A Hurricane Beta-Drift Law

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ABSTRACT

A hurricane beta-drift law is derived by applying dimensional analysis to the numerical simulations of Chan and Williams. The beta-drift speed and direction are given as simple functions of hurricane strength, radius, and the magnitude of beta. The physical significance of the drift law and its relation to previously proposed drift laws are considered.

The poleward drift of hurricanes out of the tropics is due in part to the variation of Coriolis force with latitude: the so-called beta effect. This beta drift has been simulated in full three-dimensional numerical models (e.g., Madala and Piacsek 1975; Wang and Li 1992) and in simpler two-dimensional nondivergent models (e.g., Kitade 1981; Chan and Williams 1987; Fiorino and Elsberry 1989; Peng and Williams 1990). Chan and Williams (1987, henceforth CW87) use the governing vorticity equation

\[ \frac{\partial \zeta}{\partial t} + \mathbf{V} \cdot \nabla (\zeta + f) = 0, \]

(1)

where \( f = \beta y + f_0 \), to determine the fate of an initially axisymmetric vortex

\[ V(r) = V_m \left( \frac{r}{r_m} \right) \exp \left[ \frac{1}{b} \left( 1 - \left( \frac{r}{r_m} \right)^b \right) \right], \]

(2)

where \( V_m \) and \( r_m \) are, respectively, the maximum wind and the radius at which that wind occurs. After a period of adjustment, the vortex was found to move to the northwest (for positive \( V_m \) and \( \beta \)) at a speed \( (V_d) \) and in a direction \( (\theta_d) \) dependent on the parameters of the problem; \( r_m, V_m, b, \) and \( \beta \). In this note, we extend the work of CW87 slightly by applying the principle of dimensional analysis to their numerical results.

We define a characteristic speed

\[ \bar{V} = r_m \beta \]

(3)

and a nondimensional beta as a governing parameter

\[ B = r_m \beta / V_m. \]

(4)

\[ V_d = V_d / \bar{V} \]

(5)

must, according to the principle of dimensional analysis, be given by

\[ \bar{V}_d = f(B) \]

(6)

and the angle of drift

\[ \theta_d = g(B), \]

(7)

where \( f \) and \( g \) are unknown functions. The 12 cases studied by CW87 include a variety of \( r_m \) values (50, 100, 200 km) and \( V_m \) values (2.5, 5, 10, 20, 40, 80 m s\(^{-1}\)) but with \( b = 1 \) always. In all cases, beta was set to its value at 10°N latitude:

\[ \beta = 0.225 \times 10^{-10} \text{ s}^{-1} \text{ m}^{-1}. \]

(8)

These data are plotted in Figs. 1 and 2 according to the forms specified by (6), (7). In both plots, the data collapse nicely to a single line. Since scaling is an overriding principle, the remaining scatter must be due to imperfections in the calculation such as: lack of steady state, boundary effects, or numerical errors.

The drift speed in Fig. 1 is well fit by

\[ \bar{V}_d = 0.72 B^{-0.54} \]

(9)

with a correlation coefficient \( r = 0.995 \). The drift angle in Fig. 2 is fit by

\[ \theta_d = 308^\circ - 9.6 \log_{10} B. \]

(10)

If the exponent in (9) is approximated by the rational number \(-1/2\),

\[ \bar{V}_d = CB^{-1/2}, \]

(11)

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where the best value for $C \approx 0.9$. Equation (11) can be unscaled to give
\[ V_d = 0.9r_m \sqrt{r_m \beta} , \] 
the “beta-drift law” for vortex (2) with $b = 1$. This law does not exactly agree with the conclusion of CW87 that the product $r_m V_m$ is of special importance in determining hurricane drift. According to (12), the product $r_m V_m$ plays a role. The proportionality $V_d \sim r_m V_m$ is not possible in fact, as such a relation cannot be nondimensionalized by introducing beta (with units m$^{-1}$ s$^{-1}$) to any power. That is, it cannot be written in the form of (6).

The drift law proposed by Wang and Li (1992) has a similar problem. Their conclusion that the drift speed is nearly proportional to the square root of the relative angular momentum is equivalent, in our notation, to $V_d \sim (r_m V_m)^{1/2}$. This proportionality cannot be written in the form of (6). It should be remembered, however, that the full 3D simulations of Wang and Li have other nondimensional control parameters (such as the Rossby number), and thus they need not necessarily obey (6).

