# NOTES AND CORRESPONDENCE

# On the Inclusion of Compressibility Effects in the Scorer Parameter

LOUISA B. NANCE

*Department of Atmospheric Sciences, University of Washington, Seattle, Washington*

13 September 1995 and 24 June 1996

## 1. Introduction

The vertical structure of steady, two-dimensional, linear mountain waves depends on the Scorer parameter profile of the background flow. The Scorer parameter for a compressible fluid is a complex expression that depends on the static stability $N$, wind speed $\overline{u}$, vertical wind shear, and curvature in the wind speed profile of the background flow. In the past, instrument limitations made it difficult to obtain upstream profiles of temperature and wind speed at vertical resolutions that would merit considering the compressibility or curvature terms in the Scorer parameter expression. Consequently, early studies of observed mountain waves used only the leading-order term in the Scorer parameter expression ($N^2/\overline{u}^2$) to represent the propagation characteristics of the background flow. Recent technological advances in observational instrumentation and computer hardware have made the consideration of these terms more feasible. On the other hand, using the full vertical structure equation for a compressible fluid to analyze observed mountain waves would still be somewhat cumbersome.

Over the years, a number of differential equations similar to the full vertical structure equation for a compressible fluid have appeared in the mountain wave literature. These differential equations are the products of simplifications that make analytical treatment of topographic disturbances in a compressible fluid more manageable, but the full implications of these simplifications are not always clear, making it difficult to determine which simplified vertical structure equation will produce the most accurate linear solution. More importantly, two of these simplified vertical structure equations are identical except for the signs of two terms in their Scorer parameter expressions, so the simplifications associated with at least one of these vertical structure equations must be flawed. This note briefly reviews the approximations associated with each simplified equation, discusses their relationship to standard filtered systems of equations, and evaluates the performance of each simplified vertical structure equation by determining the error introduced by each approximation in the calculation of the horizontal wavelength of trapped mountain lee waves. Resonant wavelength calculations were chosen because linear trapped mountain lee wave solutions can be rather sensitive to small changes in the characteristics of the background flow (Corby and Wallington 1956; Pearce and White 1967). Such a sensitivity suggests that even rather small differences between the full vertical structure equation for a compressible fluid and the simplified vertical structure equations could lead to sizable discrepancies between their trapped lee wave solutions, which, in turn, would highlight the differences between the various simplified vertical structure equations.

## 2. Compressible vertical structure equation

Queney et al. (1960) showed that the two-dimensional, steady-state equations for an inviscid, compressible fluid ($f = 0$) linearized about a horizontally uniform, hydrostatically balanced basic state with a mean horizontal wind can be combined to form the second-order partial differential equation

$$\overline{M}\tilde{w}_{xx} + \tilde{w}_{zz} + l^2\tilde{w} = 0, \tag{1}$$

where

$$\tilde{w} = \left(\frac{M_0}{\overline{M}}\frac{\overline{\rho}}{\rho_0}\right)^{1/2} w, \tag{2}$$

$$\overline{M} = 1 - \frac{\overline{u}^2}{\overline{c}^2}, \qquad \overline{\beta} = \frac{d}{dz}[\ln(\overline{M}\overline{\theta})],$$

$$\overline{S} = \frac{d}{dz}\left[\ln\left(\frac{\overline{M}}{\overline{\rho}}\right)\right], \qquad \overline{c}^2 = \frac{c_p}{c_v}R\overline{T}, \tag{3}$$

and

*Corresponding author address*: Dr. Louisa B. Nance, National Weather Service. 7600 Sand Point Way NE Seattle, WA 98115.
E-mail: nance@seawfo.noaa.gov

$$l^2 = \frac{\bar{\beta}g}{\bar{u}^2} + \frac{\bar{S}\bar{u}_z}{\bar{u}} - \frac{1}{4}\bar{S}^2 + \frac{1}{2}\bar{S}_z - \frac{\bar{u}_{zz}}{\bar{u}}. \qquad (4)$$

In the above, $w$ represents vertical velocity, $\rho$ represents density, $\theta$ represents potential temperature, $\rho_0$ and $M_0$ represent a reference state, $c$ is the speed of sound, $g$ is the gravitational constant, overbars depict horizontally homogeneous basic-state values and spatial coordinates used as subscripts indicate differentiation with respect to that coordinate.

