 and θ_0 one obtains

$$\left(k^2 - \frac{\delta_1}{c_s^2}\omega^2\right)\pi_0 + \left[\omega m + i\omega\left(\Gamma + \delta_2\frac{N^2}{g}\right)\right]w_0 = 0, \tag{39}$$

$$(N^2 - \omega^2)w_0 + (\omega m - i\omega\Gamma)\pi_0 = 0. \tag{40}$$

First consider the complete solution (the case $\delta_1 = 1$, $\delta_2 = 0$). Equations (39) and (40) yield the dispersion relation

$$\omega^2 = \frac{c_s^2}{2}\left\{\left(k^2 + m^2 + \Gamma^2 + \frac{N^2}{c_s^2}\right) \pm \left[\left(k^2 + m^2 + \Gamma^2 + \frac{N^2}{c_s^2}\right)^2 - \frac{4N^2k^2}{c_s^2}\right]^{1/2}\right\}. \tag{41}$$

The frequencies for sound waves are obtained by selecting the positive sign in (41). The choice of the negative sign yields the dispersion relation for gravity waves, which can be accurately approximated as

$$\omega^2 = \frac{N^2k^2}{k^2 + m^2 + \Gamma^2 + N^2/c_s^2}. \tag{42}$$

The dispersion relation obtained using the pseudo-incompressible equation ($\delta_1 = \delta_2 = 0$) is

$$\omega_{pi}^2 = \frac{N^2k^2}{k^2 + m^2 + \Gamma^2}. \tag{43}$$

Note that sound waves have been filtered out by the pseudo-incompressible approximation. The error in (43) will be negligible whenever $m \gg N/c_s$. In an isothermal atmosphere this requires the vertical wave-

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length to be much less than 100 km, a condition easily satisfied in most meteorological applications.

The dispersion relation associated with the modified anelastic continuity equation ($\delta_1 = 0$, $\delta_2 = 1$) is

$$\omega_{ma}^2 = \frac{N^2k^2}{k^2 + m^2 + \Gamma^2 + \Gamma N^2/g + imN^2/g}. \quad (44)$$

This equation contains a more serious error. There will be a spurious growth in the wave amplitude with time because ω_{ma}^2 is complex. How rapid is this erroneous growth rate? One way to assess the practical importance of the error is to consider steady-state waves forced by air flowing over topography. Then the spurious growth with time appears as a spurious increase in wave amplitude with height. If the basic-state cross-mountain windspeed U_0 is constant, the dispersion relations for steady waves may be obtained by replacing ω with U_0k in (41)–(44).

The correct expression for the vertical wavenumber,

$$m^2 = \frac{N^2}{U_0^2} - k^2\left(1 - \frac{U_0^2}{c_s^2}\right) - \frac{g^2}{4R^2T_0^2}, \quad (45)$$

may be obtained from (41) using the fact that, in an isothermal atmosphere,

$$N^2 = \frac{g^2}{c_pT_0}, \quad \Gamma = \frac{g}{RT_0}\left(\frac{c_v - R}{2c_p}\right). \quad (46)$$

The vertical wavenumber associated with the modified anelastic continuity equation, obtained from (44) is

$$m_a = -\frac{ig}{2c_pT_0} \pm \left[m^2 - \frac{U_0^2k^2}{c_s^2} - \frac{g^2}{4Rc_p^2T_0^2}(c_p + c_v)\right]^{1/2}. \quad (47)$$

According to (47), the increase in wave amplitude with height is

$$\bar{\rho}^{-1/2} \exp\left(\frac{gz}{2c_pT_0}\right). \quad (48)$$

The wave amplitude in the correct solution increases with height like $\bar{\rho}^{-1/2}$. When $T_0 = 250$ K, the error, in the modified anelastic amplitude, grows exponentially to a value of 22% at 10 km. No spurious growth is produced if one uses the pseudo-incompressible equation [although there is a slight error in the vertical wavelength as discussed in connection with (42) and (43)].

