

## The Slow Manifold of a Five-Mode Model

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### ABSTRACT

The slow manifold of an inviscid five-mode model introduced by Lorenz is investigated. When the influence of the gravity modes on the Rossby modes is neglected, the analytical solution given by Lorenz and Krishnamurthy is generalized. When gravity–Rossby coupling is included, direct numerical solutions are computed by solving a nonlinear boundary value problem. In all cases, the slow manifold has gravity mode oscillations that mimic free gravity waves and whose amplitude is proportional to the exponential of the reciprocal of the Rossby number  $\epsilon$ .

### 1. Introduction

The goal of initialization schemes for numerical forecasting is to adjust the flow onto the slow manifold, which is a hypothetical submanifold of parameter space of the model in which the flow evolves slowly on Rossby wave time scales. Without initialization, the forecast is degraded by relatively large amplitude gravity waves (Lynch 1992; Daley 1990). These unphysical high-frequency oscillations arise because observational errors are random and therefore unconstrained by geostrophic balance.

Operational initialization methods adequately suppress these spurious oscillations. In recent years, however, it has become clear that important theoretical issues about the slow manifold are still poorly understood (Boyd 1994a,b). Remarkably, strong evidence has accumulated that the slow manifold does not actually exist in the strict sense of a manifold that is *completely*, rather than merely *approximately*, free of high-frequency gravity mode oscillations.

Our goal is to shed light on these issues by explicitly computing the slow manifold of a simple model. Our hope is that through such idealized solutions, we may come to understand the true mathematical character of the slow manifold.

The simplest model that captures the mathematical essence of the slow manifold—and also shows its connection with nonlocal solitons—is that introduced by Lorenz (1986) and Lorenz and Krishnamurthy (1987). The model is obtained from a normal-mode expansion of the flow (“Hough” function series: Kasahara 1976,

1978) by truncating to just five modes: three Rossby modes and two gravity waves.

A similar philosophy of “maximum truncation” for convection gave the set of three equations now known as the Lorenz system, which is a star attraction of any text on chaos and dynamical systems theory. The five-mode model studied here plays a similar role in understanding the slow manifold. To distinguish this Rossby and gravity model from its better-known convective brethren, we shall dub the former the “LK quintet.”

After nondimensionalization and coordinate rescaling, which eliminate all explicit parameters except the nondimensional parameters shown below, the model is the system of five ordinary differential equations

$$\begin{array}{l}
 U_t = -VW \\
 \quad + b'Vz - aU \\
 V_t = UW \\
 \quad - b'Uz - aV + F \\
 W_t = -UV - aW \\
 x_t = -z - ax \\
 z_t = bUV + x - az
 \end{array}
 \left. \begin{array}{l}
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 \\
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 \\
 \end{array} \right\} \text{Rossby triad} \\
 \left. \begin{array}{l}
 \\
 \\
 \end{array} \right\} \text{Gravity dyad}
 \end{array} \right\} \text{LK Quintet,} \quad (1.1)$$

where  $a$  is the damping coefficient,  $b$  and  $b'$  are nonlinear coupling coefficients, and  $F$  is the forcing, a constant independent of time. We denote the Rossby amplitudes by uppercase letters ( $U, V, W$ ) and the gravity mode amplitudes, which are much smaller, by lowercase ( $x, z$ ). Although (1.1) is derived from fluid mechanics, the unknowns are the *amplitudes* of a generalized Fourier series, not coordinates or velocities.

The approximation of zero forcing and dissipation,  $a = F = 0$ , gives

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$$\left. \begin{aligned}
 U_t &= -VW + b'Vz \\
 V_t &= UW - b'Uz \\
 W_t &= -UV \\
 x_t &= -z \\
 z_t &= bUV + x
 \end{aligned} \right\} \begin{array}{l} \text{Rossby triad} \\ \text{Gravity dyad} \end{array} \left. \vphantom{\begin{aligned} U_t \\ V_t \\ W_t \\ x_t \\ z_t \end{aligned}} \right\} \text{Inviscid LK quintet.} \tag{1.2}$$

The gravity dyad can be simplified, without approximation, to a single second-order equation.

