The Effects of Spherical Geometry on the Evolution of Baroclinic Waves

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ABSTRACT

The effects of spherical geometry on the nonlinear evolution of baroclinic waves are investigated by comparing integrations of a two-layer primitive equation (PE) model in spherical and Cartesian geometry. To isolate geometrical effects, the integrations use basic states with nearly identical potential vorticity (PV) structure.

Although the linear normal modes are very similar, significant differences develop at finite amplitude. Anticyclones (cyclones) in spherical geometry are relatively stronger (weaker) than those in Cartesian geometry. For this basic state, the strong anticyclones on the sphere are associated with anticyclonic wrapping of high PV in the upper layer (i.e., high PV air is advected southward and westward relative to the wave). In Cartesian geometry, large quasi-barotropic cyclonic vortices develop, and no anticyclonic wrapping of PV occurs. Because of their influence on the synoptic-scale flow, spherical geometric effects also lead to significant differences in the structure of mesoscale frontal features.

A standard midlatitude scale analysis indicates that the effects of sphericity enter in the next-order correction to β-plane quasigeostrophic (QG) dynamics. At leading order these spherical terms only affect the PV inversion operator (through the horizontal Laplacian) and the advection of PV by the nondiabatic wind. Scaling arguments suggest, and numerical integrations of the barotropic vorticity equation confirm, that the dominant geometric effects are in the PV inversion operator. The dominant metric in the PV inversion operator is associated with the equatorward spreading of meridians on the sphere, and causes the anticyclonic (cyclonic) circulations in the spherical integration to become relatively stronger (weaker) than those in the Cartesian integration.

This study demonstrates that the effects of spherical geometry can be as important as the leading-order ageostrophic effects in determining the structure of evolution of dry baroclinic waves and their embedded mesoscale structures.

1. Introduction

Much of our current knowledge pertaining to the structure and evolution of nonlinear baroclinic waves has been gained through idealized simulations that ignore the complicating physics of moisture, topography, radiation, and the planetary boundary layer. These simulations are performed in a zonally periodic domain by prescribing initial conditions consisting of a zonal jet in thermal wind balance plus a small perturbation, generally with the structure of the most unstable normal mode of that jet. Typically, the initial disturbance grows exponentially at all levels for the first few days and then enters a nonlinear regime. Some aspects of the nonlinear behavior that have been studied using idealized simulations include frontogenesis (e.g., Mudrick 1974; Hoskins and West 1979; Takayabu 1986; Hines and Meehosu 1991), occlusion (Polavarapu and Pelletier 1990), and the life cycles of baroclinic waves (Simmons and Hoskins 1978, 1980). A further idealization included in many of these simulations is that of flat, Cartesian geometry—an attractive simplification for finite-difference models because it avoids numerical problems associated with the polar singularity. It is not clear, however, to what extent the curvature of latitude and longitude lines in spherical geometry can affect the life cycles of baroclinic waves. We address this question through a combination of scaling analyses and direct comparison of spherical and Cartesian simulations of baroclinic waves.

There is considerable evidence that relatively weak processes such as sphericity can strongly influence the long-term evolution of baroclinic waves. Though concentrating on different aspects of the problem, Simmons and Hoskins (1980) and Davies et al. (1991) both demonstrate that the addition of a weak barotropic shear to an otherwise symmetric baroclinic jet will drastically alter the development of the unstable waves. In addition, Snyder et al. (1991) compare the baroclinic wave development in semigeostrophic and primitive equation simulations and show that small terms (order Rossby number compared to the basic, quasigeostrophic dynamics) neglected in the semigeostrophic equations lead to a steady divergence of the solutions. The dynamical effects of sphericity can be estimated

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through a brief scale analysis (to be expanded in section 5); for now we consider only the contribution of spherical effects to the vertical component of relative vorticity. The isentropic-coordinate divergence equation in spherical geometry is

\[
\frac{\partial D}{\partial t} - k \cdot \nabla \times (\xi + 2\Omega \sin \phi) v + \nabla^2 (M + v \cdot v/2) = 0,
\]

where \( D \) is horizontal divergence, \( \xi \) is relative vorticity, \( \phi \) is latitude, and \( v = (u, v) \) is the horizontal velocity. The Montgomery potential, \( M \), takes the role of geopotential in pressure coordinates, and is defined by

\[
M = \pi \theta + g z,
\]

where \( \pi = c_p (p/p_0)^{K/\gamma} \) is the Exner function. Following Pedlosky (1987, chapter 6) we assume that motion occurs in a midlatitude region near a central latitude \( \phi_0 \), so that convenient coordinates are

\[
x = a \lambda \cos \phi_0, \quad y = a(\phi - \phi_0);
\]

where \( \lambda \) is longitude and \( a \) is the radius of the earth. Equation (1.1) is written in nondimensional form by letting \( (x, y) = L(x^*, y^*), v = UV^*, M = U_0LM^*, \xi = (U/L)\xi^*, \) and \( D = (ROU/L)D^* \), where the asterisks indicate nondimensional quantities, \( f_0 = 2\Omega \times \sin \phi_0 \), and \( RO = U/f_0L \) is the Rossby number. Assuming \( RO = L/a \ll 1 \) and expanding the spherical trigonometric factors in Taylor series about \( y = 0 \), (1.1) becomes

\[
\xi = M_{xx} + M_{yy} + RO \left\{ \cot \phi_0 \left[ y(M_{xx} + M_{yy}) + M_y \right] - 2 \frac{\partial (M_x, M_y)}{\partial (x, y)} + \tan \phi_0 (2yM_{xx} - M_y) \right\} + O(RO^2).
\]

In (1.3), asterisks have been omitted for clarity and the leading-order velocities have been evaluated from the geostrophic relation, \( k \times v = -\nabla M + O(RO) \). Beyond geostrophy, the vorticity (1.3) contains \( O(RO) \) corrections associated with the variation of \( f \) with latitude (the term proportional to \( \cot \phi_0 \)), ageostrophic effects due to flow curvature (the Jacobian term; see Snyder et al. (1991) for a discussion of its importance), and the curvature of latitude and longitude lines in spherical geometry (the term proportional to \( \tan \phi_0 \)). This implies that geometric effects associated with the convergence of meridians on the sphere are of the same order as the leading-order contributions of ageostrophy and variable Coriolis parameter in the vertical component of relative vorticity.

