Scaling Analysis of Curved Fronts: Validity of the Balance Equations and Semigeostrophy

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ABSTRACT

Strongly curved fronts occur when the ratios of cross-front to alongfront horizontal velocities and scales are comparably small for an order one distance in the alongfront direction. This means that all three velocity terms in the continuity equation contribute at leading order. The balance equations are shown to be formally accurate through two orders in powers of the small parameter in this situation; however, the semigeostrophic equations are accurate only at leading order. Both approximate models are accurate through two orders of magnitude if the front is more weakly curved.

1. Introduction

Scaling analysis has been a very useful tool in the study of atmospheric and oceanic flows. In the early work of Charney (1947), it was used to justify the quasigeostrophic (QG) equations as the leading-order approximation to the primitive equations (PE) in the limit of small Rossby number. The conservative PE are defined by applying the hydrostatic, Boussinesq, inviscid, and adiabatic approximations to the three-dimensional form of the Navier–Stokes equations with rotation. The balance equations (BE) were originally advocated because they are accurate through two orders in Rossby number in the QG scaling of the first type used by Charney (Lorenz 1960; Charney 1962). Further analyses of the regimes of validity of balanced models for different degrees of stratification and relative changes in Coriolis frequency were carried out by Gent and McWilliams (1983). McWilliams (1985) used scaling analysis to show that the BE are also accurate through two orders in the limit of small Froude number, whatever the size of the Rossby number.

Hoskins (1975) introduced the semigeostrophic (SG) equations with an application to two-dimensional frontal flow and justified them by a scaling analysis in the limit of small Lagrangian Rossby number. McWilliams and Gent (1980) used the more usual scaling analysis of small Eulerian Rossby number to show that the SG equations are accurate only at leading order in QG scaling of the first type. The relation between the Lagrangian and Eulerian scalings was discussed in appendix B of Snyder et al. (1991), which reassessed the Hoskins (1975) analysis. McWilliams and Gent (1980) also considered three-dimensional fronts and showed that the SG equations are accurate only at leading order in strongly curved fronts. These fronts occur when the ratios of cross-front to alongfront horizontal velocities and scales are comparably small for an order one distance in the alongfront direction. This means that all three velocity terms in the continuity equation contribute at leading order. The SG approximation is valid through two orders of accuracy if the front is more weakly curved, or even straight (i.e., independent of the alongfront coordinate).

McWilliams and Gent (1980) also concluded that the BE are "incorrect at leading order in all frontal flows" (see their pages 1668 and 1671 and Table 3), based upon an incomplete understanding that a small Rossby number is not the only basis for BE validity. Snyder first pointed out to the other two authors that this conclusion is wrong. The purpose of this note is to correct this published error and to show that the BE are valid through two orders of accuracy in curved frontal flows. To address the balances in curved frontal flow, the equations are given in cylindrical coordinates in the next section. The frontal scaling analysis is shown in section 3, and section 4 contains the conclusions of this work.

2. Equation sets

Section 5 of McWilliams and Gent (1980) says, "We consider a set of orthogonal curvilinear coordinates such that x is always directed along the front and y is perpendicular to it." In practice, the use of such coordinates would be very complicated because the location of the axes would vary in both time and horizontal position as the front evolved in time and space. However, these coordinates were not actually used; the equations were written in terms of rectangular coordinates (x, y, z) and included curvature terms in the

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momentum and continuity equations. These curvature terms are the ones that occur when the equations are written in cylindrical coordinates, but it is inconsistent to include these curvature terms when using rectangular coordinates.

Figure 1 is a schematic diagram of strongly curved flow, and it is clear that curvature effects in the momentum balance must be taken into account in this flow geometry. The simplest way to do this for the scaling analysis, without using general front-following coordinates, is to use a diagnostic transformation into cylindrical coordinates and to assume that the front lies approximately on a circle of constant radius r_*. This approach can locally represent the curvature terms for a more general curved front whose radius of curvature coincides with r_* and thus provides a correct estimate of errors in the various equation sets in such situations. It is also more consistent than the use of rectangular coordinates in McWilliams and Gent (1980). However, the use of cylindrical coordinates here is purely diagnostic and should not be taken as a recommendation for their use in numerical calculations where other coordinate systems are preferable.