The drift law proposed by Kitade (1981) for nondivergent flow is dimensionally correct and somewhat similar to (12). Kitade found the latitudinal component of the beta drift to be, in our notation, $V_{d,y} = 0.14 r_m^{3/2} V_m^{1/2} \beta^{1/3}$. Kitade’s formula is equivalent to (9) with a coefficient of 0.14 and an exponent of $-2/3$. The exponent $-2/3$ is sufficiently different from $-0.54$ [in (9)] that it would not give a good fit to the data plotted in Fig. 1. This difference could be due to the fact that only the latitudinal drift was considered by Kitade or that Kitade’s initial vortex was not of the form (2). Data scatter and uncertainty in curve fitting could also contribute to the difference.

It is interesting to compare this drift law (12) with the propagation of nondivergent Rossby waves. The dispersion relation for small amplitude Rossby waves is
\[ \omega = -\beta k/(k^2 + l^2) , \] 
where $k$ and $l$ are the eastward and northward components of the wavenumber vector. The phase speed components are
\[ C_{p_x} = -\beta/(k^2 + l^2) \] 
\[ C_{p_y} = -\beta k/l(k^2 + l^2) . \]

One cannot use (14, 15) to explain hurricane drift as the waves are dispersive; the vortex would quickly decompose into its Fourier components. Nonetheless, if we make the natural choice $k = l = r_m^{-1}$, we have from (14)
\[ C_{p_x} = -\beta r_m^2/2 . \] 
A similar formula could be derived for the group velocity.

According to (16) the disturbance would propagate at a speed proportional to $\beta r_m$. The proportionality $V_d \sim \beta r_m$ in (16) is the only dimensional possibility because in linear theory the amplitude of the disturbance ($V_m$) is not allowed to interact with the other variables.

The meaning of (12) is now a little clearer. The proportionality $V_d \sim r_m V_m \beta$ in (12) is fundamentally different from linear Rossby wave propagation (16), as it takes into account the nonlinear advection in (1) related to $V_m$. It is also interesting to note that using dimensional analysis, we are able to deduce the dependence of $V_d$ on $\beta$ in (12), even though $\beta$ was not varied in any of the numerical simulations of CW87.

It is difficult to develop a more general drift law.
Table 1. Drift results from Fiorino and Elsberry.

<table>
<thead>
<tr>
<th>b</th>
<th>$V_m$</th>
<th>$r_m$</th>
<th>$V_d$</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.21</td>
<td>50</td>
<td>100</td>
<td>2.66</td>
<td>0.79</td>
</tr>
<tr>
<td>1</td>
<td>35</td>
<td>100</td>
<td>2.55</td>
<td>0.91</td>
</tr>
<tr>
<td>0.4</td>
<td>20</td>
<td>100</td>
<td>5.1</td>
<td>2.4</td>
</tr>
</tbody>
</table>

without additional data from numerical simulations. If we assume that the exponent (1/2) in (11) is independent of $b$, then we only require a few more data points to determine how the value of $b$ would modify the coefficient in (12). Using three cases from Fiorino and Elsberry (1989, henceforth FE89) shown in Table 1, we can compute $V_d$ and $B$, and from (11) compute $C$.

The relationship between $b$ and $C$ in the table can be fit with $C = 0.9/b$. A possible drift law for variable $b$ is then $V_d = (0.9/b) r_m V_m$.

The form of the beta-drift law (12) can be misleading as it is written in terms of $r_m$ and $V_m$, which describe the inner part of the vortex. According to FE89, when a variety of vortex shapes are considered, the strength and radius of maximum wind have little influence on the drift vector. It is the outer parts of the vortex that act with beta to create a steering current to advect the inner vortex. Equation (12) works perfectly as long as we stay with (2) and $b = 1$, as the inner features $r_m$ and $V_m$ are proxies for the more important outer structure of the vortex. To apply (12) to vortices other than (2), the outer portion of the vortex should be fit by (2) using the three adjustable constants: $b$, $r_m$, $V_m$. Alternatively, only a few numerical simulations are needed to allow new drift laws to be deduced for any new vortex family using the dimensional analysis methods illustrated here and by Kitade. The numerical method of Ross and Kurihara (1992) could be used for this purpose.

The variables $V_m$ and $r_m$ in (12) should be considered proxies for another reason. By the time steady beta drift is established, the vortex wind profile no longer exactly satisfies (2). Thus, the initial values of $V_m$ and $r_m$ are relevant, not because they accurately describe the drifting vortex, but rather because they enter the formal statement of the entire physical problem.

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**REFERENCES**