This simple scale analysis suggests that it is important to understand how the errors produced by using channel geometry alter the evolution of nonlinear baroclinic waves. We address this problem by performing comparative simulations in spherical and Cartesian geometry using a two-layer primitive equation model. It appears that such a comparison has not appeared before in the literature, although the effects of sphericity on the linear modes of a two-level model have been examined by Hollingsworth (1975), Hollingsworth et al. (1976), and Moura and Stone (1976). These authors attributed most of the differences between spherical and Cartesian normal modes to the increase in planetary vorticity gradient toward the equator on the sphere. Here we attempt to isolate the role of purely geometric effects associated with the curvature of latitude and longitude lines in spherical geometry by using the full variation of \( f \) in the Cartesian model so that the basic-state potential vorticity (PV) structure is nearly identical in the two models. We show that although the differences between the Cartesian and spherical linear modes of midlatitude baroclinic jet are small, the errors in the Cartesian model grow during the nonlinear evolution, producing significant differences in wave structure. The differences between the Cartesian and spherical simulations are interpreted from a PV perspective and the consequences of using channel models to study the nonlinear evolution of baroclinic waves are discussed.

2. The Models

We consider a fluid system with two shallow, stably stratified layers, each with constant potential temperature. This is perhaps the simplest model that retains the basic mechanism for baroclinic instability. The Boussinesq primitive equations in isentropic coordinates are given by McWilliams and Gent (1980). In vorticity–divergence form they are given by (1.1) and

\[
\frac{\partial \xi}{\partial t} + \nabla \cdot (\xi + 2\Omega \sin \phi) v = 0,
\]

and

\[
\frac{\partial}{\partial t} \left( \frac{\partial \pi}{\partial \theta} \right) + \nabla \cdot \left( \frac{\partial \pi}{\partial \theta} v \right) = 0.
\]

Using the definition of \( M \) and the relation \( \partial \pi / \partial z = -g / \theta \), it is easily shown that the Montgomery potential is independent of \( z \) wherever \( \theta \) is independent of \( z \). Therefore, in the two-layer model, \( M \) is constant through the depth of each layer. Evaluating the definition of \( M \) at \( z = 0 \) yields

\[
M_1 = \pi_1/\theta_1,
\]

and integrating (2.2) across the interface gives

\[
M_2 = M_1 + \Delta \theta \pi_{3/2},
\]

where \( \Delta \theta = \theta_2 - \theta_1 \), the subscript 1 (2) denotes a lower-(upper-) layer quantity, and the subscript 1/2 (3/2) indicates evaluation at the surface (interface). Since
$M$ is constant through the depth of each layer, $u$ and $v$ will be independent of $z$ within each layer if they are so initially, and (1.1) and (2.1) can be replaced by identical equations for $\pi_k$ and $\Delta \pi_k$, $k = 1, 2$. Similarly, the continuity equation (2.3) can be replaced by a discretized version:

$$\frac{\partial \Delta \pi_k}{\partial t} + \nabla \cdot (\Delta \pi_k \mathbf{v}_k) = 0, \quad k = 1, 2$$

where $\Delta \pi_k = \pi_k - \pi_{k+1/2}$.

At the upper boundary the pressure is assumed constant (200 mb), so that $\pi_{5/2} = c_p0.2R/c_p$, while the lower boundary is assumed isentropic (i.e., $\Delta \pi_{1/2}$ never vanishes). The latter condition implies that the basic-state PV gradient must change sign within the fluid for there to be baroclinic instability (Charney and Stern 1962). For the integrations presented here, $\theta_1 = 280$ K and $\theta_2 = 315$ K.

In order to isolate the effects of the metric terms associated with the curvature of latitude and longitude lines in spherical coordinates, we will compare integrations of spherical and "Cartesian" models. The spherical model uses the complete equations with, of course, all differential operators written in their appropriate spherical coordinate forms. The Cartesian model is obtained from the spherical model by transforming the equations to the $(x, y)$ coordinates defined in (1.2) and expanding all trigonometric terms in Taylor series about $y = 0$ ($\phi = \phi_0$). For all the results presented here, $\phi_0 = 45^\circ$. In order to separate purely geometric effects from the effects of differential rotation, the full variation of $f = 29 \sin(y/a + \phi_0)$ is retained, but only the leading-order terms are kept in the differential operators. This is equivalent to replacing the differential operators with their Cartesian forms in the variables $x$ and $y$. For example, the horizontal divergence of an arbitrary vector $F$ in the two models is

$$\nabla \cdot F = \left\{ \begin{array}{ll}
\frac{1}{a \cos \phi} \frac{\partial F_1}{\partial \lambda} + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (F_2 \cos \phi), & \text{spherical} \\
\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}, & \text{Cartesian}.
\end{array} \right.$$  

While the spherical model represents a physically realizable system, the Cartesian model does not and is used only for comparative purposes.

Both models use similar domains and lateral boundary conditions. Hemispheric symmetry and wavenumber 6 zonal symmetry are assumed in all of the spherical simulations. The domain of the Cartesian model is $0 \leq x \leq 2\pi a \cos \phi_0/\delta$, $-a\pi/4 \leq y \leq 3\pi a/4$, and periodic boundary conditions are applied in both horizontal directions.

3. Numerical methods

We integrate the equations in spherical geometry using a spectral transform technique similar to that described by Browning et al. (1989) for the shallow-water equations. Briefly, the prognostic variables are represented as truncated series of spherical harmonic functions of the form

$$f(\lambda, \phi, t) = \sum_{m=0}^{N} \sum_{k=0}^{N} \left( \hat{f}_{km} e^{imk} P_{km}(\phi) + \text{c.c.} \right),$$

where $P_{mn}$ are the associated Legendre polynomials. Nonlinear terms are evaluated at grid points (Orszag 1970), although it was found that the removal of aliasing error by using an enhanced Gaussian grid was unnecessary at the resolution employed ($N = 96$). Hemispheric symmetry is enforced by evaluating the inner summation in (3.1) only for $n - m$ odd (even) in the expansions of vorticity (divergence and layer thickness). Similarly, sixfold zonal symmetry is enforced by evaluating the outer summation only at $m = 6j, j = 1, \ldots, N/6$, which is equivalent to applying periodic boundary conditions at the longitudes $\lambda_0$ and $\lambda_0 + \pi/3$. We use an explicit, third-order Adams-Bashforth time integration scheme (Durran 1991) with a time step of 2 minutes.

The equations in Cartesian geometry are solved using a spectral method similar to that implemented in the spherical model, except that the dependent variables are represented by truncated double Fourier series. We use a resolution of 16 Fourier wavenumbers in the $x$ direction and 72 Fourier wavenumbers in the $y$ direction.

Both the spherical and Cartesian models resolve the fundamental wave and 15 higher harmonics. Due to differences in the spectral truncation, however, the shortest resolvable scale is approximately 413 km in the spherical model and 292 km in the Cartesian model. We have verified that the results presented here are not sensitive to small differences in the spectral resolution between the Cartesian and spherical models.

The cascade of enstrophy to unresolved scales in frontal zones is controlled in both models by a $\nu^6$ diffusion on vorticity and divergence. The coefficient is chosen so that the smallest resolvable wave is damped with an $\epsilon$-folding time of one hour.