The parameter $l^2$, which is a function of $z$ only, is the Scorer parameter for a compressible fluid. Fourier decomposition of its horizontal structure reduces (1) to the second-order ordinary differential equation

$$\hat{w}_{zz} + (l^2 - \bar{M}k^2)\hat{w} = 0, \qquad (5)$$

where $k$ represents the horizontal wavenumber of the disturbance and $\hat{w}(k, z)$ represents the amplitude of the $k$th component of the Fourier decomposition of $\tilde{w}(x,z)$. When this equation is applied to flow over topography, one finds that the vertical structure of steady, two-dimensional, linear mountain waves is determined by the relationship between $l^2$ and $\bar{M}k^2$. Except for the special case of an isothermal, constant-wind-speed basic state, both $\bar{M}$ and $l^2$ are functions of $z$, making it difficult to solve (5) for realistic atmospheric profiles in its unapproximated form.

## 3. Approximate vertical structure equations

All of the simplified vertical structure equations found in the mountain wave literature use the fact that the cross-mountain flow $\bar{u}$ is typically much less than the speed of sound in the undisturbed fluid $\bar{c}$ to rewrite (5) as

$$\hat{w}_{zz} + (l^2 - k^2)\hat{w} = 0, \qquad (6)$$

where $\hat{w}$ now represents the amplitude of the $k$th component of the Fourier decomposition of the density-weighted vertical velocity:

$$\tilde{w} = \left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2} w. \qquad (7)$$

The various simplified differential equations differ only with respect to their definitions of $l^2$.

### *a. Boussinesq approximation*

The most widely used expression for $l^2$, which was originally put forth by Scorer (1949), is

$$l_b^2 = \frac{N^2}{\bar{u}^2} - \frac{\bar{u}_{zz}}{\bar{u}}, \qquad (8)$$

where $N^2 = gd\ln\bar{\theta}/dz$ (e.g., Pearce and White 1967; Reynolds et al. 1968; Starr and Browning 1972; Brown 1983; Cox 1986; Mitchell et al. 1990). The parameter $l_b^2$, which neglects all compressibility effects, also appears in the vertical structure equation describing steady, two-dimensional, linear internal gravity waves in a Boussinesq fluid. In fact, the only difference between (6) with $l^2$ defined by $l_b^2$ and the Boussinesq vertical structure equation is that $\hat{w}(k, z)$ in the Boussinesq version is defined in terms of the vertical velocity $w$ instead of the density-weighted vertical velocity $\tilde{w}$.

The impact of compressibility effects on trapped lee wave solutions was evaluated by comparing the wavelengths of stationary trapped lee waves described by (5) and the Boussinesq version of (6) for a family of idealized basic-state profiles. Since trapped lee waves commonly occur when the upstream tropospheric profile has an elevated stable layer and the mean wind speed increases with height, the profiles considered in this study were based on the following configuration:

$$N(z) = \begin{cases} 0.01\mathrm{s}^{-1} & 0 \le z \le 1 \text{ km} \\ N_s & 1 \text{ km} < z \le 2 \text{ km} \\ 0.01\mathrm{s}^{-1} & 2 \text{ km} < z \le 12 \text{ km} \\ g/(c_p\bar{T}(12 \text{ km}))^{1/2} & 12 \text{ km} < z \le 15 \text{ km}, \end{cases}$$

$$\bar{u}(z) = \begin{cases} 10\mathrm{ms}^{-1} + \sigma z & 0 \le z \le 12 \text{ km} \\ \bar{u}(12 \text{ km}) & 12 \text{ km} \le z \le 15 \text{ km}. \end{cases}$$

In the above, $N_s$ represents the buoyancy frequency in an elevated stable layer and $\sigma$ represents the vertical shear in the mean tropospheric wind profile. The isothermal, constant-wind-speed layer above 12 km represents a stratospheric layer. Resonant wavelengths were computed for values of $N_s$ ranging from 0.010 to 0.025 $\mathrm{s}^{-1}$ and values of $\sigma$ ranging from 0 to 0.005 $\mathrm{s}^{-1}$. The numerical technique used to find these resonant wavelengths is described in the appendix. These variations in the strength of the elevated stable layer and the tropospheric wind shear are based on a survey of upstream profiles linked to the formation of observed trapped lee waves (e.g., Smith 1976; Reynolds et al. 1968; Brown 1983; Shutts 1992).