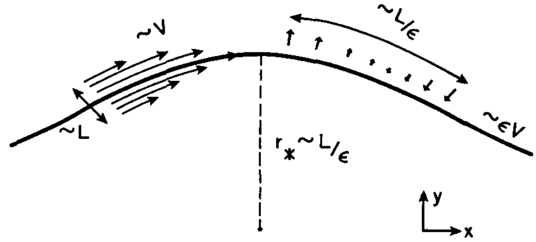

FIG. 1. Schematic drawing of a strongly curved frontal flow. The bold line is the front axis, the dashed line of length r_* is the local radius of curvature, and the alongfront and cross-front lengths and velocities are indicated by the arrows and magnitude estimates. Here ϵ is the small parameter that measures their ratios.

a. Primitive equations (PE)

The nondimensional PE momentum and continuity equations in cylindrical coordinates are

$$R\left(\frac{Du}{Dt} - \frac{v^2}{r}\right) - fv + \frac{\partial \phi}{\partial r} = 0, \qquad (1)$$

$$R\left(\frac{Dv}{Dt} + \frac{uv}{r}\right) + fu + \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0, \qquad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \qquad (3)$$

where f is the Coriolis frequency, ϕ is the appropriate potential, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}. \qquad (4)$$

The Rossby number, R, is defined by

$$R = \frac{V}{f_0 L}, \qquad (5)$$

where V, L, and f_0 are scales of alongfront horizontal velocity, cross-front horizontal length, and Coriolis frequency. Note that ϕ has been scaled by f_0VL; time, t, by the advective scale L/V; and w by VH/L, where H is a depth scale used to nondimensionalize z. The horizontal velocity vector, $\mathbf{u}$, can be written as the sum of rotational and divergent components as

$$\mathbf{u} = \mathbf{k} \times \nabla\psi + \nabla\chi$$
$$= \left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \frac{\partial \chi}{\partial r}, \frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \chi}{\partial \theta}\right), \qquad (6)$$

where $\mathbf{k}$ is the unit vertical vector and the Jacobian operator is defined by

$$J(a, b) = \frac{1}{r}\left(\frac{\partial a}{\partial r}\frac{\partial b}{\partial \theta} - \frac{\partial b}{\partial r}\frac{\partial a}{\partial \theta}\right). \qquad (7)$$

Then the PE vorticity and divergence equations are

$$\frac{D}{Dt}(f + R\nabla^2\psi) + (f + R\nabla^2\psi)\nabla^2\chi + R\left[\nabla w \cdot \nabla\left(\frac{\partial \psi}{\partial z}\right) + J\left(w, \frac{\partial \chi}{\partial z}\right)\right] = 0, \qquad (8)$$

$$R\left[\frac{D}{Dt}(\nabla^2\chi) + (\nabla^2\chi)^2 + J\left(\frac{\partial \psi}{\partial z}, w\right) + \nabla w \cdot \nabla\frac{\partial \chi}{\partial z}\right] + \nabla^2\phi = \nabla \cdot (f\nabla\psi) + J(f, \chi) + 2RJ(u, v) + \frac{R}{r}\frac{\partial}{\partial r}(\mathbf{u} \cdot \mathbf{u}). \qquad (9)$$

b. Balance equations

The BE in cylindrical coordinates can be obtained from Eqs. (8) and (9) by truncations of the usual form; that is, in (8) omit the quadratic terms in χ and w, and in (9) omit the substantial derivative and $J(f, \chi)$ terms and retain only products of ψ in all the quadratic terms. Thus, the BE are

$$\frac{D}{Dt}(f + R\nabla^2\psi) + (f + R\nabla^2\psi)\nabla^2\chi + R\nabla w \cdot \nabla\left(\frac{\partial \psi}{\partial z}\right) = 0, \qquad (10)$$

$$\nabla^2\phi = \nabla \cdot (f\nabla\psi) + 2RJ\left(\frac{\partial \psi}{\partial r}, \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) + \frac{R}{r}\frac{\partial}{\partial r}(\nabla\psi \cdot \nabla\psi). \qquad (11)$$

If Eqs. (10) and (11) are transformed back to rectangular coordinates, then they give exactly the usual BE that are written, for example, as (2.17) and (2.18) of McWilliams and Gent (1980).