4. Spherical and Cartesian simulations

We now present a comparison of spherical and Cartesian simulations of baroclinic waves evolving from small perturbations to an initial zonal flow.

The initial zonally symmetric state is defined by prescribing the interface Exner function, $\pi_{3/2}$, and assuming that the lower layer is quiescent, so that $\pi_{1/2} = c_p$. The upper-layer zonal velocity is then given by geostrophic balance in the Cartesian model,

$$f \theta_2 = -\frac{\partial \pi_{3/2}}{\partial y},$$

while in the spherical model, the upper-layer zonal velocity satisfies
\[
\left( f + \frac{u_2 \tan \phi}{a} \right) \partial u_2 = \frac{\theta_1 - \theta_2}{a} \frac{\partial \pi_{3/2}}{\partial \phi}.
\]

Figure 1 shows the interface pseudoheight \(= \theta_1 (c_\rho - \pi_{3/2})/g\) and the upper-layer zonal velocity for both the spherical and Cartesian models, using

\[
\pi_{3/2} = \pi_{\text{mid}} - \Delta \pi \alpha \int \sin^{11} (2\phi) \, d\phi,
\]

where \(\pi_{\text{mid}} = 0.5 (c_\rho + \pi_{3/2})\), \(\Delta \pi = 0.7 (c_\rho - \pi_{\text{mid}})\), and the constant \(\alpha\) is chosen so that \(\pi_{3/2} = \pi_{\text{mid}} - \Delta \pi\) at \(\phi = 0\).

Although the metric term in (4.2) produces a slightly weaker jet on the sphere, the meridional PV gradients are effectively identical in the two models as shown in Fig. 2. Because of this, we expect that differences in wave evolution will be primarily due to geometric effects associated with the curvature of latitude and longitude lines on the sphere.

Alternatively, we have prescribed the basic-state relative zonal angular momentum fields \(u \cos \phi\) in the spherical model and \(u \cos \phi_0\) in the Cartesian model) to be identical in the two models. This ensures that the basic-state zonal advection operator is identical in the two models. Since the meridional half-width of the initial zonal jets used here is on the order of \(10^6\) latitude, \(\cos \phi / \cos \phi_0\) is close to unity in the neighborhood of the jet, and there is very little difference between the evolution of baroclinic waves on initial zonal states prescribed with these two methods.

In the spherical model, the basic-state fields shown in Fig. 1 are symmetric about the equator. In the Cartesian model, the basic-state pressure field is symmetric about \(y = \pi a/4\), as is the Coriolis force, implying that \(u\) is antisymmetric. Therefore, the basic state in the Cartesian model consists of oppositely directed zonal jets centered at \(y = 0\) and \(\pi a/2\). While unrealistic, this configuration is no more artificial than applying wall boundary conditions at \(y = \pm \pi a/4\), as is usually done in channel models. Sensitivity experiments have determined that the Cartesian solutions are not affected by the meridional boundary conditions, as long as the width of the basic-state jet is small compared to the meridional extent of the domain.

a. Linear modes

The most unstable modes are obtained using the initial value technique of Brown (1969). The Cartesian mode has a growth rate of \(0.645 \text{ d}^{-1}\) and a phase speed of \(8.68 \text{ m s}^{-1}\). The spherical mode grows slightly more slowly, at \(0.61 \text{ d}^{-1}\), with a phase speed of \(1.76 \times 10^{-6} \text{ rad s}^{-1}\) (or \(8 \text{ m s}^{-1}\) at \(45^\circ\text{N}\)).

The amplitude and phase of the perturbation meridional velocity for both modes are shown in Fig. 3. In general, the modes are very similar. The spherical mode has slightly more amplitude and a more pronounced southwest–northeast phase tilt in the upper layer south of the jet axis. The modes computed here are similar to the modes computed by Warn (1976) using a two-layer incompressible PE model in spherical geometry.

b. Nonlinear evolution

Using initial conditions consisting of the most unstable normal mode scaled so that \(\max (v) = 1 \text{ m s}^{-1}\), the Cartesian and spherical models are integrated for 10 days. A sequence of maps illustrating the flow evolution is provided in Figs. 4–6.

For the first few days, the Cartesian and spherical simulations are quite similar. By day 5 (Fig. 4), both
Fig. 3. Amplitude and phase of the meridional velocity associated with the most unstable mode of the basic states shown in Figs. 1 and 2. The solid (dashed) curve denotes Cartesian (spherical) quantities.

solutions display a tendency for PV contours to be advected north and west around the cyclones in both layers. Nonlinear effects become important by day 5 and are manifested in the tendency for lower-layer cyclones (anticyclones) to migrate northward (southward). The meridional migration of the lower-layer disturbances is due to the westward tilt with height of the wave, which enables the flow induced in the lower layer by the upper-layer PV anomalies to advect the lower-layer anticyclonic (cyclonic) PV anomaly south (north).

In the upper layer, this effect tends to displace the cyclones (anticyclones) south (north). As is shown in Fig. 4, however, the meridional displacements of cyclones and anticyclones are significantly weaker in the upper layer, a fact that will be important later in our interpretation of the geometric effects. The asymmetry between the layers is largely due to the fact that, near the center of the upper-layer jet, intrinsic phase speeds ($\bar{u} - c$) are nearly 30 m s$^{-1}$, as opposed to 8 m s$^{-1}$ in the lower layer. At this point in the wave's evolution, meridional parcel displacements are, to a first approximation, given by linear theory and are inversely proportional to $\bar{u} - c$. Thus, the PV contours (and hence the cyclones and anticyclones) have larger meridional excursions in the lower layer.

Figure 5 illustrates the wave structure at day 7, approximately one day before the maximum domain-integrated eddy kinetic energy is reached. By this time, the spherical solution has stronger anticyclones and weaker cyclones, relative to the Cartesian solution. The anticyclonic bias in the spherical model is associated with significant differences in the PV fields by this time. Cyclonic wrapping of PV contours around the upper-layer cyclone in the spherical model has ceased, and the anticyclone has begun to advect high PV air southward and westward. In the Cartesian model, however, cyclonic wrapping of PV continues unabated in the upper layer.

By day 9, the spherical (Cartesian) solution is dominated by nearly barotropic anticyclonic (cyclonic) gyres (Fig. 6). As a consequence, high PV air in the upper layer of the spherical solution has been advected equatorward and then westward around the anticyclone. This behavior has been observed in other baroclinic life-cycle simulations in spherical geometry (Held and Hoskins 1985; Thorncroft and Hoskins 1990). In the lower layer of the spherical model, the PV anomaly created by cyclonic wrapping around the cyclone during the growth phase has been completely separated from the region of strongest PV gradients by the advection by the downstream anticyclone. In contrast, the large quasi-barotropic cyclone in the Cartesian simulation continues to wrap up PV contours into the vortex in both layers. This behavior is consistent with other Cartesian simulations, such as Polavarapu and Peltier (1990) and Snyder et al. (1991).