The errors introduced by assuming that $\bar{M} \approx 1$ in (5) and replacing $l^2$ with $l_b^2$ are rather small when the mean wind profile is constant with height, but the errors introduced by these simplifications increase as the wind shear increases (see Fig. 1). In fact, the resonant wavelength for the simplified equation is off by almost 30% when the elevated stable layer is absent and $\sigma = 0.005$ $\mathrm{s}^{-1}$ (see Table 2). These computations show that neglecting all compressibility effects in the expression for $l^2$ will have a significant impact on the accuracy of trapped lee wave solutions when the upstream tropospheric profile is characterized by a relatively weak elevated stable layer and relatively strong vertical wind shear. In other words, the errors associated with $l_b^2$ tend to increase as the Richardson number $(N^2/\bar{u}_z^2)$ of the flow decreases.

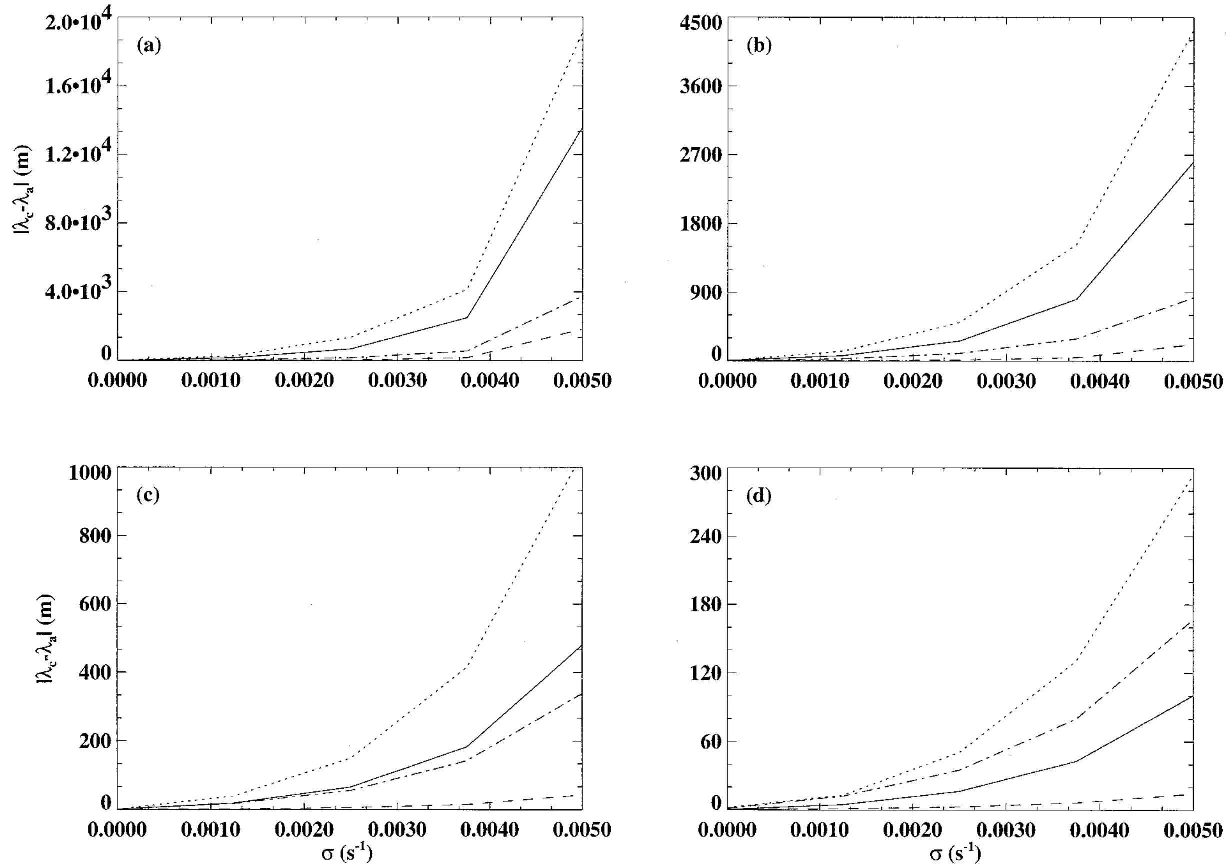


FIG. 1. The absolute value of the difference between the horizontal wavelength of the resonant mode obtained by solving (5) and the horizontal wavelength of the resonant mode obtained by solving (6) with $l^2$ defined by $l_b^2$ (solid line), $\hat{l}^2$ (dotted line), $l_{pi}^2$ (dashed line), and $l_{lh}^2$ (dot–dash line) for $\sigma$ ranging from 0 to 0.005 s$^{-1}$ and $N_s$ equal to (a) 0.01 s$^{-1}$, (b) 0.015 s$^{-1}$, (c) 0.02 s$^{-1}$, and (d) 0.025 s$^{-1}$.

