Instability of Baroclinic Currents That Are Locally Nonzonal*

IGOR V. KAMENKOVICH
MIT/WHOI Joint Program, Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

JOSEPH PEDLOSKY
Department of Physical Oceanography, Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

(Manuscript received 20 September 1993, in final form 2 February 1994)

ABSTRACT

The zonally localized instability of a zonal flow that has an added meridional component in the zonal interval $|x| < a$ is examined. The flow occurs on the infinite $\beta$ plane and is analyzed in a two-layer model. The basic flow in the lower layer is zero. The zonal flow velocity is small enough so that it is subcritical with regard to baroclinic instability. The flow is rendered unstable only by its horizontal change of direction, which introduces the meridional flow. The stability problem is solved by matching simple plane wave solutions in the regions upstream, downstream, and within the region of meridional flow.

Localized instability is found for all values of the unstable interval $a$. The growth rate diminishes as $a$ shrinks, but no critical value of $a < 0$ needs to be exceeded for instability.

For small values of $a$, the disturbance extends well beyond $a$ as a wake of slowly damped downstream radiating waves. However, the heat fluxes driving the instability are sharply confined to $|x| < a$.

Although the instability is possible only because the meridional component of the flow is different from zero, the baroclinic energy conversion of the available potential energy associated with the zonal flow is at least as large as, and usually larger than, that associated with the meridional flow.

The authors suggest that the process described in this simple model may be present whenever zonally accelerated flows are locally unstable since the region of zonal acceleration of the flow must be accompanied by meridional flow.

1. Introduction

The interest in the regional distribution of cyclogenesis has led in the last decade to a series of studies of longitudinally localized instabilities in an attempt to connect the distribution of atmospheric cyclogenesis activity to the structure of spatially confined normal modes of instability. The numerous studies differ in both their basic physics and the mathematical structure (and resulting simplifications) of each model.

The pioneering work of Pierrehumbert (1984) analyzed the zonal localization of baroclinic instability, which obtains as the consequence of zonal intervals of accelerated eastward flow and enhanced baroclinicity. To simplify the analysis, Pierrehumbert assumed the zonal variations of the flow, responsible for the localization, occurred on scales that were very large compared to the characteristic zonal scale of unstable waves (i.e., a deformation radius). While this was a remarkably illuminating study, the slowly varying assumption and the accompanying WKB analysis did not allow an investigation of the influence of the zonal interval of enhanced instability on the unstable modes' growth rates or structure. In particular, the interesting theoretical issue of the existence (or nonexistence) of a critical interval of enhanced instability length to maintain the localized instability cannot be addressed.

Subsequent numerical studies by Cai and Mak (1990, and references therein) and Finley and Nathan (1993) describe the localized normal modes on accelerated basic flows. Regions of enhanced vertical and horizontal shear are the domains of localized modes of instability. The studies of Cai and Mak clearly describe the confined structure of the normal modes and their energetics and do indicate a trend to lower growth rate as the zonal interval of the enhanced instability is reduced.

Now when the mean flow varies rapidly in the zonal direction, there are two important consequences for the stability problem. First, of course, the WKB approximation is no longer valid and the calculations must be generally numerical. From a conceptual point of view

* Woods Hole Oceanographic Institution Contribution Number 8616.

Corresponding author address: Dr. Joseph Pedlosky, Dept. of Physical Oceanography, Woods Hole Oceanographic Institution, Woods Hole, MA 02543.

© 1994 American Meteorological Society
the more important second consequence is that a substantial meridional velocity must accompany a region of strongly accelerated zonal flow since, quasigeostrophically,
\[
\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}.
\]
(1.1)

The flow is now curved rather than parallel. This change of direction of the basic flow introduces new difficulties for analysis as well as new opportunities for flow instability. Generally speaking then, domains of enhanced shear are also regions of nonparallel flow.

In an effort to distinguish between the effects of locally enhanced supercriticality of the zonal flow (as in the WKB limit) and the effects due to change in the current's direction, Samelson and Pedlosky (1990, hereafter SP) examined the localized unstable modes on a baroclinic current in a two-layer model in which the current was strictly zonal but whose supercriticality varied in the downstream direction as a result of changes in the cross-stream topographic slope. The slope was so arranged that the flow, stable in the absence of topography, was destabilized in a zonal interval of length 2a. Since the basic flow was purely zonal and parallel, the instability and the structure of the zonally confined modes could unambiguously be attributed to the enhanced local supercriticality of the flow. We define the local supercriticality as the degree to which the local flow would be unstable if the local cross section of the flow and the local ambient potential vorticity gradients were considered to be sections of a zonally uniform flow.

Samelson and Pedlosky showed that localized unstable modes were possible in their model no matter how small the interval of enhanced instability. As a went to zero, the growth rate diminished to zero, but for all a, no matter how small, an infinite number of unstable modes were supported in the unstable zone.

It is quite likely that the localized modes found by Cai and Mak (1990), for example, are also fundamentally due to the enhanced baroclinicity in the predominantly zonal portion of the accelerated flow, given the qualitative similarity of their results to the WKB results of Pierrehumbert. In the latter case, the change of direction of the current, manifested by the meridional velocity of the basic state, is certainly absent from the WKB analysis. The study of Finley and Nathan, which ignores the zonal perturbation velocity, consequently also neglects the effect of the meridional velocity of the basic flow. Nevertheless, it is difficult to rule out a priori unstable modes associated with just the change of direction of the current. The study by Gill (1974) of the instability of Rossby waves exemplifies the instability of weak currents that are unstable only because they are not unidirectional.