The differences in the large-scale structure between the two model geometries strongly modify the subsequent development of smaller-scale features within the wave. In the spherical model, the tongues of high PV advected around the upper-layer anticyclone separate from the high PV reservoir to the north and form small isolated cyclonic disturbances at about 30°N (Fig. 7). These isolated PV anomalies initiate weak disturbances on the lower-layer PV front in a manner strikingly similar to that described by Thorncroft and Hoskins (1990). Since the dominance of cyclonic vortices prevents any such development in the Cartesian model, it is clear that, at least for this particular basic-state jet, the effects of spherical geometry are ultimately responsible for the shedding of PV from the upper-layer trough and the subsequent frontal cyclogenesis. In addition, the lower-layer PV fronts in the spherical model acquire a strong southwest to northeast tilt, which is not evident in the Cartesian solution (Fig. 5).

In summary, there are major differences between the evolution of nonlinear baroclinic waves in two-
T = 5 DAYS

FIG. 4. Potential vorticity (heavy solid) and perturbation streamfunction (zero line light solid, others dashed) at day 5 in each layer from the Cartesian and spherical simulations. “Perturbation” means deviation from the initial zonal state shown in Fig. 1. Two wavelengths of the disturbance are shown for clarity. The “H” marks the center of the anticyclone. Each major tick mark on the y axis represents 10° of latitude. The contour intervals for potential vorticity are $3.2 \times 10^{-7}$ m$^2$ s$^{-1}$ K kg$^{-1}$ for the lower layer and $8 \times 10^{-7}$ m$^2$ s$^{-1}$ K kg$^{-1}$ for the upper layer. The contour intervals for streamfunction are $4.5 \times 10^6$ m$^2$ s$^{-1}$ in the lower layer and $3 \times 10^6$ m$^2$ s$^{-1}$ in the upper layer.

Layer primitive equation models formulated in spherical or Cartesian geometry. Although the linear modes have very similar structures and growth rates, as nonlinear effects become important, the cyclones (anticyclones) dominate the wave evolution in the Cartesian (spherical) model. The dominance of anticyclones in the spherical model is associated with anticyclonic wave breaking during the decay stage, while in the Cartesian model, no such wave-breaking event occurs and cyclonic vortices persist long after the wave stops growing. Because the basic states are nearly identical in the two models, and differences in numerics have been verified to be negligible, these differences in wave evolution must be due to geometric effects associated with the curvature of latitude and longitude lines in the spherical model.

5. Analysis

a. Scale analysis

To the extent that gravity waves are unimportant in the dynamics, the adiabatic, inviscid evolution of baroclinic waves is described by the conservation of PV,
\[
\frac{dQ}{dt} = 0, \quad Q = \frac{f + \frac{\zeta}{M_{th}}}{M_{th}};
\]

where the Boussinesq approximation has been made in the definition of \(Q\). In the following, we will analyze the role of spherical geometry in the PV dynamics, using a scale analysis [which assumes \(Ro = O(L/a) \ll 1\) as in the Introduction] to order various terms and identify the primary spherical effects. The simplest approach proves to be to transform the spherical equations to the coordinates \((x, y)\) defined in (1.2) and to expand the trigonometric terms in powers of \(L/a\); the scaled equations can then be written in a form identical to that for planar geometry, except for spherical corrections of order \(L/a\). To aid comparison with the Cartesian equations, we will retain the Montgomery potential as the principal dependent variable and express all others in terms of \(M\).

First, let us examine how sphericity enters the “inversion” relation between \(M\) and \(Q\). Before nondimensionalizing the PV, it is convenient to split \(M\) into a horizontally uniform reference state, \(M(\theta)\), and deviations from that state, \(M'(\lambda, \phi, \theta)\). We next introduce the scalings \(M_{th} = \Sigma \Lambda(\theta), \; Q = (f_0/\Sigma) Q^*, \; \text{and} \; \partial/\partial \theta = \Delta \theta^{-1} \partial/\partial \theta^*\), in addition to the scalings given in the Introduction, where asterisks again denote nondimensional quantities. The nondimensional PV is

\[
Q = \frac{\sin \phi / \sin \phi_0 + \text{Ro} \zeta}{\Lambda(\theta) + S \text{Ro} M_{th}}, \quad (5.1)
\]
where both asterisks and primes have been dropped and \( S = \frac{\alpha^2 L^2}{\sum \Delta \theta^2} \) is the Burger number. For the scales under consideration, \( S \) is \( O(1) \) and will be assumed to equal unity.

Although (5.1) will have an identical form for the Cartesian model (since the denominator on the right-hand side contains no spherical terms and the full variation of \( f \) is included in the Cartesian model) geometric effects are implicit in the relation of the vorticity to \( M \). This relation may be diagnosed from the divergence equation, (1.1), as was briefly outlined in the Introduction. After transforming to \((x, y)\) coordinates and nondimensionalizing, the divergence equation becomes

\[
\frac{\sin \phi}{\sin \phi_0} \zeta = \nabla^2 M - Ro \frac{\cos \phi}{\sin \phi_0} u - 2 \frac{\cos \phi_0}{\cos \phi} \frac{\partial (u, v)}{\partial (x, y)} + O(Ro^2),
\]

where \( L/a = Ro \) has been used, \( \phi = \phi_0 + (L/a)y \) and

\[
\nabla^2 = \frac{\cos^2 \phi_0}{\cos^2 \phi} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - Ro \tan \phi \frac{\partial}{\partial y}.
\]

Note that the coefficients of \( \zeta \) and \( u \) above arise from the variation of \( f \), and so also appear in the Cartesian model. All other trigonometric coefficients in (5.2),

Fig. 6. As in Fig. 4 but for day 9 with contour intervals for the streamfunction of \( 1.1 \times 10^7 \) m\(^2\) s\(^{-1}\) in the lower layer and \( 9 \times 10^6 \) m\(^2\) s\(^{-1}\) in the upper layer.
however, including those in $\nabla^2$, are due to spherical metrics and will have a different form (or be absent) in the Cartesian model.

The importance of the various spherical terms may now be assessed by expanding the trigonometric coefficients near $\nu = 0$ in powers of $\theta$. When such an expansion is made in (5.2), differences between the Jacobian term and its Cartesian form do not appear until $O(\theta^2)$, since the Jacobian term is multiplied by $\theta$. [In addition, (5.2) remains correct through $O(\theta)$ when the horizontal velocity is replaced by the geostrophic wind in Cartesian form, that is, $u = -M_y$ and $v = M_x$.] Thus, through $O(\theta)$, the divergence equation in the Cartesian model is identical in form to (5.2), except for the Laplacian term, which may be written

$$\nabla^2 M = M_{xx} + M_{yy} + \theta \tan \phi_0 (2yM_{xx} - M_y) + O(\theta^2).$$  \quad (5.3)

The inversion of $Q$ for $M$ on the sphere therefore differs at $O(\theta)$ from the inversion in the Cartesian model only because of metric effects in the Laplacian.