TABLE 1. Summary of the various approximate Scorer parameter expressions found in the mountain wave literature and the articles in which these expressions appeared. See (10) and (12) for definitions of $\bar{s}$ and $\Gamma$.

| Notation | Scorer parameter | Reference |
|---|---|---|
| $l_b^2$ | $\frac{N^2}{\bar{u}^2} - \frac{\bar{u}_{zz}}{\bar{u}}$ | Scorer (1949) |
| $\hat{l}^2$ | $\frac{N^2}{\bar{u}^2} + \frac{\bar{s}\bar{u}_z}{\bar{u}} - \frac{1}{4}\bar{s}^2 + \frac{1}{2}\bar{s}_z - \frac{\bar{u}_{zz}}{\bar{u}}$ | Berkshire and Warren (1970),[a] Sawyer (1960)[b] |
| $l_{pi}^2$ | $\frac{N^2}{\bar{u}^2} - 2\Gamma\frac{\bar{u}_z}{\bar{u}} - \Gamma^2 - \Gamma_z - \frac{\bar{u}_{zz}}{\bar{u}}$ | Vergeiner (1971), Danielsen and Bleck (1970)[c] |
| $l_{lh}^2$ | $\frac{N^2}{\bar{u}^2} - \frac{\bar{s}\bar{u}_z}{\bar{u}} - \frac{1}{4}\bar{s}^2 - \frac{1}{2}\bar{s}_z - \frac{\bar{u}_{zz}}{\bar{u}}$ | Smith (1979),[d] Keller (1994) |

[a] Berkshire and Warren (1970) use the incompressible form of static stability. The errors introduced by using their expression in resonant wavelength computations are significantly larger than those introduced by using (9).

[b] Sawyer (1960) neglects the term $\bar{s}\bar{u}_z/\bar{u}$. Sawyer's expression can be derived from (12) of Scorer (1949).

[c] Danielsen and Bleck (1970) neglect the term $\Gamma^2$ and reduce $\Gamma_z$ to $(N^2/g)_z$.

[d] Smith actually presented a derivation for the expression $\hat{l}^2$, but, due to a sign error in (2.21), the expression that appears in Smith (1979) is equivalent to $l_{lh}^2$.

TABLE 2. Resonant wavelengths associated with (5), $\lambda_c$, for four values of $N_s$ and $\sigma = 0.005$ s$^{-1}$, and the difference between $\lambda_c$ and the resonant wavelengths associated with the approximate vertical structure equation (6) for $l^2$ defined by the expressions in Table 1.

| $N_s$ (s$^{-1}$) | $\lambda_c$ (m) | $\lambda_c - \lambda_b$ (m) | $\lambda_c - \hat{\lambda}$ (m) | $\lambda_c - \lambda_{pi}$ (m) | $\lambda_c - \lambda_{lh}$ (m) |
|---|---|---|---|---|---|
| 0.010 | 45 395 | 13 589 | 19 151 | 1829 | −3754 |
| 0.015 | 23 219 | 2613 | 4339 | 219 | −833 |
| 0.020 | 12 882 | 480 | 1041 | 43 | −339 |
| 0.025 | 8207.7 | 100.6 | 294.8 | 14.3 | −167.1 |

### b. *Compressible approximations*

The relationship between the accuracy of the Boussinesq version of (6) and the magnitude of $\underline{\sigma}$ suggests that retaining at least a portion of the term $\bar{S}\bar{u}_z/\bar{u}$ in the simplified expression for $l^2$ might significantly improve the accuracy of (6). Since $\bar{M} \approx 1$, it seems logical to also assume that $\bar{M}_z \approx 0$. When these simplifications are applied to (4), $l^2$ becomes

$$\hat{l}^2 = \frac{N^2}{\bar{u}^2} + \frac{\bar{s}\bar{u}_z}{\bar{u}} - \frac{1}{4}\bar{s}^2 + \frac{1}{2}\bar{s}_z - \frac{\bar{u}_{zz}}{\bar{u}}, \tag{9}$$