At the beginning of this section, diagnostic calculations for w_{ma} and w_{pi} were compared by integrating the modified anelastic continuity equation and the pseudo-incompressible equation through a 10 km depth in an isothermal atmosphere. The modified anelastic result exceeded the pseudo-incompressible result by 26%. The preceding linear analysis, in which the modified anelastic system gave rise to a 22% overestimate, suggests that most of this 26% difference can be attributed to errors in the modified anelastic formulation. Now consider the original anelastic system, and recall that the diagnostically calculated w_{oa} exceeded w_{pi} by 23.6%. Since w_{oa} and w_{ma} differed by less than 3%, and since w_{pi} is approximately correct while w_{ma} is at least 20% too large, it appears that the pseudo-incompressible result is also more accurate than the result obtained from the original anelastic continuity equation.

The linearized versions of the original anelastic equations may be obtained by specifying a constant value for $\bar{\theta}$ in (28)–(30), and modifying (31) by setting $\delta_1 = \delta_2 = 0$ and replacing c_s with c_{s_a}. After these changes, (28)–(30) have coefficients that are independent of z, but (31) contains c_{s_a}, which has a polynomial dependence on z. As a consequence of this polynomial dependence, the linear solution to the original anelastic system will not exhibit the simple combination of exponential and periodic vertical structure found in the previous solutions. The exact magnitude of the error arising from this incorrect vertical structure is hard to determine because it is difficult to obtain closed form analytic solutions for the original anelastic equations. The previously discussed diagnostic calculations suggest, however, that the error will be similar to that obtained using the modified anelastic approximation.

5. Energy conservation

Before examining the fully nonlinear case, consider the energy conservation properties of the linearized equations (28)–(31). The perturbation energy equation for the complete compressible system ($\delta_1 = 1$, $\delta_2 = 0$) may be expressed

$$\frac{\partial E'}{\partial t} + \frac{\partial p'u'}{\partial x} + \frac{\partial p'w'}{\partial z} = 0, \quad (49)$$

where E' is the total perturbation energy

$$E' = \frac{\bar{\rho}}{2}\left(u'^2 + w'^2 + \frac{g^2}{N^2}\frac{\theta'^2}{\bar{\theta}^2}\right) + \frac{1}{2\bar{\rho}}\frac{p'^2}{c_s^2}, \quad (50)$$

and $p' = c_p\bar{\rho}\bar{\theta}\pi'$. Use of the pseudo-incompressible equation ($\delta_1 = \delta_2 = 0$) leads to an identical perturbation energy equation except that E' is replaced by

$$E'_{pi} = \frac{\bar{\rho}}{2}\left(u'^2 + w'^2 + \frac{g^2}{N^2}\frac{\theta'^2}{\bar{\theta}^2}\right). \quad (51)$$

On the other hand, E'_{pi} is not conserved if one uses the modified anelastic continuity equation ($\delta_1 = 0$, $\delta_2 = 1$), in which case

$$\frac{\partial E'_{pi}}{\partial t} + \frac{\partial p'u'}{\partial x} + \frac{\partial p'w'}{\partial z} = c_p\bar{\rho}\pi'w'\frac{d\bar{\theta}}{dz}. \quad (52)$$

Energy conservation can be recovered if we return to the original anelastic equations, from which one may

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obtain a perturbation energy equation of the form (49), except that E' is replaced by

$$E'_{oa} = \frac{\rho_a}{2}\left(u'^2 + w'^2 + \frac{g^2}{N^2}\frac{\theta'^2}{\Theta^2}\right), \qquad (53)$$

where $\rho_a(z)$ is the density in an adiabatic reference state with potential temperature Θ. In addition, the perturbation pressure appearing in the energy flux in (49) is replaced by the less accurate approximation $p' = c_p\rho_a\Theta\pi'$. As shown by Lipps and Hemler (1982), energy conservation can also be obtained using the modified anelastic continuity equation if the pressure gradient terms in (28) and (29) are written as $c_p\partial(\bar{\theta}\pi')/\partial x$ and $c_p\partial(\bar{\theta}\pi')/\partial z$. This system conserves an energy form identical to E'_{pi}. One of the major reasons that the system proposed by Lipps and Hemler conserves energy is that it's derived under the assumption that $d\bar{\theta}/dz$ is small, thereby eliminating the troublesome energy source/sink term in (52).