The spherical and Cartesian models also potentially differ in the form of the conservation relation for PV. Again using $(x, y)$ coordinates, the nondimensional form of this conservation relation in isentropic coordinates is

$$Q + \frac{\cos \phi_0}{\cos \phi} \frac{\partial (\psi, Q)}{\partial (x, y)} + \theta \frac{\partial Q}{\partial y} = 0,$$  \quad (5.4)

where $\psi$ is the streamfunction for the rotational wind and $x$ is the potential for the divergent wind. If the trigonometric terms in (5.4) are also expanded about $y = 0$, the advection of $Q$ by the rotational wind becomes

$$(1 + \theta \tan \phi_0) \frac{\partial (\psi, Q)}{\partial (x, y)} + O(\theta^2),$$  \quad (5.5)

and hence, because of the term proportional to $\tan \phi_0$, the spherical model differs in form from the Cartesian at $O(\theta)$. Through the same order, the advection of PV by the divergent wind has Cartesian form in (5.4), since it is $O(\theta)$ compared to the rotational advection.

Although sphericity clearly affects the advection of PV by the rotational wind, it remains to check how sphericity might change the relation between $M$ and the velocities used to advect $Q$. If we are interested in the form of (5.4) through $O(\theta)$, then the relation between $M$ and the rotational wind (or $\psi$) must be calculated through $O(\theta)$ as well, while only leading-order accuracy is necessary for the relation between $M$ and the divergent wind (or $x$). After nondimensionalization and use of the hydrostatic equation, the continuity equation (2.3) relates $M$ and $x$:

$$\nabla^2 x = \frac{1}{\Lambda + \theta M_{x0}} \frac{dM_{x0}}{dt}.$$  \quad (5.4)

At leading order, $d/dt$ is replaced by its geostrophic, Cartesian form and the above relation will be identical to that in the Cartesian model. The rotational streamfunction has a more subtle connection with $M$ that is revealed by again considering the scaled divergence equation (5.2), but with the vorticity replaced by $\nabla^2 \psi$. The same arguments as following (5.2) apply, except in this case there are additional terms on the sphere that arise from the expansion of $\nabla^2 \psi$ [as in (5.3) with $M$ replaced by $\psi$]. However, since $\psi = M + O(\theta)$ and $\sin \phi_0 = 1 + O(\theta)$, these additional terms and the spherical terms in $\nabla^2 M$ (5.3) cancel, except for a residual of $O(\theta^2)$, and so the relation between $M$ and $\psi$ through $O(\theta)$ is the same in the spherical and Cartesian models.

Given the relation of $\psi$ and $x$ to $M$, differences between the spherical- and Cartesian-model PV dynamics at leading order are limited to metric terms in the inversion of $Q$ for $M$ [through the spherical Laplacian, as in (5.3)] and in the advection of the PV by the rotational wind [as in (5.5)]. Unfortunately, the scale analysis does not elucidate how these spherical effects alter the wave evolution, nor can it predict the relative importance of the terms in the inversion and in the advection of $Q$. 

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Fig. 7. Upper-layer PV from the spherical simulation at day 10. The contour interval is $8 \times 10^{-4} \text{ m}^2 \text{ s}^{-1} \text{ K kg}^{-1}$. 

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b. Effects of sphericity on the inversion of PV

Although there are differences in the linear modes and early development between the spherical and Cartesian simulations, much larger differences appear once the development becomes strongly nonlinear. One may compare the solutions after five days (Fig. 4), when nonlinear effects are beginning to emerge, to the solutions just two days later (Fig. 5), when nonlinear effects are strong; from day 5 to day 7, significant qualitative differences between the spherical and Cartesian solutions develop. For this reason, the following analysis of how spherical metrics in the PV inversion might lead to such differences will focus on the nonlinear stage of baroclinic wave growth.

The primary signature of nonlinearity in growing baroclinic waves is the meridional migration of cyclones and anticyclones, which is visible in both Figs. 4 and 5. An alternative, but essentially equivalent, statement is that the flow in such a wave may be approximated by the linear mode at finite amplitude and a mean-flow correction (Phillips 1954; Hoskins 1976), which for our case consists of a westerly jet in the lower layer and a weaker upper-layer easterly jet. The sum of these two components then produces the northward (southward) displacement of cyclones in the lower (upper) layer and the opposite displacement of anticyclones.

To better understand how spherical metrics modify the PV inversion, we begin with the barotropic problem of deducing the streamfunction from the vertical vorticity by inverting a horizontal Laplacian operator. Barotropic inversions of vorticity include the salient feature of the baroclinic PV inversions, since the dominant spherical metrics in the baroclinic problem enter the PV inversion operator through the horizontal Laplacian [see (5.3)]. We will concentrate on the inversion of vorticity distributions that are representative of nonlinear baroclinic waves in that positive and negative vorticity anomalies are displaced meridionally in opposite directions.

The dimensionless vorticity in spherical geometry is

\[ \zeta = \frac{\cos^2 \phi_0}{\cos^2 \phi} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \text{Ro} \tan \phi \frac{\partial \psi}{\partial y}, \]  

(5.6)

where, as previously, we have set \( L/a = \text{Ro} \). Expanding the trigonometric factors in (5.6) in Taylor series about \( y = 0 \) (\( \phi = \phi_0 \)) and performing a power series expansion of \( \psi \) in \( \text{Ro}(\psi = \psi^0 + \text{Ro} \psi^1 + \cdots) \) yields

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi^0 = \zeta, \]  

(5.7a)

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi^1 = \tan \phi_0 \left( \frac{\partial}{\partial y} - 2y \frac{\partial^2}{\partial x^2} \right) \psi^0. \]  

(5.7b)

The leading-order streamfunction \( \psi^0 \) is identical to the Cartesian solution, and \( \psi^1 \) is the leading-order correction due to the spherical metrics in the Laplacian.