where

$$\bar{s} = -\frac{d \ln \bar{\rho}}{dz}.^{1} \tag{10}$$

The parameter $\hat{l}^2$ characterizes the basic format of the Scorer parameter expressions that appear in Berkshire and Warren (1970) and Sawyer (1960). When $l_b^2$ is replaced by $\hat{l}^2$ in the resonant wavelength calculations, the errors introduced by using the simplified equation actually increase substantially for nonzero $\sigma$ (see Fig. 1 and Table 2). Hence, neglecting compressibility effects entirely would be preferable to using this ''compressible'' version of the Scorer parameter. To understand why retaining this particular set of compressibility terms adversely affects the performance of the simplified equation, it is helpful to rewrite (4) in the form

$$l^2 = \frac{\bar{M}N^2}{\bar{u}^2} - 2(\Gamma - \Phi)\frac{\bar{u}_z}{\bar{u}} - (\Gamma - \Phi)^2 - (\Gamma - \Phi)_z - \frac{\bar{u}_{zz}}{\bar{u}}, \tag{11}$$

where

$$\Gamma = -\frac{N^2}{g} + \frac{1}{2}\bar{s} = -\frac{1}{2}\bar{s} + \frac{g}{\bar{c}^2};$$
$$\Phi = \frac{1}{2}\frac{d \ln \bar{M}}{dz}. \tag{12}$$

[1] Some authors define $\bar{s}$ with a negative sign, while others do not include the negative sign in their definition of $\bar{s}$. This inconsistency probably stems from a typographical error in Eq. (8) of Queney et al. (1960), which should read $\bar{s}=-d \ln \bar{\rho}/dz=(g/R - \gamma)/\bar{T}$, instead of $\bar{s}=d \ln \bar{\rho}/dz=(g/R - \gamma)/\bar{T}$.

The quantity $g/\bar{c}^2$ that appears in the compressibility terms of (11) stems from the vertical differentiation of $\bar{M}$. This expression for $l^2$ reduces to $\hat{l}^2$ when $\Phi \ll \Gamma$, $\Phi_z \ll \Gamma_z$, and $\Gamma \approx -\bar{s}/2$. It can be shown that $\Phi$ and $\Phi_z$ are generally at least an order of magnitude smaller than their counterparts $\Gamma$ and $\Gamma_z$, but $g/\bar{c}^2$ is actually greater than $\bar{s}/2$. Simplifying the Scorer parameter expression by applying the simplification $\bar{M}_z \approx 0$ to (4) neglects a significant contribution to the wind shear term. Neglecting this contribution to the wind shear term causes the sign of wind shear term in $\hat{l}^2$ to be opposite that in the full expression for the Scorer parameter. The large errors associated with the approximate ''compressible'' Scorer parameter $\hat{l}^2$ illustrate the importance of following a systematic scaling procedure.

The energy-conserving ''soundproof'' system proposed by Durran (1989), which is referred to as the pseudo-incompressible system, and that proposed by Lipps and Hemler (1982) were both derived by applying systematic scaling techniques to the equations governing the dynamics of a compressible fluid. The vertical structure equations for these filtered systems are equivalent to (6) when $l^2$ is defined by either $l_{pi}^2$ or $l_{lh}^2$ (see Table 1). When $l_b^2$ is replaced by $l_{pi}^2$, the errors introduced by using the simplified equation to compute resonant wavelengths for profiles with nonzero $\sigma$ are considerably smaller (see Fig. 1 and Table 2). When the elevated stable layer is characterized by weak to moderate static stability, using $l_{lh}^2$ instead of $l_b^2$ also reduces the resonant wavelength errors, but the reduction is significantly less than that achieved by using $l_{pi}^2$. In addition, the errors for $l_{lh}^2$ actually exceed those for $l_b^2$ when the elevated stable layer is characterized by a relatively strong static stability ($N_s = 0.025$ s$^{-1}$). Hence, the resonant wavelength computations for this family of idealized profiles indicate trapped lee wave solutions derived using $l_{pi}^2$ to represent the Scorer parameter for a compressible fluid will be more accurate than those derived using $l_{lh}^2$. In their study of the accuracy of four soundproof systems, Nance and Durran (1994) concluded that the pseudo-incompressible system should be more accurate than the Lipps and Hemler system when the disturbance is nonhydrostatic. Since trapped lee waves are a nonhydrostatic phenomenon, the performance of $l_{pi}^2$ over that of $l_{lh}^2$ in these resonant wavelength computations is consistent with Nance and Durran's earlier analysis. On the other hand, the difference between the errors associated with $l_{pi}^2$ and $l_{lh}^2$ and those associated with $l_b^2$ is rather small when the background wind speed is constant with height, so the quadratic and vertical tendency terms in these approximate Scorer parameter expressions can be neglected without significantly affecting the performance of these expressions.