Although physically  $b = b'$ , the terms that couple gravity wave and Rossby wave in the Rossby triad are small in comparison to the other terms. A useful and reasonable approximation is therefore  $b' = 0$ . The inviscid quintet is then “maximally simplified” to the Rossby triad, *uncoupled* from the gravity waves, plus a second-order *linear* equation for a gravity mode that is forced by the Rossby waves:

$$\left. \begin{aligned}
 U_t &= -VW \\
 V_t &= UW \\
 W_t &= -UV \\
 z_{tt} + z &= b(UV)_t
 \end{aligned} \right\} \begin{array}{l} \text{Rossby triad} \\ \text{second-order gravity} \end{array} \left. \vphantom{\begin{aligned} U_t \\ V_t \\ W_t \\ z_{tt} + z \end{aligned}} \right\} \text{Simplified inviscid LK quintet.} \tag{1.3}$$

Lorenz and Krishnamurthy (1987) observed that the Rossby triad has an exact solution:

$$U = \epsilon \operatorname{sech}(\epsilon t), \quad V = \epsilon \tanh(\epsilon t), \quad W = -\epsilon \operatorname{sech}(\epsilon t), \tag{1.4}$$

where  $\epsilon$  is an arbitrary constant, the Rossby number. In practical applications,  $\epsilon \ll 1$  so that the time scale for the Rossby dynamics is a “slow” scale in comparison to the unit oscillation frequency of the gravity waves. The three modes have a single episode of transience and interaction, centered on  $t = 0$ , and then settle back into steady values for both large positive and large negative time, as shown in Fig. 1a.

The gravity wave dyad is then

$$z_{tt} + z = -b\epsilon^3 \{ 2 \operatorname{sech}^3(\epsilon t) - \operatorname{sech}(\epsilon t) \}. \tag{1.5}$$

This same sort of equation describes the so-called far field of nonlocal solitary wave except that the coordinate is space rather than time (Boyd 1989a, 1994a). The crucial point is that the asymptotic form of a particular solution to (1.5) is

$$z(t) \sim -b\pi \exp\left(-\frac{\pi}{2\epsilon}|t|\right) \sin(|t|), \quad |t| \rightarrow \infty \tag{1.6}$$

(Fig. 1b). By adding multiples of the homogeneous solutions,  $\cos(t)$  and  $\sin(t)$ , we can modify the phase of the oscillation and even completely suppress it for

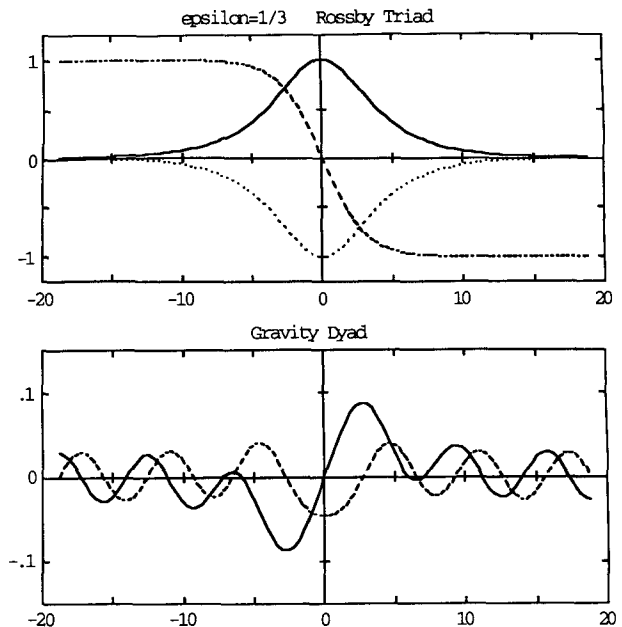


FIG. 1. (Top) Rossby wave amplitudes for the Lorenz–Krishnamurthy solutions with  $\epsilon = 1/3$  and  $b = 1$ . Solid: U. Dashed: V. Dotted: W. (Bottom) Gravity wave amplitudes. Solid:  $x(t)$ . Dashed:  $z(t)$ .

either large positive or large negative time—but not both.

To put it another way, the interaction of the three modes in the Rossby triad has an effect on the gravity waves that is *nonlocal in time*. The gravity modes oscillate forever even when the Rossby modes have become steady.

Following the usage of nonlocal soliton theory (Boyd 1994a), we will denote the amplitude of the nonlocal gravitational oscillations by  $\alpha(\epsilon)$ . The small parameter  $\epsilon$  is the Rossby number and measures the ratio of gravity wave time scales ( $O[1]$ ) to the Rossby time scale,  $O(1/\epsilon)$ . A crucial point is that  $\alpha$  is proportional to the exponential of the inverse of  $\epsilon$ , that is  $\exp(-\pi/(2\epsilon))$ .