Now consider energy conservation in a finite-amplitude inviscid, adiabatic flow. The original anelastic equations, as presented by Ogura and Phillips (1962), may be expressed as

$$\frac{Du}{Dt} + c_p\Theta\frac{\partial\pi'}{\partial x} = 0, \qquad (54)$$

$$\frac{Dv}{Dt} + c_p\Theta\frac{\partial\pi'}{\partial y} = 0, \qquad (55)$$

$$\frac{Dw}{Dt} + c_p\Theta\frac{\partial\pi'}{\partial z} = g\frac{\theta'}{\Theta}, \qquad (56)$$

$$\frac{D\theta'}{Dt} = 0, \qquad (57)$$

$$\nabla\cdot(\rho_a\mathbf{V}) = 0, \qquad (58)$$

where π' is the deviation of the Exner function pressure from its value in the adiabatically stratified reference state. As shown by Ogura and Phillips, these equations conserve the energy form

$$E_{oa} = \rho_a\left[\frac{1}{2}(u^2 + v^2 + w^2) - gz\frac{\theta'}{\Theta}\right]. \qquad (59)$$

Unless the actual mean state is close to adiabatic, the use of a constant Θ in the coefficient of the pressure gradient can introduce significant errors in (54)–(56). Wilhelmson and Ogura (1972) addressed this problem by replacing Θ with the vertically varying mean state $\bar{\theta}(z)$ in (54)–(56), by including the mean stratification in (57), and by replacing (58) with the modified anelastic continuity equation. As discussed by Wilhelmson and Ogura, however, there is no known energy form that is conserved by the resulting modified anelastic system.

The pseudo-incompressible system, consisting of (13)–(16) together with either (2) or (17), has the advantage of treating the pressure gradient terms without approximation while conserving "total pseudo-incompressible energy":

$$E^* = \rho^*\left[\frac{(u^2 + v^2 + w^2)}{2} + gz\right] + c_v\bar{\rho}\bar{T}. \qquad (60)$$

Here E^* differs from the total energy in the complete compressible system in that ρ^* replaces the actual density in the expressions for the kinetic and potential energy, and the internal energy of the mean-state replaces the actual internal energy. The pseudo-incompressible system requires that, in the absence of diabatic processes,

$$\frac{\partial E^*}{\partial t} + \nabla\cdot[(E^* + p^*)\mathbf{V}] = 0, \qquad (61)$$

where $p^* \equiv \bar{p} + c_p\bar{\rho}\bar{\theta}\pi' \approx \bar{p} + p'$ (p' is formally equal to the deviation of the pressure from its mean value only when the pressure perturbation is small).

6. Practical implications

There are two major uses for the anelastic continuity equation. It is often used as a diagnostic relation to compute the vertical velocity from the horizontal windfield, and it is used, together with the complete set of approximate anelastic equations, in the numerical simulation of small-scale atmospheric circulations (particularly for deep-convection and gravity waves). In this section, we will discuss the applicability of the pseudo-incompressible approximation to each of these activities.

First, consider the problem of diagnosing the vertical velocity. The comparisons presented in section 4 suggest that the pseudo-incompressible equation should be used to diagnose the vertical velocities associated with adiabatic motion in deep, strongly stable regions such as the stratosphere. In the more weakly stratified troposphere, the difference between the two equations is greatly reduced, and probably dwarfed by other data analysis errors. In particular, if each equation is integrated through a 10 km deep layer, one might expect 25% differences in the isothermal stratosphere, but only 8% differences in the less stable troposphere.