It is interesting to note that the expressions $l_{pi}^2$ and $l_{lh}^2$ actually appeared in the mountain wave literature prior to the introduction of the pseudo-incompressible and Lipps and Hemler systems. On the other hand, the parameters $l_{pi}^2$ and $l_{lh}^2$ can also be derived by applying a systematic scaling analysis directly to (11), so the appearance of these expressions prior to the introduction of these soundproof systems is not really all that surprising. As noted earlier, $\Phi \ll \Gamma$ and $\Phi_z \ll \Gamma_z$. For these conditions and $\overline{M} \approx 1$, (11) simplifies to $l_{pi}^2$. The parameter $l_{pi}^2$ is equivalent to the approximate Scorer parameter expression used by Vergeiner (1971) and an expression closely related to $l_{pi}^2$ appeared in Danielsen and Bleck (1970). Note that the simplifications $\Phi \ll \Gamma$ and $\Phi_z \ll \Gamma_z$ are equivalent to assuming $\overline{M}_z \approx 0$, the same simplification that was used to derive $\hat{l}^2$. By rewriting the expression for $l^2$ so that the compressible effects are grouped according to wind shear, quadratic, and vertical tendency terms, the subtle scaling mistake that led to an inaccurate compressible Scorer parameter expression can be avoided.

The parameter $l_{lh}^2$ takes the simplification of (11) one step further by applying the additional approximation $N^2/g \ll \overline{s}/2$ or $\Gamma \approx \overline{s}/2$. The approximation $\Gamma \approx \overline{s}/2$ is only appropriate when the basic state is characterized by weak to moderate static stability. Hence, using $l_{lh}^2$ in the resonant wavelength computations introduced relatively large errors when the basic state contained layers with strong static stability because the characteristics of these layers violated the assumptions used to derive this expression. The expression for $l^2$ found in Smith (1979), which also recently appeared in Keller (1994), is equivalent to the parameter $l_{lh}^2$. On the other hand, a careful review of Smith's derivation reveals that equation (2.21) in Smith (1979) should read $\overline{s} \equiv -d\ln\overline{\rho}/dz$, instead of $\overline{s} \equiv d\ln\overline{\rho}/dz$. With this correction, Smith's expression for $l^2$ is equivalent to the parameter $\hat{l}^2$, which differs from $l_{lh}^2$ only with respect to the sign of two terms: the wind shear and vertical tendency terms. Ironically, this sign error produced a more accurate representation of the Scorer parameter for a compressible fluid. The performance of $l_{lh}^2$ over that of $\hat{l}^2$ in the resonant wavenumber computations for nonzero $\sigma$ illustrates the importance of maintaining the correct sign for the wind shear term when simplifying the Scorer parameter for a compressible fluid.

## 4. Conclusions

The compressibility terms in the Scorer parameter for a compressible fluid serve as small corrections to the Boussinesq version of the Scorer parameter. A comparison between the resonant wavelengths for the full compressible vertical structure equation and the various simplified equations showed that these small corrections are important when the background flow is characterized by relatively strong vertical wind shear. This comparison also showed that the inclusion of compressibility effects in the Scorer parameter can actually degrade the performance of (6) if the simplification of $l^2$ does not follow a systematic scaling procedure. The review of the approximations associated with each approximate equation and the resonant wavenumber computations presented in this article showed that the approximate Scorer parameter associated with the pseudo-incompressible system ($l_{pi}^2$) provides the most accurate and versatile representation of the leading-order compressibility effects in the Scorer parameter for a compressible fluid.

*Acknowledgments.* I would like to thank Dr. Dale R. Durran for his helpful suggestions on the presentation of this material and Rajul Pandya for his assistance in coding the resonant wavelength calculations. This research was supported by the National Science Foundation Grants ATM-8813971, ATM-9218376, and ATM-9322480.