c. Semigeostrophic equations

The SG equations result from making the geostrophic momentum approximation in Eqs. (1) and (2). In cylindrical coordinates they are

$$R\left(\frac{Du_g}{Dt} - \frac{vv_g}{r}\right) - fv + \frac{\partial \phi}{\partial r} = 0, \qquad (12)$$

$$R\left(\frac{Dv_g}{Dt} + \frac{vu_g}{r}\right) + fu + \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0, \qquad (13)$$

where the geostrophic velocity, $\mathbf{u}_g$, is defined in the usual way by

$$\mathbf{u}_g = \mathbf{k} \times \nabla \phi. \qquad (14)$$

The SG vorticity and divergence equations are

$$\frac{D}{Dt}(f + R\nabla^2\phi) + (f + R\nabla^2\phi)\nabla^2\chi + R\left[\nabla w \cdot \nabla\left(\frac{\partial \phi}{\partial z}\right) + J(v, v_g) + J(u, u_g) + \frac{1}{r}\frac{\partial}{\partial r}(\mathbf{u} \cdot \mathbf{k} \times \mathbf{u}_g)\right] = 0, \qquad (15)$$

$$\nabla^2\phi = \nabla \cdot (f\nabla\psi) + J(f, \chi) + R\left[J\left(w, \frac{\partial \phi}{\partial z}\right) + J(u, v_g) + J(u_g, v) + \frac{1}{r}\frac{\partial}{\partial r}(\mathbf{u} \cdot \mathbf{u}_g)\right]. \qquad (16)$$

3. Scaling analysis

A schematic diagram of a strongly curved front is shown in Fig. 1. It shows that the cross-front velocity u is small compared to the alongfront velocity v. The ratio of these two velocities is $O(\epsilon)$, where ϵ is a small parameter expressing the frontal character of the flow. Flow variables change much more quickly across the front than along it, so that the ratio of cross-front to alongfront scales is also taken to be $O(\epsilon)$; that is, $1/r\, \partial/\partial\theta$ is $O(\epsilon)$ compared to $\partial/\partial r$. These ratios are assumed to hold for an $O(1)$ distance in the alongfront direction. If the vertical velocity w is also taken to be $O(\epsilon)$, then all three velocity terms in the continuity equation (3) are $O(\epsilon)$ and contribute at the first nontrivial order in that equation. This means that the horizontal divergence, $\nabla^2\chi$, and χ itself are also $O(\epsilon)$. In contrast, the vorticity, $\nabla^2\psi$, and ψ itself are $O(1)$. Thus, when R is $O(1)$, the strongly curved front scaling is

$$v, \frac{\partial}{\partial r}, \frac{\partial}{\partial z}, \psi, \phi = O(1), \qquad (17)$$

$$u, \frac{1}{r}\frac{\partial}{\partial \theta}, \chi, w, \frac{1}{r}, \frac{\partial}{\partial t} = O(\epsilon). \qquad (18)$$

For the scaling (18) to be correct when applied to the Coriolis frequency requires that gradients in f are small compared to f itself; that is, ∇f is $O(\epsilon)$ compared to f, which is generally an excellent assumption in frontal dynamics.

The fact that the divergence is small compared to the vorticity in this scaling indicates that the BE should be a good approximation to the PE because this is precisely the criterion on which the BE truncation is based. The BE vorticity equation (10) omits the Jacobian of w and $\partial\chi/\partial z$ term compared to the PE vorticity equation (8). This term is $O(\epsilon^2)$ compared to the other terms in (8). The BE divergence equation (10) also only omits $O(\epsilon^2)$ terms compared to the PE divergence equation (9), since u itself is $O(\epsilon)$ and the $O(1)$ alongfront velocity v is given correctly through two orders by its rotational component; that is,

$$v = \frac{\partial \psi}{\partial r} + O(\epsilon^2). \qquad (19)$$

This shows that the BE are correct through two orders in ϵ in strongly curved fronts.

In contrast, the cross-front momentum equation (1) shows that v is only given correctly at leading order by its geostrophic component; that is,

$$v = \frac{\partial \phi}{\partial r} + O(\epsilon), \qquad (20)$$

so that the alongfront ageostrophic velocity component is $O(\epsilon)$. This suggests that the geostrophic momentum approximation will only be accurate at leading order. The SG divergence equation (16) is correct through $O(\epsilon^2)$ in this strongly curved frontal scaling, compared to the PE divergence equation (9). However, Eq. (16) shows that $\nabla^2\phi$ and $\nabla^2\psi$ differ at $O(\epsilon)$. Thus, the SG vorticity equation (15), which contains the substantial derivative of $\nabla^2\phi$, differs from the PE vorticity equation (8), which contains the substantial derivative of $\nabla^2\psi$, by $O(\epsilon)$. Thus, the SG equations are only accurate at leading order in strongly curved fronts.