We now consider analytic solutions of (5.7) for a simple distribution of vorticity that is the sum of a single zonal harmonic and zonally independent part, that is,

\[ \zeta = A \sin k x \cos l y + B \sin 2ly. \]

The leading-order, Cartesian streamfunction associated with this vorticity distribution is

\[ \psi^0 = -\frac{A}{\mu^2} \sin k x \cos l y - \frac{B}{4l^2} \sin 2ly, \]  

(5.8)

where \( \mu^2 = l^2 + k^2 \). This streamfunction, which mimics the lower-layer pattern in the early nonlinear stages of our baroclinic wave solutions, is shown in Fig. 8a for the parameter setting \( \text{Ro} = 0.25, A = 1, B = 0.25, \) and \( k = 2l = \pi/2 \). In the solution of (5.7b), \( \psi^0 \) will force contributions to \( \psi^1 \); these contributions are given by

\[ \psi^1_a = \frac{2A}{\mu^4} \tan \phi_0 \frac{k^2 \sin k x}{\cos l y} \left( y \cos l y - \frac{2l}{\mu^2} \sin l y \right), \]  

(5.9a)

\[ \psi^1_b = \frac{A}{\mu^4} \tan \phi_0 \frac{l \sin k x}{8l^3} \sin l y + \frac{B}{8l^3} \cos 2ly. \]  

(5.9b)

Here, \( \psi^1_a \) is forced by the term \( \partial^2 \psi^0/\partial x^2 \) in (5.7b), while \( \psi^1_b \) is forced by \( \partial \psi^0/\partial y \).

In addition to \( \psi^0 \), Fig. 8 shows the corrected streamfunction, \( \psi^0 + \text{Ro} \psi^1 \), and the two contributions to \( \psi^1 \). Relative to \( \psi^0 \), the corrected streamfunction shows stronger anticyclones and weaker cyclones, and the wave amplitude has been increased to the south of \( y = 0 \) and decreased to the north.

The largest part of the spherical correction to \( \psi^0 \) is \( \psi^1_a \), the part forced by the term \(-2y \tan \phi_0 \partial^2 \psi^0/\partial x^2 \) in (5.7b), which arises from the convergence of longitude lines on the sphere. To see the effects of this correction, note that, since \( \partial^2 \psi^0/\partial x^2 \) has the same sign as \( \zeta \), the quantity \(-y \partial^2 \psi^0/\partial x^2 \) has the same sign as \( \zeta \) south of \( y = 0 \), but the opposite sign north of \( y = 0 \). The \( \psi^1_a \) field therefore reinforces (opposes) the Cartesian streamfunction (\( \psi^0 \)) south (north) of \( \phi = \phi_0 \), so that a vorticity distribution symmetric about \( \phi_0 \) produces relatively more response in the \( \psi \) field south of \( \phi_0 \).

Finally, if \( \psi^1_a \) is combined with \( \psi^0 \), the anticyclones (which have migrated south of \( \phi_0 \)) strengthen, while the cyclones (which have migrated north of \( \phi_0 \)) weaken. Consideration of Fig. 8ac shows that this is the case.

A simple interpretation of this effect is based on the fact that the solution of the Laplacian is proportional to the total wavelength of the forcing. Since the distance between meridians increases toward the equator, a vorticity distribution with a given zonal wavenumber and latitudinal structure has a larger total wavelength at lower latitudes and induces a larger response in the \( \psi \) field. Broadly speaking, one may think of the lower-
layer anticyclones intensifying and the cyclones weakening as they move southward and northward, respectively, on the sphere.

The correction $\psi_k$, shown in Fig. 8d, is a manifestation of the term proportional to $\partial \psi^0 / \partial y$ on the right-hand side of (5.7b). This metric, which arises from the curvature of latitude lines on the sphere, reflects the fact that a zonal flow on a sphere is actually a radially symmetric flow about the rotation axis of the sphere and has nonzero curvature vorticity. For the parameter setting used here ($k \sim l$), the effect of this metric is small, and tends to augment (diminish) the strength of the anticyclone (cyclone) in the center of the domain, while having the opposite effect along the meridional edges of the domain. Other parameter settings, such as $k \ll l$, could make this term relatively more important; however, the choice $k \sim l$ is probably the most relevant for the waves of interest here.

Thus, spherical corrections in this case lead to more intense anticyclones and weaker cyclones, and produce a southward bias in the wave amplitude. If the idealized vorticity distribution had been chosen, however, to be representative of the upper-layer pattern in a nonlinear wave, with an easterly mean-flow correction ($B < 0$ in (5.8)) and cyclones and anticyclones displaced southward and northward, respectively, the spherical corrections would have had the opposite effect. Although the largest spherical correction (Fig. 8c) would be unaltered by such a change in the sign of the mean-flow correction, its superposition with the Cartesian streamfunction $\psi^0$ would intensify the cyclones and diminish the anticyclones. Since, as was explained in
section 4b, the presence of a strong zonal jet inhibits
the initial meridional migration of cyclones and anti-
cyclones (or equivalently, the mean-flow modification)
in the upper layer, we expect that the asymmetry be-
 tween cyclones and anticyclones due to spherical met-
rics will be largest in the lower layer.

As indicated by the scale analysis, these barotropic
results should also be relevant for a stratified fluid. To
verify this, and to check that the cyclone–anticyclone
asymmetry is controlled by the lower-layer PV in our
simulations, we have performed PV inversions using
both the spherical and Cartesian version of the Charney
balance equations (Charney 1962; Davis and Emanuel
1991). The algorithm solves the balance equation si-
multaneously with the definition of PV in isentropic
coordinates to obtain the nondivergent wind and
Montgomery potential fields. Using the PV from the
spherical model at day 6, inversions including the
spherical metrics produced weaker cyclones and stron-
ger anticyclones than Cartesian inversions, as expected.

The spherical metrics may also influence the vertical
penetration of the induced flow above or below a PV
anomaly. The penetration “depth” for a small-ampli-
tude PV anomaly in isentropic coordinates is, in terms
of dimensional variables, \( \Delta \theta = fL \Phi^{1/2} \) (Holzberg et
al. 1985), where \( L \) is a horizontal length scale char-
acteristic of the anomaly. For a PV anomaly of a given
horizontal wavenumber located at various latitudes, \( L \)
will be fixed in the Cartesian model but will vary as
\( \cos \phi \) in the spherical model. Taking \( M_{\phi} \) to be the same
in both models, the ratio of the spherical to the Carte-
sian penetration depth is \( \cos \phi / \cos \Phi_{0} \), since \( f = 2\Omega \)
\( \sin \phi \) in both models. Thus, in the spherical model there is
enhanced (diminished) penetration of the circulation
induced by a wavelike PV anomaly south (north) of
\( \phi_{0} \), relative to the Cartesian model. Numerical inver-
sions of zonal wavenumber 6 PV distributions using
spherical and Cartesian versions of the PV inversion
equations confirm that this is the case in the two-layer
model. Changes in \( \Delta \theta \) on the sphere are therefore due
to the equatorward spreading of meridians, which in-
creases the zonal wavelength of a disturbance that is
displaced southward. This effect derives from the
spherical metrics associated with the \( \nabla^{2} \) operators in
the PV inversion equations, and is simply a general-
ization to a stratified fluid of the effect just discussed in
the context of the barotropic vorticity inversion.

\[ \text{c. Effects of sphericity on the advection of PV} \]

The effect of spherical geometry on the advection of
PV by the rotational wind can be deduced from (5.5).
The term proportional to \( y \tan \phi_{0} \) represents the part
of the horizontal advection of PV that is associated with
the curvature of latitude and longitude lines on the
sphere. Since the factor \( y \tan \phi_{0} \) is negative (positive)
south (north) of \( \phi_{0} \), the advection of PV on the sphere
is diminished (enhanced) relative to the Cartesian re-
sult south (north) of \( \phi_{0} \), implying that the spherical
metrics in the advection operator tend to oppose the
main spherical effects from the PV inversion operator.
Geometrical effects are likely to be larger for PV in-
version than for PV advection, since the inversion op-
erator involves two \( \alpha \) derivatives instead of one and
has a larger numerical coefficient on the leading-order
spherical corrections, which are proportional to \( y \tan \phi_{0} \)
in either case.