APPENDIX

### Numerical Method for Solving the Eigenvalue Problem

Replacing the vertical derivatives of $\hat{w}$ with centered finite differences transforms the eigenvalue problems described by (5) and (6) into linear equations of the form

$$a\hat{w}_{j+1} + b\hat{w}_j + a\hat{w}_{j-1} = 0, \qquad \text{(A1)}$$

where $\hat{w}_j$ represents the value of $\hat{w}$ at the $j$th grid point, $a = \Delta z^{-2}$, $b = L^2 - 2\Delta z^{-2}$, $L^2 = l^2 - \overline{M}k^2$ or $l^2 - k^2$, and $\Delta z$ represents the vertical grid spacing. For this study, the discretization was done over a grid 15 km deep with a grid resolution of 100 m. To complete the formulation of this problem, boundary conditions must be specified at the surface and the top of the domain. The ground is a physical boundary for which nonresonant mountain waves will satisfy the boundary condition $\hat{w}_0 = 1$ and resonant modes will satisfy the boundary condition $\hat{w}_0 = 0$. The top of the numerical domain is a nonphysical boundary that requires a numerical formulation that will not distort the interior solution. By placing an isothermal, constant-wind-speed layer in the top 3 km of each idealized profile, analytic solutions of the form $e^{\pm iLz}$ can be obtained for both (5) and (6) at the top of the domain. This study used the positive root of the analytic solution, which represents either the exponentially decaying mode or the mode characterized by upward energy propagation, to write $\hat{w}$ at the point just beyond the numerical domain in terms of the point at the top of the domain (i.e., $\hat{w}_{nz+1} = \hat{w}_{nz}e^{iL\Delta z}$). This relationship allows the centered finite difference for the vertical derivative at the top boundary to be written in terms of points within the numerical domain. The numerical formulation of this boundary condition assumes that the energy associated with the disturbance generated by flow over a mountain ridge is completely absorbed at some level above the numerical domain.

When the mode is nonresonant, this finite-difference formulation of the eigenvalue problems described by (5) and (6) generates a system of equations of the form

$$\mathbf{A}\hat{\mathbf{w}} = \mathbf{d}, \qquad \text{(A2)}$$

where **A** is a tridiagonal matrix, $\hat{\mathbf{w}}$ is a vector whose components are $\hat{w}_j$ for $j = 1$ to $nz$, and **d** is a vector with one nonzero element. A nontrivial solution to this heterogeneous system can be obtained by specifying the horizontal wavenumber of the nonresonant mode and applying basic methods from linear algebra, such as LU decomposition, to solve for $\hat{\mathbf{w}}$ (Press et al. 1986). When the mode is resonant, the finite-difference formulation described above generates a system of equations of the form

$$\mathbf{A}\hat{\mathbf{w}} = 0. \tag{A3}$$

A nontrivial solution to this homogeneous system can only be obtained when the determinant of the tridiagonal matrix **A** is zero (Kaplan 1981). The determinant of **A** will only be zero when the horizontal wavenumber $k$ corresponds to that of the resonant mode. Since the horizontal wavenumber of the resonant mode is not known a priori, solving the eigenvalue problem for resonant modes is slightly more complicated.

By dividing the numerical domain into two layers ($0 \leq z < 1$ km and 1 km $< z \leq 15$ km) and specifying $\hat{w}_i = 1$, where $i$ represents the grid point at $z = 1$ km, the homogeneous system for resonant modes can be transformed into two heterogeneous systems of the form (A2), where the vector indexes now run from 1 to $i - 1$ and $i + 1$ to $nz$. Specifying a nonzero value for $\hat{w}$ at a point above the boundary simply determines the scaling factor for the nontrivial solution to the homogeneous system (A3). Since the maximum vertical velocity of a resonant mode generally lies between 1 and 3 km, an interface at 1 km basically normalizes the amplitude of the resonant mode to order one. A solution for $\hat{\mathbf{w}}$ is then obtained for a given horizontal wavenumber by applying LU decomposition to these heterogeneous systems. The solution obtained in this manner will correspond to that of the resonant mode when $\hat{w}_{i+1}$ from the top layer and $\hat{w}_{i-1}$ from the bottom layer satisfy the finite difference form of the governing equation at the interface

$$a(\hat{w}_{i+1} + \hat{w}_{i-1}) + b = 0. \tag{A4}$$

The resonant wavenumber for each idealized profile was determined by applying an iterative scheme that terminates when the input horizontal wavenumber generates values of $\hat{w}_{i+1}$ and $\hat{w}_{i-1}$ that satisfy (A4) to within $10^{-10}$. Two initial guesses for $k$ were used to initiate this iterative scheme, and then the formula

$$k_{n+1} = k_n - e_n\left(\frac{k_n - k_{n-1}}{e_n - e_{n-1}}\right), \tag{A5}$$

where $k_n$ represents the $n$th guess and

$$e_n = a(\hat{w}(k_n)_{i+1} + \hat{w}(k_n)_{i-1}) + b, \tag{A6}$$

was used to generate successive values of $k$ that converge to the resonant wavenumber; (A5) and (A6) are based on Newton's method, which is quadratically convergent (Kaplan 1981).