Because this function is not an analytic function of  $\epsilon$  but rather has an essential singularity at  $\epsilon = 0$ , the Baer and Tribbia (1977) initialization series diverges for all  $\epsilon$ . However, the  $\epsilon$  series is *asymptotic* as  $\epsilon \rightarrow 0$ . (This expansion is a power series in  $\epsilon$  derived by the singular perturbation scheme known as the method of multiple scales.) For fixed  $\epsilon$ , the terms of the series first diminish with degree and then begin an inexorable rise. If we truncate the series with the smallest term (for a given  $\epsilon$ ), the so-called optimal truncation, then we may obtain a useful answer. It turns out, as shown in Boyd (1994a), that the error in the optimally truncated Baer–Tribbia series and related initialization schemes is  $O(\alpha)$ . Because this is tiny for reasonable values of the Rossby number,  $\epsilon \approx 1/10$ , one may obtain excellent

initializations by using a few terms of the series (or equivalent iterations), as is now done with great success in operational forecasting.

Unfortunately, our understanding of the reasons for this success is poised on a slender reed. Equations (1.4)–(1.6) do not give the *general* solution to (1.3), but only represent a very special portion of the slow manifold in which the Rossby modes have but a single episode of interaction throughout all eternity. The real atmosphere, of course, is recurrent and Rossby modes rise and fall in mutual interaction again and again. In the next section, we shall analytically generalize the Lorenz and Krishnamurthy solution.

**2. The general analytical solution to the simplified Lorenz–Krishnamurthy quintet**

The special one-parameter family of solutions found by Lorenz and Krishnamurthy (1987) can be extended in a generalization presented here for the first time.

*Theorem 1. General solution to the simplified Lorenz–Krishnamurthy quintet*

The set of ordinary differential equations (2.3):

$$\left. \begin{aligned} U_t &= -VW \\ V_t &= UW \\ W_t &= -UV \end{aligned} \right\} \text{Rossby triad} \left. \begin{aligned} & \\ & \\ & \end{aligned} \right\} \text{Simplified inviscid LK quintet}$$

$$z_u + z = b(UV)_t \text{ 2d order gravity}$$

$$x_t = -z \tag{1.3 bis}$$

has the general five-parameter family of solutions where the parameters are the set  $(\epsilon, L, \varphi, \gamma, \delta)$  where  $\epsilon$  is the “pseudowavenumber,”  $L$  is the “half-period” in time,  $\varphi$  is the “phase shift,” and  $\gamma$  and  $\delta$  are the “seiche amplitudes”:

$$U = \epsilon \sum_{m=-\infty}^{\infty} \text{sech}(\epsilon[\tau - mL]) \tag{2.1a}$$

$$V = -\epsilon \sum_{m=-\infty}^{\infty} (-1)^m \tanh(\epsilon[\tau - mL]) \tag{2.1b}$$

$$W = -\epsilon \sum_{m=-\infty}^{\infty} (-1)^m \text{sech}(\epsilon[\tau - mL]) \tag{2.1c}$$

$$x = -b \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1) \frac{\pi}{L} - \frac{L}{\pi(2n+1)}} \times \text{sech}\left(\left(n + \frac{1}{2}\right) \frac{\pi^2}{\epsilon L} \tau\right) \sin\left((2n+1) \frac{\pi}{L} \tau\right) - \gamma \sin(\tau) + \delta \cos(\tau) \tag{2.1d}$$

$$z = b \sum_{n=-\infty}^{\infty} \frac{1}{1 - \frac{L^2}{\pi^2(2n+1)^2}} \text{sech}\left(\left(n + \frac{1}{2}\right) \frac{\pi^2}{\epsilon L} \tau\right) \times \cos\left((2n+1) \frac{\pi}{L} \tau\right) + \gamma \cos(\tau) + \delta \sin(\tau), \tag{2.1e}$$

where

$$\tau \equiv t + \varphi. \tag{2.1f}$$

*Proof:* The general solution to the resonant triad equations is given in Boyd (1991c, section 14). To compute the gravity wave components, note that the third triad equation implies that the second-order equation for  $z(t)$  can be written as

$$z_u + z = -bW_u. \tag{2.2}$$

Fortunately, each of the elliptic functions has an analytical Fourier series as well as the so-called imbricate series (2.1a–c):

$$U = \frac{\pi}{L} \left\{ 1 + 2 \sum_{n=1}^{\infty} \text{sech}\left(n \frac{\pi^2}{\epsilon L}\right) \cos\left(\frac{2n\pi}{L} \tau\right) \right\} \tag{2.3a}$$