We have investigated the relative importance of
spherical effects in the PV inversion and advection op-
erators by performing comparative integrations of the
barotropic vorticity equation. Barotropic flow is again
considered as a simpler model problem that retains the
same leading-order geometric effects as baroclinic flow.
Three models were used: a Cartesian model (obtained
by neglecting the spherical metrics in both the advect-
on and vorticity inversion operators, but retaining the
full variation of \( f \)), a fully spherical model, and a
spherical model with the effects of sphericity suppressed
only in the vorticity inversion. The initial conditions
consist of a wavenumber 6 vorticity perturbation (with
an amplitude of \( 5 \times 10^{-5} \) s\(^{-1} \) and a \( \sin^{12} \phi \) meridional
structure) superimposed upon a zonal jet of the form
\( u = 40 \sin^{3} \phi m s^{-1} \). The resolution, dissipation, and
numerical techniques used are identical to that de-
scribed for the two-layer model in section 3.

Figure 9 shows the initial vorticity distribution and
the solutions for the three models at day 2. Cyclones
“migrate” north and anticyclones migrate south in all
the models; the resulting configuration is similar to
that in the lower layer of the baroclinic wave simu-
lations, although it is driven by a distinct dynamical
process. In the Cartesian model (Fig. 9b), the vorticity at
day 2 is nearly symmetric about \( \phi_{0} = 45^\circ \) and the
cyclones and anticyclones are of roughly equal mag-
nitude. In the spherical model (Fig. 9c), however, an-
ticyclones soon dominate the solution and an anticly-
clone wave-breaking pattern develops in the vorticity.
The spherical model with the metric terms suppressed
in the inversion operator produces a vorticity field
similar to that of the Cartesian model, but with slightly
stronger cyclonic vorticity anomalies and weaker an-
ticyclonic vorticity anomalies (Fig. 9d). From the simi-
larity of Figs. 9b and 9d, it is clear that the dominant
spherical metrics are in the inversion operator.

6. Discussion

In section 5, we argued that the primary effect of
sphericity on the baroclinic wave solutions presented
here enters through the Laplacian in the PV inversion
operator and leads directly to an anticyclonic bias in
the lower-layer streamfunction once the meridional
migration of the cyclones and anticyclones has begun.
Here we offer some speculation on how these initially
small geometric effects can lead to the large differences in wave structure illustrated in Fig. 6 and briefly discuss a possible alternate description based on theories of nonlinear radiation of Rossby waves by baroclinic instability.

Our conceptual view is that the growth of nonlinear baroclinic waves results in the formation of quasi-barotropic vortices, at least in cases of rapidly growing instabilities such as ours. Our calculations suggest that even small effects that initially produce a bias toward either cyclonic or anticyclonic circulations have a large influence on the strength and sign of the vortices formed. We contend that once the wave has reached sufficient amplitude for closed streamlines to form (in frame of reference moving with the disturbance), these biases will tend to reinforce themselves as PV of the same sign as the bias is accumulated within the region of closed streamlines, while PV anomalies of the opposite sign are progressively deformed and sheared apart. In addition, in a baroclinic system the PV anomalies that are sheared to smaller scales will induce circulations with relatively less penetration into other layers of the atmosphere, thereby further reducing their influence relative to circulations induced by PV of the opposite sign. The barotropic decay phase then proceeds through the largely barotropic interaction of the vortices with each other and with the zonal mean flow.

Thus, for example, the inherent bias in the Cartesian solution toward closed cyclonic circulations in both layers (due to stretching of relative vorticity) leads to coherent cyclonic vortices. In the spherical solution, the metric terms in the PV inversion, which favor
closed streamlines in the lower-layer anticyclones as they migrate southward, eventually overwhelm the cyclonic bias in the stretching term. The spherical solution therefore begins with a tendency for stronger cyclones and for cyclonic wrapping of PV (Fig. 4) but is later dominated by quasi-barotropic anticyclones and anticyclonic "wave breaking," with its attendant wrapping and filamentsation of high PV around the anticyclones (Fig. 6). Note that this explanation for the behavior of the spherical case depends on the nonlinear migration of cyclones and anticyclones being largest in the lower layer, since for a westerly jet spherical effects will produce a cyclonic bias in the upper layer.

Similar arguments explain the results of Davies et al. (1991) and Simmons and Hoskins (1980) concerning the influence of barotropic shear on developing baroclinic waves, since the addition of barotropic shear to a given basic state will result in closed circulations developing relatively earlier in regions where the vorticity has the same sign as the basic-state shear.

Theories based on Rossby wave radiation (Held and Hoskins 1985; Hoskins et al. 1985; Feldstein and Held 1989; Simmons and Hoskins 1980) provide a different view of these processes, in particular the anticyclonic wrapping of PV in the upper layer in the spherical solutions. Such theories assume that once a baroclinic disturbance reaches its maximum amplitude, Rossby waves radiate energy (or more precisely, wave activity) meridionally into polar and subtropical regions. Within this framework, the wrapping of PV around the anticyclone at upper levels is associated with wave breaking as the waves propagating southward approach their critical latitudes. Furthermore, the differences between the spherical and Cartesian solutions might be explained by the tendency of Rossby waves on the sphere to propagate equatorward and have phase lines that tilt southwest to northeast, as shown by Hoskins et al. (1977).\(^1\) The relatively stronger equatorward propagation on the sphere would then result in a greater portion of the radiated waves moving toward the sub-tropics and more pronounced anticyclonic wave breaking in the spherical solutions. This description is likely, however, to be no more than qualitatively accurate for situations like those presented here: where the mode is rapidly growing, has a meridional scale that is comparable to that of the basic-state jet, and significantly modifies the upper-level PV before saturation.