In this study, we present an analysis of localized instability of a flow that is made possible only because the current changes direction. The situation is as shown in Fig. 1. We present this model as possibly the simplest example of a localized instability due to the change of direction of the basic flow and in the expectation that more general curved flow may display similar instabilities.

A zonal current of infinite meridional extent flows eastward with a velocity \( U \) in the upper layer of a two-layer model. The basic flow in the lower layer is zero. The flow occurs on the \( \beta \) plane, and \( U \) is chosen to be subcritical with respect to the minimum critical shear \( U_c = \beta L z \) for the Phillips model (\( L \) is the deformation radius defined below). At \( x = -a \), the flow abruptly changes direction and flows northeastward until at \( x = a \), it once again abruptly changes direction and again flows in a zonal direction. In the interval \(-a < x < a\), the basic flow has a zonal component \( U \) and a meridional component \( V \), both of which are chosen small enough so that \( (U^2 + V^2)^{1/2} < U_c \).

The meridional component of the flow can be thought of as being in balance with a potential vorticity source \( Q \) such that in \( |x| < a \),
\[
\beta V = Q.
\]
(1.2)

In an oceanographic context, \( Q \) might be thought of as a wind stress curl operating on the strip \( |x| < a \). If \( a \) were infinite, this nonzonal flow would be unstable (Pedlosky 1987) since a plane wave disturbance with crests parallel to the \( x \) axis would yield parallel trajectories across the basic density gradient (interface slope), releasing potential energy without feeling the stabilizing effects of \( \beta \). The goal of the present calculation is to see whether such instabilities can be supported for finite \( a \) (indeed for small \( a \)), so that a localized unstable mode can be attributed purely to the direction change of the basic current. If instability is found, it certainly must be localized, since no growing disturbance with real \( x \) wavenumber can be found in \( |x| > a \) since \( U \) is subcritical.

---

**Fig. 1.** The configuration of the basic flow. The figure shows the streamlines of the basic flow in a two-layer model on the \( \beta \) plane. The basic flow in the lower layer is zero. For \( |x| > a \), the flow, \( U \), is purely zonal; in \(-a < x < a\), the flow picks up a meridional component \( V \) and changes direction. The flow extends uniformly to \( y = \pm \infty \).
The analysis proceeds as in SP. In each of the three regions \( x < -a \), \(-a \leq x \leq a \), and \( x \geq a \), analytic solutions can be found since the linear equations have constant coefficients in each subregion. Employing the proper matching conditions at \( x = \pm a \) yields the eigenvalue relation and the normal modes. The reason for using the matching method as in SP is that it allows a quasi-analytical approach independent of numerical resolution. It conveniently allows us to study the stability problem over a large range of parameter settings. In particular, it allows us to carefully examine the behavior of the instability as a function of the interval length of \( a \) from large \( a \) to very small \( a \).

In section 2, the stability problem is formulated and the proper matching conditions at \( x = \pm a \) are derived. Since the region \( |x| < a \) is produced by a discontinuous (in \( x \)) addition of a meridional current of strength \( V \), it might be thought that the model will produce Helmholtz shear-layer instabilities at \( x = \pm a \). We will show, however, that the presence of the zonal advection flow, \( U \), eliminates the shear layer instabilities and an energy analysis of the unstable modes in section 4 verifies that the instability is primarily baroclinic.

Section 3 presents a discussion of the matching analysis. The limits of large \( a/L_0 \) are discussed analytically and serve to verify our results in that limit.

Section 4 is a discussion of the structure and growth rates of the unstable normal modes. We find that as \( a \) decreases, the growth rate goes to zero, but for all \( a \neq 0 \), at least one unstable mode exists. In distinction to SP, it appears that the number of unstable modes for finite \( a \) is itself finite.

Although the energy transformation from the basic flow to the unstable normal modes is largely limited to the unstable zone, for small \( a \) the disturbance field radiates beyond the unstable region by a considerable distance.

In short, we find that a baroclinic current that changes direction is inherently unstable, and the unstable modes can be found that are localized to the transition region of direction change. Since currents that experience zones where the flow speeds up or slows down must have such transition regions of direction change, where the meridional velocity is significant, we suggest that modes of instability of zonally nonuniform flows may be related to the regions of turning of the flow direction as well as to the local intensification of the zonal flow.

2. The model

a. Equations and matching conditions

We consider flow described by the standard two-layer quasigeostrophic potential vorticity equation on the \( \beta \) plane (Pedlosky 1987). The flow is laterally unbounded and the streamfunction of the basic flow whose stability we investigate is of the form

\[ \Phi_1 = -Uy + \int V(x')dx', \]

\[ \Phi_2 = 0. \]  

For the time being we imagine \( V(x) \) to be a continuous function of longitude, \( x \), although we will be eventually interested in the limiting case where

\[ V(x) = \begin{cases} V, & |x| < a \\ 0, & |x| > a. \end{cases} \]  