$$V = -\frac{\pi}{L} \sum_{n=-\infty}^{\infty} \text{cosech}\left(\left(n + \frac{1}{2}\right) \frac{\pi^2}{\epsilon L}\right) \times \sin\left(\frac{(2n+1)\pi}{L} \tau\right) \tag{2.3b}$$

$$W = -\frac{\pi}{L} \sum_{n=-\infty}^{\infty} \text{sech}\left(\left(n + \frac{1}{2}\right) \frac{\pi^2}{\epsilon L}\right) \times \cos\left(\frac{(2n+1)\pi}{L} \tau\right). \tag{2.3c}$$

The derivation of these amazing series and the general theory of imbricate series are explained in Boyd (1984, 1989a).

Substituting the Fourier series for  $W(t)$  into (2.2), assuming a similar cosine series for  $z(t)$ , and matching terms gives (2.1e) (after adding in the homogeneous solutions,  $\sin(\tau)$  and  $\cos(\tau)$ , with arbitrary coefficients). Substituting the  $z$  series into  $x_t = -z$  gives (2.1d). *Q.E.D.*

Although the general solution to a fifth-order system without explicit parameters must contain five independent degrees of freedom, the parameter  $\varphi$  is a trivial phase shift that merely reflects the translational invariance of (1.3). Similarly,  $\gamma$  and  $\delta$  are the amplitudes of free gravity wave oscillations, and the goal of every good initialization scheme is to set  $\gamma = \delta = 0$ . The two important degrees of freedom are the pseudowavenumber  $\epsilon$  and the half-period  $L$ . The accomplishment of theorem 1 is to increase the number of nontrivial parameters from one to two; the solutions of Lorenz and

Krishnamurthy (1987) are the special case that the half-period  $L \Rightarrow \infty$ .

Lorenz (1986) observed that the Rossby triad, for  $b' = 0$ , has a general solution in terms of elliptic functions, which are always spatially periodic. However, the imbricate series (2.1) are much more illuminating than the bare symbols “cn,” “sn,” and “dn,” and have not been previously applied in this context.

One striking feature of (2.1) is that the slow manifold, which is the three-parameter special case  $\gamma = \delta = 0$ , is periodic in time. The general five-parameter solution, which allows for the superposition of free gravity waves on top of the slow manifold, is quasi-periodic with two independent periods: one is that of the unforced gravitational oscillations ( $=2\pi$ ) and the other is the period of the Rossby–Rossby interactions,  $2L$ .

The second striking feature is that the slow manifold solutions are similar to the limiting case of Lorenz and Krishnamurthy except that the Rossby wave interactions are now repeated periodically every  $2L$  time units instead of happening just once, as illustrated in Fig. 2. There are small gravitational oscillations for all  $t$ , and these dominate the solution for the gravitational modes in the otherwise quiescent periods between Rossby mode interactions.

Figure 2 is powerful evidence for the assertion that temporal periodicity has left unaltered the qualitative nature of the solutions from the limiting solution found by Lorenz and Krishnamurthy (1987). Further arguments are given in Boyd (1991a, 1994a,b). These sources also show, using methods from the theory of weakly nonlocal solitary waves and their periodic generalizations (Boyd 1989a, 1991b), that the amplitude of these between-Rossby-peaks gravitational oscillations is *smallest* for  $L = n\pi$ ,  $n$  even. For other values of  $L$ , the stronger near-resonance between the free gravity frequency (one) and the nearest component of the Fourier series (2.1d,e) produces even larger oscillations in  $x(t)$  and  $z(t)$ . The generalization from infinite period to finite temporal period does not alter the physics.

### 3. Direct numerical solutions: Solving the nonlinear boundary value problem

The analytic solution of the previous section still has two limitations even in the small universe of the five-mode model: (i)  $b' = 0$ , which means that the gravitational modes are passively forced by the Rossby waves with no way to feedback or modify the Rossby waves, and (ii) neglect of forcing and dissipation. We shall discuss the latter in the next section. The goal here is to allow for nonzero  $b'$  by solving (1.2), the full inviscid LK quintet, by numerically solving a nonlinear boundary value problem.