## REFERENCES


Berkshire, F. H., and F. W. G. Warren, 1970: Some aspects of linear lee wave theory for the stratosphere. *Quart. J. Roy. Meteor. Soc.,* **96,** 50–66.

Brown, P. R. A., 1983: Aircraft measurements of mountain waves and their associated momentum flux over the British Isles. *Quart. J. Roy. Meteor. Soc.,* **109,** 849–865.

Corby, G. A., and C. E. Wallington, 1956: Airflow over the mountains and lee-wave amplitudes. *Quart. J. Roy. Meteor. Soc.,* **82,** 266–274.

Cox, K. W., 1986: Analysis of the Pyrenees lee wave event on 23 March 1982. *Mon. Wea. Rev.,* **114,** 1146–1166.

Danielsen, E. F., and R. Bleck, 1970: Tropospheric and stratospheric ducting of stationary mountain lee waves. *J. Atmos. Sci.,* **27,** 758–772.

Durran, D. R., 1989: Improving the anelastic approximation. *J. Atmos. Sci.,* **46,** 1453–1461.

Kaplan, W., 1981: *Advanced Mathematics for Engineers.* Addison-Wesley, 928 pp.

Keller, T. L., 1994: Implications of the hydrostatic assumptions on atmospheric gravity waves. *J. Atmos. Sci.,* **51,** 1915–1929.

Lipps, F., and R. Hemler, 1982: A scale analysis of deep moist convections and some related numerical calculations. *J. Atmos. Sci.,* **39,** 2192–2210.

Mitchell, R. M., R. P. Cechet, P. J. Turner, and C. C. Elsum, 1990: Observation and interpretation of wave clouds over Macquarie Island. *Quart. J. Roy. Meteor. Soc.,* **116,** 741–752.

Nance, L. B., and D. R. Durran, 1994: A comparison of the accuracy of three anelastic systems and the pseudo-incompressible system. *J. Atmos. Sci.,* **51,** 3549–3565.

Pearce, R. P., and P. W. White, 1967: Lee wave characteristics derived from a three-layer model. *Quart. J. Roy. Meteor. Soc.,* **93,** 758–772.

Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, 1986: *Numerical Recipes: The Art of Scientific Computing.* Cambridge University Press, 818 pp.

Queney, P., G. Corby, N. Gerbier, H. Koschmieder, and J. Zierep, 1960: The airflow over mountains. World Meteorological Organization Tech. Note 34, 135 pp. [Available from WMO, Case Postale 2300, CH-1211 Geneva 2, Switzerland.]

Reynolds, R. D., R. L. Lamberth, and M. G. Wurtele, 1968: Investigation of a complex mountain wave situation. *J. Appl. Meteor.,* **7,** 353–358.

Sawyer, J. S., 1960: Numerical calculation of the displacements of a stratified airstream crossing a ridge of small height. *Quart. J. Roy. Meteor. Soc.,* **86,** 326–345.

Scorer, R., 1949: Theory of waves in the lee of mountains. *Quart. J. Roy. Meteor. Soc.,* **75,** 41–56.

Shutts, G. J., 1992: Observations and numerical model simulation of a partially trapped lee wave over the Welsh mountains. *Mon. Wea. Rev.,* **120,** 2056–2066.

Smith, R. B., 1976: The generation of lee waves by the Blue Ridge. *J. Atmos. Sci.,* **33,** 507–519.

——, 1979: The influence of mountains on the atmosphere. *Advances in Geophysics,* Vol. 21, Academic Press, 87–230.

Starr, J. R., and K. A. Browning, 1972: Observations of lee waves by high-power radar. *Quart. J. Roy. Meteor. Soc.,* **98,** 73–85.

Vergeiner, I., 1971: An operational linear lee wave model for arbitray basic flow and two-dimensional topography. *Quart. J. Roy. Meteor. Soc.,* **97,** 30–60.