In standard notation (Pedlosky 1987), the perturbation form of the potential vorticity equation for each layer becomes

\[ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} + U \frac{\partial}{\partial y} + V \frac{\partial}{\partial y} \right) [\nabla^2 \Phi_1 - (\Phi_1 - \Phi_2)] \]

\[ + \frac{\partial \Phi_1}{\partial x} (\beta + U) \frac{\partial \Phi_1}{\partial y} \left( -\frac{\partial^3 V}{\partial x^2} + V \right) = 0, \]

\[ \frac{\partial}{\partial t} \left[ \nabla^2 \Phi_2 - (\Phi_2 - \Phi_1) \right] + \frac{\partial \Phi_2}{\partial x} (\beta - U) \]

\[ + \frac{\partial \Phi_2}{\partial y} (-V) = 0, \]  

where \( n = 1, 2 \) refers to the upper and lower layers, respectively. The above equations are written in nondimensional form. The length scale has been chosen to be the deformation radius

\[ L_D = \sqrt{g' H_1 / f_0}, \]

and \( H_1 \) has been chosen to be equal to \( H_2 \) for simplicity. The velocities have been scaled by the dimensional zonal velocity so that \( U \) in (2.3a,b) is, in fact, unity. The time is scaled by \( L_0/U_* \), and hence the complex frequency is scaled by \( U_* / L_D \). The parameter

\[ \beta = \beta_* \frac{L_D}{U_*}, \]

where \( \beta_* \) is the dimensional planetary vorticity gradient and \( U_* \) is the magnitude of the constant zonal velocity of the basic flow. By presumption,

\[ \beta > U, \]  

so that the zonal flow is stable; that is, \( V = 0 \), no instability would be possible.

Since \( V \) is a function of \( x \) but is independent of \( y \) and \( t \), we can find normal mode solutions of (2.5) in the form

\[ \Phi_n = R_n A_n(x) e^{i(t - \omega t)}, \]

where the \( A_n \) satisfy
\[ -i\omega \left\{ \frac{d^2A_1}{dx^2} - (l^2 + l)A_1 + A_2 \right\} + il \frac{\partial}{\partial x} \left\{ V \frac{dA}{dx} - A \frac{dV}{dx} \right\} + U \frac{\partial}{\partial x} \left\{ \frac{d^2A_1}{dx^2} - l^2A_1 - (A_1 - A_2) \right\} + \frac{dA_1}{dx} (\beta + U) = 0, \]
\[ -i\omega \left\{ \frac{d^2A_2}{dx^2} - (l^2 + l)A_2 + A_1 \right\} + \frac{dA_1}{dx} (\beta - U) = 0. \] (2.7a,b)

In the above equations, \( V(x) \) is continuous. However, we want to consider the case where \( V(x) \) is discontinuous and constant in each of the three regions as in Eq. (2.2). To derive the proper matching conditions at the points \( x = \pm a \) where the solutions are discontinuous, we proceed as follows. We imagine \( V(x) \) to be continuous but rapidly varying at \( x = \pm a \) and integrate (2.7a, b) across each transition point as shown in Fig. 2. For example, at \( x = -a \), we integrate from \( x = -a - \epsilon \) to \( x = -a + \epsilon \) in which interval \( V \) changes from zero to a constant value \( V_0 \). We then take the limit \( \epsilon \to 0 \) to deduce the proper jump conditions at \( x = -a \).

Thus, integrating (2.7a) across the \( \epsilon \) neighborhood of \( x = -a \) yields, as \( \epsilon \to 0 \), and ignoring terms of \( O(\epsilon) \),
\[ \delta \left\{ U \left\{ \frac{d^2A_1}{dx^2} - (l^2 + l)A_1 + A_2 \right\} + il \left\{ (V - c) \frac{dA_1}{dx} - A_1 \frac{dV}{dx} \right\} + A_1(\beta + U) \right\} = 0, \] (2.8)
where \( \delta G = G(-a + \epsilon) - G(-a - \epsilon) \), and \( c = \omega l \).

That is, the function in the square bracket is a continuous function of \( x \) at \( x = -a \). That being the case, that continuous function may be integrated again across the \( \epsilon \) neighborhood of \( x = -a \).

First we note that
\[ V \frac{dA_1}{dx} - A_1 \frac{dV}{dx} = \frac{d}{dx} (VA_1) - 2A_1 \frac{dV}{dx}. \] (2.9a)

Let \( A_1^+, V^+, \) and \( A_1^- \) and \( V^- \) be the values of \( A_1 \) and \( V \) at \( x = -a + \epsilon \) and at \( x = -a - \epsilon \), respectively. As \( \epsilon \to 0 \), we may represent \( A_1 \) and \( V \) in the neighborhood of \( x = -a \) as
\[ A_1 = \frac{A_1^+ + A_1^-}{2} + \frac{A_1^+ - A_1^-}{2} \tanh \left( \frac{x + a}{\epsilon} \right), \]
\[ V = \frac{V^+ + V^-}{2} + \frac{V^+ - V^-}{2} \tanh \left( \frac{x + a}{\epsilon} \right). \] (2.9b,c)

Thus, as \( \epsilon \to 0 \)
\[ \int_{-a - \epsilon}^{-a + \epsilon} \left( \frac{d}{dx} (VA_1) - 2A_1 \frac{dV}{dx} \right) dx \to V^+ A_1^+ - V^- A_1^- - (V^+ - V^-)(A_1^+ + A_1^-), \] (2.10)
where \( r \gg 1 \) and \( r_0 \to 0 \) as \( \epsilon \to 0 \).