The critical assumption we shall make is that the slow manifold solution is simply periodic in time, just

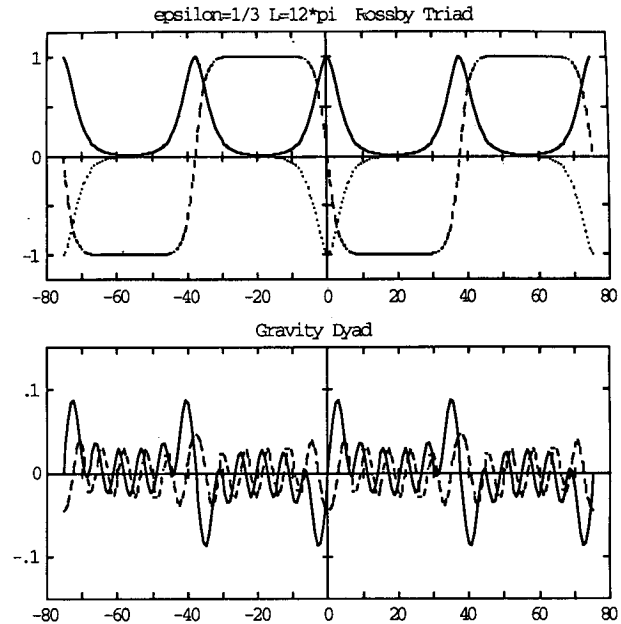


FIG. 2. Same as Fig. 1 except that the solution is temporally periodic with a half-period  $L = 12\pi$ .

as for  $b' = 0$ . The general solution with  $b' \neq 0$  still depends upon five parameters, excluding  $b'$  itself. The translational degree of freedom can be eliminated by the further assumption that each unknown has definite parity (symmetric or antisymmetric) with respect to some point in time that we shall choose as the origin. A second degree of freedom may be taken to be the temporal half-period  $L$ . The third degree of freedom will be chosen to be  $U(0)$ . Although the solution for  $U(t)$  is no longer given exactly by the imbricate series (2.1a), we can still define a parameter  $\epsilon'$  in terms of  $U(0)$  through the implicit relation

$$U(0) = \epsilon' \sum_{m=-\infty}^{\infty} \operatorname{sech}(\epsilon' mL) \quad (3.1)$$

so as to compare the numerical solutions more easily with the analytical solutions for  $b' = 0$ .

The two remaining degrees of freedom are  $x(0)$  and  $z(0)$ . These will be varied by our boundary value solver to force the solution to have the property we associate with the slow manifold: periodicity in time.

Two warnings are needed. First, although we always found such periodic solutions for the inviscid LK quintet, the manifolds of more realistic models are not periodic and indeed are probably chaotic. Second, periodicity defines a manifold that we may reasonably dub “slow” in the sense that oscillations with unit frequency, the frequency of free gravity waves, are suppressed. Periodicity, however, merely defines a three-parameter manifold; it does not prove that this is equiv-

alent to the manifolds computed by alternative initialization schemes reviewed in Boyd (1994a,b).

The quintet can be reduced to a system of three equations in the Rossby amplitudes only by using (2.3),  $z_{tt} + z = -bW_{tt}$ , and introducing the operator

$$Q \equiv \left( \frac{d^2}{dt^2} + 1 \right)^{-1} \frac{d^2}{dt^2}. \quad (3.3)$$

The unknowns are approximated by truncated Fourier series with assumed time period  $2L$  and assumed symmetries:

$$\begin{aligned} U(t) &\approx \sum_{j=1}^{N+1} a_j \cos(2[j-1]\pi t/L), \\ V(t) &\approx \sum_{j=1}^N a_{j+N+1} \sin([2j-1]\pi t/L), \\ W(t) &\approx \sum_{j=1}^N a_{j+2N+1} \cos([2j-1]\pi t/L). \end{aligned} \quad (3.4)$$

It is trivial to show that

$$\begin{aligned} QW(t) &\approx \sum_{j=1}^N \left\{ \frac{1}{1 - L^2/([2j-1]^2\pi^2)} \right\} \\ &\quad \times a_{j+2N+1} \cos([2j-1]\pi t/L). \end{aligned} \quad (3.5)$$