If the front is less strongly curved, then the curvature terms in the momentum equations (1) and (2) will be smaller with $1/r$ being $O(\epsilon^2)$. In this situation the momentum equations (1) and (12) differ at $O(\epsilon^2)$, so that the SG equations are accurate through two orders in ϵ. Equation (19) remains the same in less strongly curved flow, so that the BE remain accurate through two orders in ϵ in this situation. Finally, if the front is straight, then (19) and (20) still have errors of $O(\epsilon^2)$, so that both BE and SG equations remain accurate through only two orders.

The scaling analysis above assumes that the Rossby number, R, is $O(1)$. It breaks down near frontal collapse when R becomes large even though the divergence remains $O(\epsilon)$ compared to the vorticity. (In the absence of limiting dissipation, frontogenesis forced by large-scale deformation rapidly reduces the vertical and cross-front length scales while leaving the other length and velocity scales fixed; we define frontal collapse to be this process.) In this situation, the BE vorticity equation (10) remains accurate because the neglected term is still $O(\epsilon^2)$ compared to terms retained. However, the BE divergence equation (11) becomes invalid because it omits terms of $O(R\epsilon^2)$, which are now not small. The SG equations also become invalid because both the vorticity equation (15) and the divergence equation (16) omit terms of $O(R\epsilon^2)$.

4. Conclusions

The main conclusion is that the BE are accurate through $O(\epsilon)$ in frontal situations ranging from straight fronts to strongly curved fronts that occur when all three velocity terms in the continuity equation contribute at leading order. The conclusion in McWilliams and Gent (1980) in Table 3 that the BE "are incorrect at leading order in all frontal flows" is wrong. The BE validity in frontal flows is assured once it is recognized that the divergence is small compared to the vorticity. This is also the case in the familiar QG scaling of the first type used for midlatitude flow where the BE are accurate through two orders in the Rossby number. McWilliams (1985) showed that the BE are also accurate through two orders in flows where the Froude number is small and the Rossby number is large. Also, Craig (1991) and Shapiro and Montgomery (1993) showed that balanced approximations are valid for nearly axisymmetric, rapidly swirling vortices. Thus, the BE have high accuracy in a wide range of circumstances found in atmospheric and oceanic flows of interest.

In contrast, the SG equations are accurate only at leading order in QG scaling of the first type and in strongly curved fronts. Their accuracy increases to two orders in frontal situations that are less strongly curved. The SG equations do not necessarily have accuracy when the Froude number is small and the Rossby number is large.

This work on the asymptotic validity of the BE was provoked by the results in Snyder et al. (1991). They calculated nonlinear baroclinic instability solutions from an initial condition prescribed in the work of Hoskins and West (1979). Snyder et al. repeated Hoskins and West's calculations using the SG equations, but then found that these solutions did not compare very well with PE solutions using the same initial conditions. They then calculated solutions that were accurate through two orders in Rossby number and found that they corresponded very closely to the PE solutions for as long as they could be calculated. The baroclinically unstable flow produces strongly curved fronts, and the fronts become so strong that blowup occurs in a finite time using SG equations; see Hoskins and West (1979). The second-order accurate solutions fail well before this, but it seemed inconsistent that these solutions would follow the PE so well compared to SG solutions if the asymptotic validity of BE (which is also accurate through two orders in Rossby number) were much poorer. The work in this note explains this inconsistency by showing that the BE have better asymptotic validity than the SG equations in strongly curved fronts.

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APPENDIX

Additional Corrections to McWilliams and Gent (1980)

(a) On page 1659 it is stated that θ for the ocean is density; in fact, it should be minus the density. This makes no difference to any of the results of the paper.
(b) On page 1674 it is stated that appendix A contains results for type 2 geostrophic asymptotics. The scaling used there is a valid asymptotic regime for large domains in which the Rossby number is assumed to be small, but it is not the QG scaling of type 2 described in Phillips (1963), where the acceleration terms are deleted from the momentum equations.

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