Thus, integrating the continuous function in (2.8) across the \( \epsilon \) neighborhood of \( x = -a \) yields the second matching condition:
\[ U \left( \frac{dA_1}{dx} \right)^+ + il(V^+ - c)A_1^+ = U \left( \frac{dA_1}{dx} \right)^- + il(V^- - c)A_1^-. \] (2.11)

If \( U \) were identically zero and \( \beta \) were neglected, (2.8) and (2.11) would reduce to the classical jump conditions used to describe the shear instabilities of broken-line profiles, that is,
\[ \frac{A}{V - c} = \text{continuous}, \]
\[ (V - c) \frac{dA}{dx} - \frac{dV}{dx} A = \text{continuous}. \] (2.12a,b)

where \( V \) or its derivative is discontinuous. However, \( U \) is not zero, and, in fact, it multiplies the highest \( x \) derivative in the problem. Its neglect would be a singular perturbation, and its presence has a profound effect on the stability problem.

We can write (2.11) as

Fig. 2. A schematic representing the limiting process by which a continuous distribution \( V(x) \) approaches the steplike profile used in the analysis as \( \epsilon \to 0 \).
\[ U \frac{dA_1}{dx} + (s(x) - c)iA_1 = R(x), \quad (2.13) \]

where \( R(x) \) is a continuous function and \( s(x) \) is a bounded function, which takes on the value \( V^- \) at \( x = -a - \epsilon \) and \( V^+ \) at \( x = -a + \epsilon \).

Integrating (2.13) across the \( \epsilon \) neighborhood of \( x = -a \) results in the condition that

\[ A_1^+ = A_1^-, \quad (2.14) \]

that is, that \( A \) is continuous. Although \( V \) is discontinuous, the presence of the advecting flow in the \( x \) direction eliminates the discontinuity in the perturbation streamfunction that (2.12a) would imply were \( U \) identically zero.

Collecting and using our results we find that at \( x = \pm a \), where \( V \) is discontinuous, our continuity conditions are

\[ A_1^+ = A_1^- \]

\[ U \left( \frac{dA_1^+}{dx} - \frac{dA_1^-}{dx} \right) = il(V^+-V^-)A_1, \]

\[ U \left( \frac{d^2A_1^+}{dx^2} - \frac{d^2A_1^-}{dx^2} \right) = -il \left( (V^+-c) \frac{dA_1^+}{dx} - (V^- - c) \frac{dA_1^-}{dx} \right). \]

(2.15a,b,c)

For the lower layer in which the basic flow is zero, a similar treatment shows that \( A_2 \) and \( \frac{dA_2}{dx} \) are continuous at \( x = \pm a \), that is:

\[ A_2^+ = A_2^- \]

\[ \frac{dA_2^+}{dx} = \frac{dA_2^-}{dx}. \]

(2.16a,b)

We must now solve (2.7a,b) where \( V \) is given by (2.2) subject to the matching conditions [(2.15) and (2.16)] as well as the localization condition

\[ \lim_{|x| \to \infty} A_n = 0. \]

(2.17)

b. Nonexistence of Helmholtz instability

The discontinuous jump in \( V \) at either \( x = \pm a \) raises the question of whether a shear-layer instability will result as a consequence. Certainly if \( U \) were zero and (2.12) applied, shear-layer instability would result. To examine this question, we consider here the one-layer barotropic model without \( \beta \) in which \( V \) is given by (2.2) and \( U \) is a constant different from zero. For a single layer, that is, the upper layer in our model in the limit of large \( I \) for which the interface appears rigid and \( \beta \) is negligible, the limiting form of (2.7a) is

\[ \left( U \frac{\partial}{\partial x} - il \right) \left[ \frac{d^2A_1}{dx^2} - l^2A_1 \right] = 0 \quad x < -a, \]

\[ \left( U \frac{\partial}{\partial x} + il \right) \left[ \frac{d^2A_1}{dx^2} - l^2A_1 \right] = 0 \quad x > -a. \]

(2.18a,b)

Let us further consider the case where the only discontinuity in \( V \) is at \( x = -a \). If shear-layer instability is possible, this should be manifest in this situation.

The general solution of (2.18a), which is finite at \( x \to -\infty \) for \( \Im \omega > 0 \), is

\[ A_1^- = b_1 e^{i(x+a)}, \quad (2.19a) \]

while for \( x > -a \)

\[ A_1^+ = b_2 e^{-i(x+a)} + a_2 e^{i(-v)(x+a)U}, \quad (2.19b) \]

where \( c = \omega/|U| \).

The matching conditions (2.15a,b,c) imply

\[ b_1 = a_2 + b_2 \]

\[ U \left[ (b_2 - b_1) - \frac{(V-c)^2}{U^2} a_2 \right] + \frac{(V-c)^2}{U} \frac{dA_1^+}{dx} = 0 \]

\[ U \left[ a_2 \frac{(c-V)}{U} - b_2 \right] = Ub_1 + iVb_1. \]

(2.20a,b,c)

from which it follows that

\[ c = V - iU, \quad (2.21) \]

which implies that no unstable normal mode exists in the one-layer limit. Thus, the introduction of the discontinuous \( V \) profile will simplify the calculations without introducing spurious shear-layer instabilities. The Helmholtz instability is expunged by the presence of the crossflow \( U \).