Similarly the Fourier series for the gravity modes are given by the term-by-term solution of

$$z = -bQW, \quad x_t = -z. \quad (3.6)$$

Newton's method and continuation in  $b^2$  are used to solve the system of  $(3N+1)$  nonlinear algebraic equations consisting of

$$\begin{aligned} r_1(t_i; a_1, \dots, a_{3N+1}) &= 0, \quad i = 1, \dots, N \\ r_2(t_i; a_1, \dots, a_{3N+1}) &= 0, \quad i = 1, \dots, N \\ r_3(t_i; a_1, \dots, a_{3N+1}) &= 0, \quad i = 1, \dots, N \\ r_0(a_1, \dots, a_{3N+1}) &\equiv U(0) - \sum_{j=1}^{N+1} a_j = 0, \end{aligned} \quad (3.7)$$

where the last condition,  $r_0 = 0$ , ensures that  $U(t)$  has the specified value at  $t = 0$  and where the points of the pseudospectral collocation grid (Boyd 1989a) are

$$t_i = L(2i-1)/(4N), \quad (3.8)$$

and where the residuals of the three Rossby mode equations are (with  $b = b'$ )

$$\begin{aligned} r_1 &\equiv U_t + V(W + b^2QW) \\ r_2 &\equiv V_t - U(W + b^2QW) \\ r_3 &\equiv W_t + UV. \end{aligned} \quad (3.9)$$

We found that the solution for  $b' = 0$  was a good first

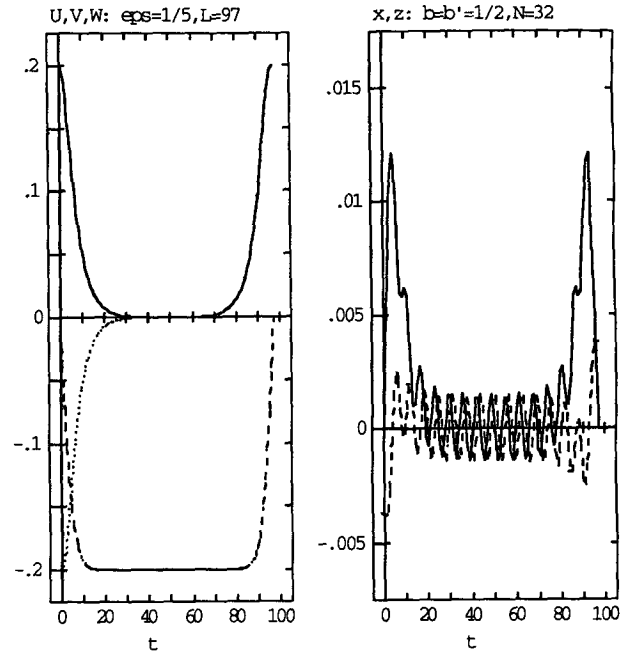


FIG. 3. Numerical solution of the full inviscid LK quintet (1.2),  $b' = b$ , for  $b = 1/2$ ,  $\epsilon = 1/5$ , and a temporal half-period  $L = 97$ . Left panel:  $U$  (solid),  $V$  (dashed), and  $W$  (dotted). Right panel:  $x$  (solid) and  $z$  (dashed).

guess for the solution with nonzero  $b'$ . Figure 3 shows the fields for a typical solution.

An important question is: How much does the solution with  $b' \neq 0$  differ from the analytical  $b' = 0$  solution of the previous section? The answer, for reasonable parameter values, is: Not much. Figure 4 compares the absolute values of the Fourier coefficients for  $b' = 0$  and  $b' = 1/2$ . The solid and dashed curves are almost indistinguishable except near the tails; the dotted curve, which is the difference, has a maximum value which is smaller than the largest Fourier coefficients by more than a factor of 1000.

The changes are  $O(\alpha)$  where  $\alpha$  is the amplitude of the oscillations of unit frequency. Figure 5 shows that  $z(t)$ , by far the smallest of the five unknowns, is changed by a factor of 2 by the change in  $b'$ . The absolute magnitude of the alterations in the other unknowns is comparable, but because  $U$ ,  $V$ ,  $W$ , and  $x$  are all much larger than  $z$ , the relative change in those other fields is very small.

#### 4. A comparison with Lorenz (1986) and Lorenz and Krishnamurthy (1987)

Lorenz (1986) showed that the Rossby triad could, when  $b' = 0$ , be solved in terms of elliptic functions. This implies that the Rossby modes are periodic in time and have poles in the complex  $t$  plane. The imbricate