One complication associated with the use of the pseudo-incompressible equation is the need to evaluate H, the heating rate per unit volume, in diabatic flows. A typical upper bound on the heating rate produced by convection in numerical simulations of midlatitude squall lines is $H/c_p\bar{\rho}\bar{T} \leqslant 10^{-4}$ s^{-1} (Fovell, personal communication). Thus, the complete neglect of the heating term in a region of moist convection might be expected to introduce inaccuracies comparable to a 0.1 m s^{-1} uncertainty in the horizontal windspeed when calculating horizontal divergence over a 1 km grid interval. While this error is not large, it is probably wiser to evaluate the vertical velocity from the conventional anelastic equation, as that equation does not require knowledge of H. Furthermore, since convection is

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generally found in regions where the background stratification is weak, there would be little advantage in using the pseudo-incompressible form even if H were known. On the other hand, less error is incurred in neglecting the heating in the stratosphere, where the maximum cooling rates might be 10 K day^{-1}, due to radiative flux divergence at cloud tops. This cooling rate is roughly equivalent to a convergence of 10^{-6} s^{-1}, or a 0.1 cm s^{-1} error in the horizontal velocity when calculating divergence over a 1 km grid interval. Therefore, it is probably best to diagnose vertical velocities, in the strongly stable stratosphere, using the adiabatic form of the pseudo-incompressible equation—even when diabatic processes are active.

Another possible diagnostic calculation that might be performed using the pseudo-incompressible equation is the evaluation of the heating rate in convective clouds from Doppler-radar measurements of the velocity field. If one could obtain direct observations of the three-dimensional velocity field, and the mean-state profiles of $\bar{\rho}$ and $\bar{\theta}$, one could calculate the net heating due to latent heat exchanges, radiation and turbulent dissipation. Fine resolution thermodynamic data are not required for this calculation, a single rawinsonde sounding would suffice. According to the preceding estimates, the velocity data would have to be sufficiently accurate to determine the three-dimensional divergence to within 10^{-5} s^{-1} in order to evaluate the strongest heating rates to within ±10%. This appears to be beyond our current capabilities, but advances in remote sensing may someday make it practical. The attractiveness of this approach lies in the fact that very few approximations are required in the derivation of the diagnostic equation [i.e., the neglect of sound-wave frequencies and the assumption $\pi' \ll \bar{\pi}(z)$].

Now consider the application of the pseudo-incompressible approximation to the numerical simulation of small-scale atmospheric circulations. As in the traditional anelastic system, the replacement of the compressible continuity equation with the pseudo-incompressible equation reduces the number of prognostic governing equations. The pressure is therefore calculated through a diagnostic equation, obtained by requiring

$$\frac{\partial}{\partial t}\nabla\cdot(\bar{\rho}\bar{\theta}\mathbf{V}) = \frac{1}{c_p\bar{\pi}}\frac{\partial H}{\partial t}. \quad (62)$$

If the Coriolis terms are neglected for simplicity, (62) can be combined with the momentum equations (13)–(15) to yield

$$c_p\left[\frac{\partial}{\partial x}\left(\bar{\rho}\bar{\theta}\theta\frac{\partial \pi'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\bar{\rho}\bar{\theta}\theta\frac{\partial \pi'}{\partial y}\right) + \frac{\partial}{\partial z}\left(\bar{\rho}\bar{\theta}\theta\frac{\partial \pi'}{\partial z}\right)\right]$$
$$= g\frac{\partial\bar{\rho}\theta'}{\partial z} - \frac{1}{c_p\bar{\pi}}\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}(\bar{\rho}\bar{\theta}\mathbf{V}\cdot\nabla u)$$
$$- \frac{\partial}{\partial y}(\bar{\rho}\bar{\theta}\mathbf{V}\cdot\nabla v) - \frac{\partial}{\partial z}(\bar{\rho}\bar{\theta}\mathbf{V}\cdot\nabla w). \quad (63)$$