3. The solutions in each region: Matching

In each of the regions \( x < -a, -a \leq x \leq a, x > a \), \( V \) is constant, so that in each subregion solutions of (2.7a, b) can be found in the form

\[ A_n = B_n e^{ikx}, \quad (3.1) \]

where \( k \) satisfies

\[ \mathcal{D}(k, \omega, \beta, U, V) = 0, \quad (3.2) \]

where \( \mathcal{D} \) is the polynomial

\[ \mathcal{D} = k^4 \omega U - k^4 \omega U^2 - k^2 \omega V U^2 + (\omega^2 V - \beta U + U^2) \]

\[ - k^2 (2\omega^2 (\beta - U) (I^2 + 1) - 2\omega V (I^2 + 1)) \]

\[ + \beta (\beta - U) (I^2 + 1) + (\omega^2 V - \beta) (I^2 + 1) \]

\[ - k(l - \omega U I^2 (I^2 + 1) + 2\omega \beta (I^2 + 1)) \]

\[ - \omega V (I^2 + 1) + 2\omega U I^2) \]

\[ - 2\omega^3 I^2 - \sqrt{3} \omega I^3. \]

(3.3)
The solutions of (3.2) will yield five roots for \( k \). In the region \( |x| > a \) where \( V \) is zero, each of these roots must be complex if \( \Omega_m(\omega) = \omega_i > 0 \). For if \( U < \beta \) (subcritical zonal shear), no real \( k \) will yield instability, \( \omega_i > 0 \). To be consistent with the condition for localization, roots with \( \Omega_m(k) = k_i > 0 \) correspond to disturbances decaying to the right of the unstable interval, while roots with \( k_i < 0 \) decay upstream. The latter will be used to construct the evanescent portion of the disturbance upstream, while the former will be used for the disturbance downstream. For \( |x| < a \), where \( V \neq 0 \), all five roots need, in general, to be used.

Typically, we find three roots representing disturbances decaying downstream and two roots for \( k \) representing the upstream disturbance. Figure 3, for example, shows the real and imaginary parts of the wavenumbers corresponding to the various component portions of the most unstable mode as a function of the half-interval length \( a \) (scaled by the deformation radius). The first two panels (Figs. 3a,b) show the real and imaginary parts of the \( k \) in the interval \( |x| < a \), while Figs. 3c,d show the wavenumbers in the external region \( |x| > a \) where \( V = 0 \). Note that for all \( a \) shown (\( 0 \leq a \leq 15 \)) in \( |x| > a \), two roots for the \( k \) have \( k_i < 0 \), while there are three roots with \( k_i > 0 \). The former are used to represent the solution for \( x < a \), while the latter are needed for \( x < -a \).

We note in passing that the results given here will be for \( V > 0 \). For \( V < 0 \), the symmetry properties of equations and matching conditions imply that if \( \omega \) and \( k \) are solutions for \( V > 0 \), the same solution will obtain for \( V < 0 \) with a change of sign of the real parts of both \( k \) and \( \omega \). Since \( l \) is fixed in this transformation, this implies only that the phase speed in the \( y \) direction is reversed when \( V \) is reversed; otherwise, the mode is unaltered.

Thus, for \( x > a \)

\[
A_1 = \sum_{n=1}^{3} B_n^{(1)} e^{i\theta_n x},
\]

while for \( x < a \)

\[
A_1 = \sum_{n=4}^{5} B_n^{(1)} e^{i\theta_n x},
\]

and for \( |x| < a \)

\[
A_1 = \sum_{n=6}^{10} B_n^{(1)} e^{i\theta_n x}.
\]

For each \( k_n \), the amplitude of the lower-layer disturbance follows from

![Fig. 3. The real and imaginary parts of the roots \( k_n \) of the dispersion relation for the most unstable normal mode. Re(\( k_n \)) and Im(\( k_n \)) for \( |x| < a \) are given in (a) and (b), respectively. The equivalent roots for \( |x| > a \) (where \( V = 0 \)) are shown in (c) and (d).](image-url)
\[ B_{n}^{(2)} = B_{n}^{(1)} \left[ (k_{n}^{2} + l^{2} + 1) + k_{n} \frac{\beta}{\omega} - \frac{IV}{\omega} - \frac{k_{n}U}{\omega} \right]^{-1}. \]  

(3.5)

The ten matching conditions (2.15) and (2.16) at \( x = \pm a \) yield ten linear homogeneous equations for the \( B_{n}^{(1)}, n = 1, 2, \cdots, 10 \). The vanishing of the determinant of that system yields the eigenvalue relation for the complex frequency

\[ \omega = \omega(l, \beta, U, V), \]  

(3.6)

and the accompanying eigenfunction structure.

4. Results

To first give an overview of the principal results, Fig. 4 shows the growth rate and frequency for the value of \( l \) yielding the most unstable mode as a function of the unstable interval length \( a \).

For large \( a \), the growth rate and frequency (Figs. 4ab) approach the result of a WKB calculation in which \( V \) is assumed to be a slowly varying function of \( x \) that attains the constant value \( V = 0.25 \) used in the matching calculation and then diminishes gradually to zero for \( |x| > a \). The solution is found by the technique outlined by Pierrehumbert (1984). The agreement is excellent for large \( a \). Figures 4d and 4e

FIG. 4. The growth rate (a) and the frequency (b) of the most unstable mode as a function of interval half-length \( a \).

For this case \( U = 1, V = 0.25 \), and \( \beta = 1.25 \). The numbered circles indicate the wave number of the most unstable mode. The dotted line, to which the curves asymptote for large \( a \), is the WKB result. In (c) we show the real part of the frequency (O) and growth rate (×) for small \( a \). Note the Re\( \omega \) remains ≠0 as \( a \to 0 \); Im\( \omega \) is always ≥0 for \( a > 0 \), although it become so small that it is hard to graphically distinguish it from zero.