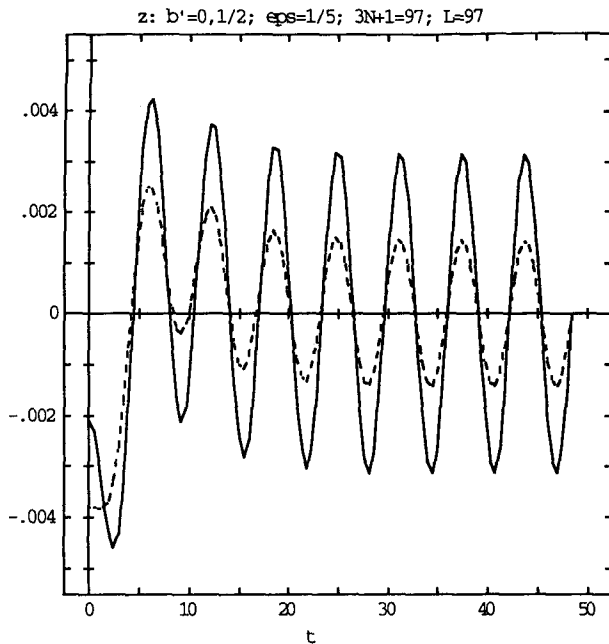


FIG. 4. A comparison of  $z(t)$  for two otherwise identical runs with  $b' = 0$  (solid) and  $b' = 1/2$  (dashed);  $\epsilon = 1/5$ ,  $L = 97$ , and  $b = 1/2$ .

series given here explicitly show both the periodicity and the poles, and make it easy to see that the general solution is merely the periodic repetition of Rossby interactions, each described by the same sech and tanh functions as in the limiting special solution (1.4) given by Lorenz and Krishnamurthy (1987).

When  $b' \neq 0$ , Lorenz and Krishnamurthy use a *different* definition of the slow manifold from ours. For the inviscid, unforced system (1.2), we define the slow manifold by its periodicity in time. Lorenz and Krishnamurthy, however, concluded that with forcing and damping, the slow manifold must pass through "point H," the steady solution which is the model's analog of a steady Hadley circulation in the real atmosphere. With forcing and damping neglected, they still chose their slow manifold to emerge from point H, thus defining their manifold not through temporal periodicity but rather through the initial conditions:  $V(0) = \epsilon$ ,  $U(0) = W(0) = x(0) = z(0)$ .

As illustrated in their Fig. 7, their manifold is non-periodic. However, the overall level of oscillations of unit frequency for their  $F = 0.195$  (equivalent to  $\epsilon = 0.195$ ) is comparable to that in our Figs. 3 to 5, for  $\epsilon = 0.2$ .

One important conclusion, stressed in Lorenz (1992) and Boyd (1994a,b) using different examples, is that the slow manifold is not unique. Rather, there are multiple definitions that lead to different manifolds. These manifolds are similar, in the sense of sharing the same initial values for the slow Rossby variables and also in

the sense that high-frequency oscillations are small. However, these various manifolds are not identical because the initial values for the fast variables are different. Sometimes the mathematical properties—here the periodicity of our manifold versus the aperiodicity of Lorenz and Krishnamurthy's—may differ, too.

Even so, most of the vast literature on the subject refers to *the* slow manifold. We have, too, and this is reasonable as long as "the" is linked with a precise definition, such as the property of temporal periodicity. But, as illustrated in Boyd (1994b), any such well-defined manifold is not an "only child," but rather has a large number of siblings.

### 5. The quintet with damping and forcing

Lorenz and Krishnamurthy explored their model thoroughly for nonzero  $F$  and  $a$ . The crucial point is that for all  $F > F_c$  where  $F_c$  is the critical forcing for a pitchfork bifurcation, there are three steady solutions. The Hadley-like solutions ( $V = F$ , all other components zero) is unstable. The other two solutions are steady Rossby waves with all the variables equal to constants. Lorenz and Krishnamurthy assert: "They [the Rossby waves] appear to be stable for all positive values of  $b$ ,  $a$ , and  $F$  so that almost all points are attracted to one or the other of them, but . . . we shall not seek a proof."