This equation forces the pressure gradient to generate time tendencies in the velocity field that maintain the pseudo-incompressible relationship (2). Equation (63) is very similar to the Poisson pressure equation used in traditional anelastic models; but there are two new features that make its solution somewhat more difficult. The first is the presence of the heating term on the right-hand side. If the entire model is integrated with leapfrog time differencing, $\partial H/\partial t$ must be approximated as a one-sided difference, backward in time, because $H^{t+\Delta t}$ is not available when calculating π^t. Even then, the governing equations must be integrated in a slightly unnatural order because H^t represents the heating associated with the leapfrog time step between $\theta^{t-\Delta t}$ and $\theta^{t+\Delta t}$. A suitable integration order would be: 1) obtain the velocities at t from the momentum equations, 2) obtain $\theta^{t+\Delta t}$ together with H^t from the thermodynamic equation, and 3) obtain π^t from (63). The second difficulty is introduced by the presence of $\theta(x, y, z, t)$ in the coefficients on the left-hand side of (63). In the traditional anelastic pressure equation, the left-hand side coefficients are only functions of z, and this specialized coefficient structure can be exploited to increase the computational efficiency of the Poisson solver. Therefore, the more general coefficient structure in (63) may necessitate the use of slower numerical algorithms for the solution of the pressure equation. These concerns notwithstanding, the equations (13)–(16) and (63) have been successfully implemented in a numerical model for the simulation of moist convection (Fovell and Durran, personal communication). At present, we are continuing to examine the question of how to best formulate the numerical algorithms, and the equations themselves (should one use π or p?), in order to produce the most efficient solution.

7. Conclusions

An alternative to the anelastic continuity equation—the pseudo-incompressible equation—has been presented. The validity of this equation rests on two assumptions. First, the Lagrangian time scale of the disturbance should be much greater than the time scale for sound wave propagation, and second, the perturbation pressure should be small in comparison with the vertically varying mean-state pressure field. Unlike the anelastic approximation, no assumptions are required concerning the magnitude of the potential temperature perturbations or the mean-state stratification. Like the anelastic approximation, the pseudo-incompressible approximation filters sound waves.

It has been shown that the physical approximation, associated with the replacement of the complete compressible continuity equation with the pseudo-incompressible form, is that a portion of the perturbation density field can be neglected in the mass balance. Those density perturbations that arise in response to fluctuations in the perturbation pressure field are ne-

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glected. Density perturbations which arise through fluctuations in the temperature field are figured into the mass balance. This may be compared with the de facto approximation in the anelastic continuity equation, in which the influence of the perturbation density field is completely neglected in the mass balance.

A quantitative comparison of the pseudo-incompressible equation with two versions of the anelastic continuity equation was presented for the case of adiabatic inviscid flow. All the approximate equations are equivalent when the mean-state stratification is adiabatic. The difference between them increases nonlinearly with increasing static stability; it is small for the relatively weak stratification typically found in deep layers of the troposphere. The pseudo-incompressible equation, however, appears to be distinctly superior to either anelastic form for application to the stratosphere, where the typical lapse rate is isothermal. The errors associated with the use of the pseudo-incompressible equations to represent linear gravity wave propagation in a 10 km deep isothermal layer are very small, whereas the errors associated with the use of either anelastic formulation can exceed 20%.

The pseudo-incompressible approximation yields a system of equations suitable for use in nonhydrostatic numerical models. It appears likely that the pseudo-compressible system can be integrated numerically with only a modest increase in computation time over that required to integrate the anelastic equations. The pseudo-incompressible system conserves an energy form that is closely related to the actual total energy conserved by the complete compressible equations. Unlike the various anelastic approximations, energy conservation is achieved independent of the mean-state stratification.

The pseudo-incompressible equation can also be used to diagnostically calculate the vertical velocity in an adiabatic inviscid flow. If diabatic processes are present, however, the need to specify the heating rate introduces uncertainties in the calculation, and therefore, the traditional anelastic form is recommended for diagnostic calculations of the vertical velocity in regions of moist convection. On the other hand, if the vertical profiles of $\bar{\rho}$ and $\bar{\theta}$ are known, together with extremely accurate measurements of the three-dimensional velocity field, the pseudo-incompressible equation could be used to compute the net heating rate. This heating calculation would not require detailed knowledge of the thermodynamic fields.

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