7. Baroclinic wave evolution on the sphere with an easterly jet

One further test of our explanation of the effects of sphericity on the baroclinic wave life cycle was performed by using a zonal jet identical to that shown in Fig. 1, except with easterly flow in the upper layer. [The jet was obtained using (4.2) and (4.3), with \(\Delta \pi = -0.7(C_p - \pi_{mid})\) and a quiescent lower layer.] An unstable wave on an easterly jet undergoes a nonlinear evolution like that of a wave on a westerly jet, except that the displacement of cyclones or anticyclones, and the accompanying changes to the zonally averaged flow, have opposite sign: in the lower layer, cyclones are displaced southward and anticyclones northward for an easterly jet and the mean-flow correction is easterly. If our interpretation of the effects of sphericity on the evolution of a baroclinic wave is correct, in the easterly jet case the circulation associated with the cyclonic PV anomalies should dominate the development as the cyclones migrate equatorward.

Nonlinear integrations of the spherical model from normal-mode initial conditions exhibit precisely this behavior. The most unstable mode of the easterly jet has a growth rate of 0.776 d\(^{-1}\) and a phase speed of \(-18.7\) m s\(^{-1}\) at 45\(^\circ\)N. Figure 10 shows the PV and perturbation streamfunction fields at day 5. The cyclones dominate the solution, producing strong cyclonic wrapping of PV contours in both layers; no anticyclonic wrapping of PV occurs in this simulation. In fact, if the panels of Fig. 10 are viewed upside down, the solution resembles the Cartesian solution for the westerly jet shown in Fig. 5.

8. Conclusions

Comparative integrations of a two-layer PE model in spherical and Cartesian geometry show that the life cycles of baroclinic waves are significantly affected by spherical geometry. Although the linear modes are similar in spherical and Cartesian geometry, significant differences begin to appear as nonlinear effects become important. Geometric effects on the sphere enhance the development of anticyclones and weaken the cyclones for a westerly jet with weak flow in the lower layer. After an initial period of cyclonic wrapping of high PV in the upper layer of both models, high PV air is advected equatorward and then westward around the upper-layer anticyclone on the sphere, finally separating from the region of large PV gradients. Weak frontal cyclogenesis occurs in a manner similar to that described by Thornicroft and Hoskins (1990) as the detached upper-layer PV anomaly interacts with the lower-layer PV front. In Cartesian geometry, large quasi-barotropic cyclonic vortices persist even after wave growth ceases and no anticyclonic wrapping of PV contours occurs. In addition, the relatively stronger anticyclones on the sphere produce significant differences in frontal structure.

Assuming \(L/\alpha = O(Ro)\), asymptotic analysis demonstrates that the effects of sphericity enter in the next-order correction to \(\beta\)-plane quasigeostrophic dynamics. Spherical effects at this order appear only in the PV

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\(^1\) Although their analysis includes the effects of both fully variable \(f\) and the spherical metrics, it can be shown that the metrics alone produce a similar tendency for waves to turn toward the equator.
T = 5 DAYS

**UPPER LAYER (SPHERE)**

**LOWER LAYER (SPHERE)**

Fig. 10. Potential vorticity (heavy solid) and perturbation streamfunction (zero line light solid, others dashed) at day 5 for a simulation in spherical geometry starting with initial easterly zonal flow in the upper layer (H marks the anticyclone center). The contour interval for PV is $5 \times 10^{-7}$ m$^2$ s$^{-1}$ K kg$^{-1}$, and the contour interval for streamfunction is $6 \times 10^4$ m$^2$ s$^{-1}$.

inversion operator and in the advection of PV by the nondivergent wind. An analysis of the PV inversion operator for the barotropic model shows that the dominant spherical metric allows a PV anomaly with a given zonal wavelength and latitudinal scale in spherical geometry to induce a relatively stronger flow as the anomaly is displaced southward. This is because the equatorward divergence of meridians on a sphere implies that the disturbance has a longer total wavelength as it is displaced southward and the induced flow is proportional to the total wavelength of the anomaly. Thus, as the baroclinic wave grows into the nonlinear regime and the lower-layer cyclones (anticyclones) migrate northward (southward), the anticyclonic (cyclonic) circulations in the spherical integration become relatively weaker (stronger) than those in the Cartesian integration. For a stratified fluid, this metric also implies that the induced velocity fields associated with a wavelike PV disturbance displaced southward penetrate vertically away from the PV disturbance more effectively in spherical geometry. Spherical metrics in the advection operator have an influence opposite to those in the PV inversion operator, but simple scaling arguments suggest, and numerical integrations of a barotropic model confirm, that the effects of sphericity in the inversion operator are probably more important.

These results confirm that the effects of spherical geometry can be as important as leading-order ageostrophic effects in determining the structure and evolution of baroclinic waves, at least at typical synoptic scales where $L/a = O(\text{Ro})$. Therefore, the errors incurred by using channel geometry to simulate baroclinic waves can be comparable to the errors inherent in using the QG equations instead of the PE. Snyder et al. (1991) have shown that the inclusion of leading-order ageostrophic effects in a channel model simulation produces stronger cyclones and more cyclonic wrapping of PV contours. We have shown that the inclusion of spherical effects opposes these ageostrophic effects and favors the development of anticyclones and anticyclonic wrapping of PV contours. Therefore, a spherical PE baroclinic wave simulation may look more like a QG channel simulation than a PE channel simulation.

It is important to emphasize that spherical simulations do not always produce an anticyclonic evolution, nor is a cyclonic evolution guaranteed in Cartesian geometry. As was demonstrated by Davies et al. (1991) and Simmons and Hoskins (1980), both types of evolution are possible in either geometry, depending sensitively on the amount of cyclonic or anticyclonic shear in the basic state. This study demonstrates that, relative to Cartesian geometry, there is a bias toward anticyclones on a sphere, and that this bias can have a profound effect on the nonlinear evolution.

This study highlights the use of “PV thinking” to understand subtle aspects of the dynamics of nonlinear baroclinic waves. As outlined by Hoskins et al. (1985), the implementation of PV thinking involves two steps: 1) the inversion of the PV distribution to obtain the velocity and temperature fields, and 2) the use of these velocity fields to understand how the PV will evolve over time. The results of this study demonstrate that small errors in step 1 can dramatically affect the nonlinear evolution.
Since these results were obtained with a two-layer model, the question remains: Are the inferences made applicable to continuously stratified flows? As was demonstrated in section 5, the effects of sphericity are contained in the horizontal derivatives in the PV inversion and advection equations. Therefore, the vertical discretization should not qualitatively affect the relative differences between the spherical and Cartesian baroclinic wave simulations. In addition, the striking similarities between the westerly jet spherical life cycle presented here and those computed using models with higher vertical resolution (see, e.g., Thornicroft and Hoskins 1990) give us further confidence in the generality of our conclusions.

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REFERENCES