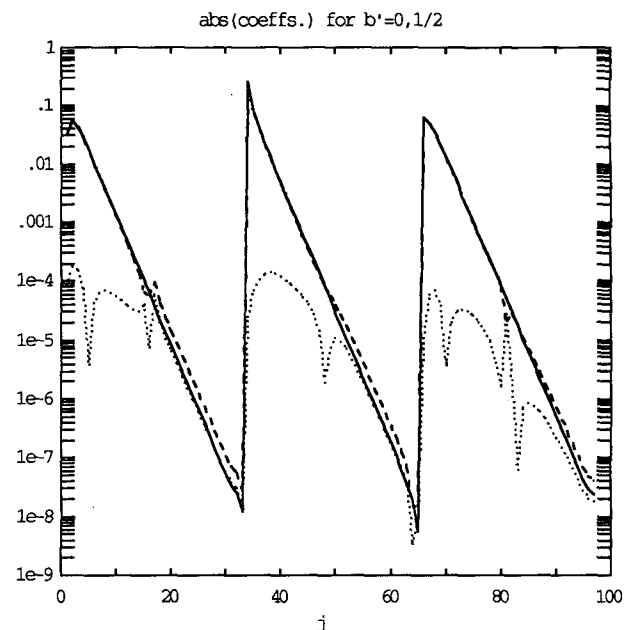


FIG. 5. Graph of  $|a_j|$  versus  $j$  for two cases:  $b' = 0$  (solid) and  $b' = 1/2$  (dashed). The dotted curve is the difference between the solid and dashed curves. The left third, middle third, and right third show the coefficients for  $U$ ,  $V$ , and  $W$ , respectively, for a numerical solution of (1.2) with  $b = 1/2$ ,  $\epsilon = 1/5$ , and  $L = 97$ .

Thus, the asymptotic solution, except perhaps in limited regions in parameter space missed by these authors, is always independent of time. This is both good and bad.

The bad is that real atmospheric flows are chaotic rather than steady. In this respect, the quintet with forcing and damping is a bad and unfaithful model.

The good is that Lorenz and Krishnamurthy show that as the flow evolves towards these steady states, the stately and slow evolution of the Rossby modes is accompanied by small oscillations of roughly unit frequency. No choice of initial conditions (other than the steady states!) is successful in eliminating these oscillations.

In models that do have chaotic attractors, experiments such as Fig. 11 of Vautard and Legras (1986) suggest that the attractor itself may have gravitational oscillations. The LK quintet studies, as pointed out by Lorenz and Krishnamurthy themselves, have shown that such oscillations are an essential part of the dynamics off the attractor, too.

An inviscid model, such as we solve here, can say nothing directly about the structure of the attractor of a forced and damped system. When the damping is weak, however, the off-the-attractor dynamics is inviscid to a first approximation, and then inviscid solutions are useful.

This in turn sheds light, albeit indirectly, on a chaotic attractor because the essential point about such an attractor is that it is never steady. Rather, the flow slowly moves toward or away from the stationary points on the long time scale of the forcing and damping (assumed weak). On a shorter time scale, the dynamics on the attractor is inviscid.

## 6. Summary

In this work, we have made a couple of useful extensions to the five-mode model of the slow manifold. First, we derived the analytical general solution to the undamped model with  $b' = 0$ , that is, with the Rossby dynamics unaffected by the gravity modes, in terms of infinite series of hyperbolic functions. These series almost trivially show the close connection to the special solution derived in Lorenz and Krishnamurthy (1987). For both the special and general solutions, there are gravitational oscillations with an amplitude proportional to  $\exp(-\pi/2\epsilon)$  where  $\epsilon$  is the Rossby number, that is, the maximum amplitude of the Rossby modes.

The second extension is to numerically compute a temporally periodic solution when  $b' \neq 0$  so that the gravity mode amplitudes modify the evolution of the Rossby modes. Differences from the solutions with  $b' = 0$  are small. It is interesting, however, that for arbitrary initial values of the three Rossby modes, there is always a temporally periodic solution. ("Always"

means for all parameter values we explored including cases not explicitly described in the paper; an open problem is to rigorously *prove* this.) The periodic solution qualifies as a slow manifold because the *general* solution to the inviscid five-mode model, with the gravity modes allowed to assume arbitrary initial values, is quasi-periodic with *two independent* temporal periods, the second being the  $2\pi$  period of free gravity waves.

Another striking conclusion is that the mathematics of the periodic slow manifold for the five-mode model is identical with that of the weakly nonlocal solitary waves ("nanopterons") and their periodic generalizations ("nanopteroidal waves"), which arise in a wide variety of physical contexts (Boyd 1994b).

The probability is that for more complicated models, as for the Lorenz and Krishnamurthy quintet studied here, there is a slow manifold. In fact, there are likely to be a lot of manifolds that can be given precise mathematical definitions and merit the label "slow." However, the slow manifold is a manifold with gravitational "wings." Perhaps it would be more truthful to say that for the quintet, we have computed a "slow, with an inescapable minimum of fast, manifold."

The challenge for more realistic models is to characterize the structure of their slowest manifolds. These, too, are likely to resemble the classical slow manifold except for a dressing of gravity waves whose amplitude is an exponential function of the reciprocal of the Rossby number.

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