In (d) and (e) we show the growth rates and frequency of the first three unstable modes versus \( l \) for \( a = 10 \). Note the excellent agreement with the WKB indicated with circles.
Fig. 5. The growth rate and frequencies of the first three unstable modes (when all three exist) as a function of meridional wavenumber for different values of $a$. In each case $U$, $V$, and $\beta$ are as in Fig. 4. (a) $a = 5$, (b) $a = 3.3$ and 3.4, (c) $a = 2$, (d) $a = 0.25$ (solid), 0.15 (dashed), and 0.05 (dot-dashed).
show the results for the growth rate and frequency for the first three unstable modes as a function of $l$ for $a = 10$. The circles show the result of the WKB calculation, and its agreement with the most unstable of the modes over the entire range of $l$ is impressive. Truthfully, the excellence of the agreement is somewhat mysterious to us since our numerical profile is not slowly varying, nor is the matching solution composed of a single $x$ wavenumber at each $x$ location as in the WKB method. It is perhaps significant that the complex frequency of both our matching solution and the WKB solution also approaches the complex frequency for the most unstable plane wave for the $V = constant, a \to \infty$ problem as long as $l$ is equal to or greater than the $l$ corresponding to the most unstable mode.
As $a$ diminishes, the growth rate is reduced and vanishes as $a \to 0$ (Fig. 4b). Instability exists, however, although weakly, for all $a > 0$. For the parameters chosen, the $x$ wavelength for maximum instability for the plane wave solution as $a \to \infty$ is about 20. Thus, for $a$ considerably less than this zonal scale, substantial growth rates remain. The growth rate falls to half its value when $2a \sim 6$.

As $a$ goes to zero, the frequency becomes numerically small but does not vanish. This, unfortunately, prevents us from developing a useful asymptotic theory for small $a$ as in SP. Indeed, the frequency approached a limit discovered analytically by requiring that two real coalescing roots exist for $\partial \omega / \partial k = 0$ as $V \to 0$, which is equivalent to looking for the marginally stable mode adjacent to the absolutely unstable mode as $a \to 0$.
For each value of $\alpha$ several unstable modes will exist if $\alpha$ is large enough. For example, at $\alpha = 5$, three unstable modes have been found whose growth rates and frequencies are shown in Fig. 5a. As $\alpha$ is reduced, the number of unstable modes is reduced. Figure 5b shows the growth rates and frequencies of the two remaining modes at $\alpha = 3.3$ and $\alpha = 3.4$ as dashed and solid curves, respectively. Note that between these two values of $\alpha$ there is an interchange of branches of the dispersion curves. For $\alpha = 2$, the second mode is barely unstable as shown in Fig. 5c. For very small $\alpha$, only a single mode survives whose dispersion relations for $\alpha = 0.05$, 0.15, and 0.25 are shown in Fig. 5d. We note the excellent agreement of the real part of the frequency with the asymptotic result in the limit $\alpha \to 0$ described above. In all cases, the maximum growth rate occurs
for \( l \) slightly greater than unity and is nearly equal to the value of \( l \) for the most unstable wave in the plane wave problem as \( a \to \infty \).

Once the dispersion relation yields \( \omega \), the eigenfunction can be constructed in each region using (3.4a,b,c). The normal modes are nontrivially time dependent since \( \omega_r \neq 0 \), and the \( B_{ij}^{(p)} \) are complex. Hence, in each region the solution \( \phi_n \) will be the sum of terms of the form \( e^{i|B_{ij}^{(p)}| \cos(k_n x - \omega t + \phi_n)} \), where \( \phi_n \) is a constant phase angle. Figure 6 shows the streamfunction at \( y = 0 \) at each quarter-period of the oscillating normal mode in which the overall amplitude has been rescaled by removing the growth factor \( e^{\omega t} \). Restoring this factor would not alter the shape of the normal mode at each instant. In Fig. 6, the streamfunction is given for the flow \( U = 1, V = 0.25, \omega = 1.25 \) when the interval is fairly large, that is, \( a = 10 \). From Fig. 4 we see that the growth rate is also fairly large (\( \sim 0.09 \)), and nearly equal to the growth rate for the flow in the infinite interval. As a result, the normal mode decays rapidly outside the unstable interval. Within the interval, the solution is nearly a plane wave although a detailed analysis of the solution shows that, in fact, the solution is composed of two components, as shown by Fig. 7.

For smaller \( a \), the growth rate is considerably reduced and the decay of the solution in the stable zone \( |x| > a \) is much slower. Figure 8 shows the normal mode at four quarter-periods for the same flow as in Fig. 6 but where \( a \) is now \( 1 \) (8a) and 2 (8b). It is immediately apparent that the unstable normal mode extends far beyond the unstable interval. In particular, there is substantial wave activity downstream of the unstable zone. It is important to note that the disturbance in the two layers is largely in phase for \( |x| > a \), which hints at the fact that the energy transformation from the mean to the unstable normal mode is occurring nearly entirely in \( |x| < a \).

For very small \( a \), the normal mode is composed of a mixture of widely varying scales. Figure 9a shows the eigenfunction for \( a = 0.05 \) (i.e., \( 1/20 \) of a deformation radius). The dominant streamfunction is a long wave in the upper layer. The upper panel shows \( A_1(x) \) and \( 5 \times A_2(x) \) (at \( t = 0 \)). The second panel shows that \( A_n \) is essentially constant over the unstable interval. A very small contribution, however, is present from a wave component with a much higher wavenumber. Figure 9b shows the real and imaginary wavenumbers and corresponding real and imaginary amplitudes for the solution in \( |x| > a \), as a function of \( a \). Note that as \( a \to 0 \) one wavenumber (marked by the open circle) becomes very large (and has \( k_i > 0 \) representing a downstream decay). Its amplitude becomes very small as \( a \to 0 \). Yet in constructing the meridional velocity, which is the derivative of \( A_n \), this term dominates in the lower layer, yielding a rapid oscillation in the meridional velocity as shown in the third panel of Fig. 9a, and leads to significant meridional velocities in the stable region \( |x| > a \). A similar result occurs for \( |x| < a \) where one component has a large \( x \) wavenumber (see Fig. 3a), and this yields to a substantial meridional velocity for \( |x| < a \) even as \( |a| \to 0 \), with important consequences for the energy flux. For small \( a \) the asymptotic solution implies that \( \omega(k) \) has a double real root, and we see from the third and fourth panels of Fig. 9b that the wavenumbers of two long waves coalesce as \( a \to 0 \), as do their amplitudes, as shown in panels 5 and 6.

The ability of weakly unstable modes to radiate disturbance energy far from the mode's energy source suggests an importance for the weakly growing modes in spite of their slow growth. Small unstable intervals can support normal modes that can send disturbance energy over large distances and might excite other zones of instability, building global modes by a process of step-by-step instability and radiation.

The unstable normal modes have several potential sources of energy. There is the available potential energy associated with the vertical shear of the zonal flow, \( U \), which is present over the entire \( x \) interval. There is also the available potential energy associated with the meridional flow \( V \), limited, of course, to the interval \( |x| < a \). Finally, there is the energy available in the horizontal shear of the meridional flow, which is limited to the regions of abrupt change in \( V \) at \( x = \pm a \).

If the linearized potential vorticity equations (2.3a,b) are each multiplied by \( \phi_1 \) and \( \phi_2 \), respectively, and then integrated over a wavelength in \( y \) and over the entire \( x \) axis, we obtain

\[
\frac{\partial E}{\partial t} = V \int_0^{2\pi/l} dy \int_{-a}^a dx \phi_1 \frac{\partial \phi_2}{\partial y} + \int_0^{2\pi/l} dy \int_{-a}^a dx \phi_1 \frac{\partial \phi_2}{\partial x} + U \int_0^{2\pi/l} dy \int_{-a}^a dx \phi_1 \frac{\partial \phi_2}{\partial x} + \frac{U}{2} \int_0^{2\pi/l} \left[ \left( \frac{\partial \phi_1}{\partial x} \right)^2 + \left( \frac{\partial \phi_2}{\partial y} \right)^2 \right] dy,
\]

(4.1)

where \( E \) is the total perturbation energy, namely,

\[
E = \int_0^{2\pi/l} dy \int_{-\infty}^\infty dx \left[ \frac{1}{2} \left( \frac{\partial \phi_1}{\partial x} \right)^2 + \left( \frac{\partial \phi_2}{\partial y} \right)^2 + \left( \frac{\partial \phi_2}{\partial x} \right)^2 + \left( \frac{\partial \phi_1}{\partial y} \right)^2 \right].
\]

(4.2)

The third term in (4.1) involves the product of \( U \) with the jump in \( (\partial \phi_1/\partial x)^2 \) at \( x = a \) and \( x = -a \). If we use the notation

\[
\frac{\partial \phi_1^+}{\partial x} = \frac{\partial \phi_1}{\partial x} (a+) \quad \text{and} \quad \frac{\partial \phi_1^-}{\partial x} = \frac{\partial \phi_1}{\partial x} (a^-)
\]

at \( x = a \), for example, then the third term in (4.1) may be rewritten as

\[
\frac{\partial \phi_1^+}{\partial x} = \frac{\partial \phi_1}{\partial x} (a+), \quad \frac{\partial \phi_1^-}{\partial x} = \frac{\partial \phi_1}{\partial x} (a-)
\]

(4.3)
Fig. 9a. The eigenfunction (at \( t = 0 \)) for the case \( U = 1, V = 0.25, \beta = 1.25, \) and \( a = 0.05 \). The upper panel shows \( A_1 \) (solid) and \( 5 \times A_2 \) (dashed). The second panel shows the eigenstructure for small \( x \), for example, \( A_{11} \) (solid) and \( A_{21} \) (dashed). The third and fourth panels show \( v \) (solid) and \( 5 \times v_2 \) (dashed), while the lower panel shows \( u_1 \) and \( u_2 \) (dashed) for small \( x \).

\[
T = \frac{U}{2} \int_0^{2\pi/l} dy \left[ \left( \frac{\partial \phi_1^+}{\partial x} + \frac{\partial \phi_1^-}{\partial x} \right) \left( \frac{\partial \phi_1^+}{\partial x} - \frac{\partial \phi_1^-}{\partial x} \right) \right]_{x=a} \\
+ \left( \frac{\partial \phi_1^+}{\partial x} + \frac{\partial \phi_1^-}{\partial x} \right) \left( \frac{\partial \phi_1^+}{\partial x} - \frac{\partial \phi_1^-}{\partial x} \right) \right]_{x=0} \\
= \int_0^{2\pi/l} dy \left\{ \frac{1}{2} \left[ \left( \frac{\partial \phi_1^+}{\partial x} + \frac{\partial \phi_1^-}{\partial x} \right) \frac{\partial \phi_1}{\partial y} (-V) \right]_{x=a} \\
+ \frac{1}{2} \left[ \left( \frac{\partial \phi_1^+}{\partial x} + \frac{\partial \phi_1^-}{\partial x} \right) \frac{\partial \phi_1}{\partial y} V \right]_{x=0} \right\},
\]

if the matching condition (2.11) is used. This, in turn, can be rewritten as

\[
T = \int_0^{2\pi/l} dy \int_{-\infty}^{\infty} dx \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_1}{\partial y} \frac{\partial V}{\partial x}
\]

if \( \partial V/\partial x \) is interpreted as

\[
\frac{\partial V}{\partial x} = V(\delta(x + a) - \delta(x - a)),
\]

and \( \partial \phi_1/\partial x \) is written in the form of (2.9b) at both \( x = \pm a \).
The two baroclinic fluxes,

$$H_x = \int_0^{2\pi/x} dy \phi_1 \frac{\partial \phi_2}{\partial y}$$  \hspace{1cm} (4.5a)$$

and

$$H_y = \int_0^{2\pi/x} dy \phi_1 \frac{\partial \phi_2}{\partial x}.$$  \hspace{1cm} (4.6b)$$

are shown in Fig. 10 for the cases $a = 1, 3, 5,$ and $10$. In each case, the fluxes are contained within the unstable interval $|x| < a$. This is especially striking for the case of small $a$, for example, $a = 1$. Reference to Fig. 8b shows that the unstable mode has a significant amplitude out to $x \sim 30$ (i.e., 30 times the interval length of local instability), but the heat fluxes both remain localized within $|x| < a$. 

**Figure 9b.** The first two panels show the real and imaginary parts of the five wavenumbers comprising the outer solution as a function of $a$. One wavenumber marked by the open circle is much larger than the other. The third and fourth panels show the four small wavenumbers. Note the coalescence of two of these as $a \to 0$ to form the WKB solution. Panels 5 and 6 show the real and imaginary amplitudes of the corresponding plane wave components of the solution.
Although the zonal velocity is subcritical and unable by itself to support an energy-releasing instability, it does embody a source of available potential energy, and it is obvious from Fig. 9 that the heat flux in the meridional direction, $H_x$, will release that energy. The presence of $V$ not only produces a source of available potential energy in the meridional shear, but acts catalytically. The instability provoked by the presence of $V$ allows the release of energy present in the zonal flow. Figure 11 shows the ratio of the second and third terms on the right-hand side of (4.1) to the first term for values of $a$ between 1 and 15. The open circles represent the ratio of the transformation of the available potential energy of the zonal flow to that associated with the meridional flow. This ratio turns out to be always greater than one for the values of $U$, $V$, and $\beta$ of 1, 0.25, and 1.25, respectively. Thus, although the instability is only possible because of the presence of the meridional flow, the largest energy release to the perturbation is still the meridional density flux in the north–south density gradient. The stars in Fig. 11 represent the ratio of the third term on the right-hand side of (4.1), that is, the barotropic conversion, with respect to the first baroclinic conversion term. As we might anticipate from the results of section 2b, this conversion term is negligible for all $a$.

5. Summary and conclusions

We have presented a simple example of a zonal current with subcritical shear, $U$, which becomes unstable when it changes direction introducing a meridional flow, $V$, component over an interval $2a$. The instability is localized to the neighborhood of the direction change and exists for all $a \neq 0$. Calculations not presented here show that the instability persists for all $V/U$. We are
led to suggest that localized instabilities associated with the zonal acceleration of currents may involve locally provoked instabilities of this direction change type as well as the localization induced by zonally limited zones of enhanced baroclinicity described by the WKB theory of Pierrehumbert (1984). It certainly would be of interest to examine the stability properties of realistically configured jets that have regions of enhanced baroclinicity but remain subcritical considered as purely zonal flows. We intend to carry our investigations further in that direction.

The instabilities described in this study share many of the features of the study of Samelson and Pedlosky (1990). In addition to instability for all interval lengths \( a \), the realized normal mode can extend significantly beyond the source region of instability by the radiation of waves. For the westerly flow studied here, the radiated instability extends considerably downstream, and we believe this energy is the manifestation of stable Rossby waves energized by the instability. However, technical difficulties associated with the finite value of \( \text{Re}\omega \) as \( a \to 0 \) prevent us from producing an asymptotic theory that would firmly substantiate our intuition.

As in Samelson and Pedlosky, the energy flux terms are sharply contained within the source region where \( V \neq 0 \). Our calculations show that while it is the meridional shear that produces the instability, the normal mode is able, with greater efficiency, to release the available potential energy embodied in the shear of the zonal flow.

In the matching method utilized in this study, the meridional velocity \( V(x) \) has the form of a "top hat" profile. Were it present alone, Helmholtz shear instabilities on its edges at \( x = \pm a \) would be anticipated. A curious feature of the analysis shows that the presence of the zonal flow, \( U \), no matter how weak, which carries fluid elements across the shear layers in \( V \) at \( \pm a \), completely vitiates the barotropic shear instability. This is manifested by the change in the continuity conditions at \( x = \pm a \) produced by \( U \). Since \( U \) multiplies the highest \( x \) derivative in the governing equation, its neglect, leading to the classical Helmholtz shear instability, is a singular perturbation.

It would also be of interest to construct global normal modes in currents experiencing periodic direction changes to make contact with the earlier work of Gill (1974) on the instability of Rossby waves, in which instabilities can also exist regardless of the strength of the basic flow.

Acknowledgments. This work was supported in part by a grant from the National Science Foundation (ATM 89-03890).

REFERENCES


