The Brewer–Dobson Circulation: Dynamics of the Tropical Upwelling

R. ALAN PLUMB
Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, Massachusetts

JANUSZ ELUSZKIEWICZ
Atmospheric and Environmental Research, Inc., Cambridge, Massachusetts

(Manuscript received 16 September 1997, in final form 27 May 1998)

ABSTRACT

Recent advances in our understanding of the dynamics of the stratospheric circulation have led to the concepts of “downward control” and the “extratropical pump.” However, under the assumptions on which these concepts are based, midlatitude wave driving cannot explain the fact that mean stratospheric upwelling is located in the Tropics. Nevertheless, using a nonlinear two-dimensional model it is shown here that a steady and (in the lower stratosphere) linear circulation with a qualitatively reasonable upwelling can be produced, provided the wave drag extends to within about 20° of the equator. In a linear analysis of the problem, it is shown that the effects of weak model viscosity (some 50 times weaker than thermal relaxation) are crucial in permitting flow across angular momentum contours within a tropical boundary layer whose width is of order $L_R^{1/4}$, where $L_R$ is the equatorial Rossby radius and $P$ a Prandtl number (the ratio of radiative to viscous relaxation times). Provided the wave drag extends into this boundary layer, upwelling is distributed across the Tropics. These considerations put limits on the generality of the concepts of the extratropical pump and downward control and, inter alia, open the possibility that diabatic heating alone can drive a meridional circulation within the Tropics. On the basis of simple representations of wave drag and diabatic heating in a nonlinear, zonally symmetric model, it is found that, although driving by wave drag is the dominant mechanism, stratospheric (and perhaps tropospheric) heating may make a significant contribution to the net upwelling and may help explain its structure. Just what, in reality, might play a role analogous to that of viscosity in the model is an open question.

1. Introduction

Since the pioneering work of Brewer (1949) and Dobson (1956) it has been recognized that the meridional and seasonal distributions of stratospheric tracers imply that tropospheric air ascends into and within the tropical stratosphere, and spreads poleward and downward from there into the winter hemisphere. This pattern of tropical diabatic upwelling and winter diabatic downwelling has been confirmed by calculations of diabatic heating rates using satellite observations of global temperature distributions [e.g., see the recent papers by Rosenlof (1995), Eluszkiewicz et al. (1996), Eluszkiewicz et al. (1997), and references therein].

Although it has in the past been common to regard this circulation as being “driven” by diabatic heating in the Tropics and diabatic cooling in middle and high winter latitudes, it has been recognized at least since the work of Dickinson (1971)—with subsequent refinements by Held and Hou (1980), WMO (1985), and, in particular, by Haynes et al. (1991)—that global-scale, axisymmetric circulations of this kind cannot exist without some kind of drag acting on the flow. This statement follows from the angular momentum budget, which is

$$ \frac{\partial m}{\partial t} + \mathbf{v} \cdot \nabla m = a \cos \varphi F, $$

(1)

where $\mathbf{v} = (u, w)$ is the meridional wind vector and $m = \Omega a^2 \cos^2 \varphi + ua \cos \varphi$ is the angular momentum density, $\Omega$ being the planetary rotation rate, $a$ the planetary radius, $u$ the zonal wind, $\varphi$ latitude, and $F$ the force per unit mass, from frictional or other forces. In the presence of waves, (1) is valid for the zonally averaged angular momentum, provided $\mathbf{v}$ is interpreted as the residual circulation, and the divergence of the Eliassen–Palm flux (the wave drag) is incorporated into $F$ (Andrews and McIntyre 1976). If the angular momentum distribution is steady (either in a time average or near solstice when the zonal winds reach their seasonal extremes) and its gradient nonzero, then, in the absence of wave stresses and of friction, (1) implies that there can be no flow across angular momentum surfaces. Since these surfaces depart little from the vertical in...
extratropical latitudes, it follows that there can be no meridional circulation if $F = 0$. [See Haynes et al. (1991) for a full discussion.] As shown by Held and Hou (1980) for the tropospheric Hadley cell, and extended to the stratospheric case by Dunkerton (1989), a circulation can exist in the case $F = 0$ only in low latitudes where nonlinear effects become important through the elimination of the weak background angular momentum gradient by the flow.

The primary driving for the midlatitude stratospheric circulation is wave drag associated with breaking Rossby waves in the winter hemisphere, together with contributions by synoptic waves in the very low stratosphere. Gravity wave drag, the primary forcing in the mesosphere, is thought to be of secondary importance in much of the stratosphere. The dominant contribution in the stratosphere, from large-scale Rossby waves, is concentrated within the “surf zone” (McIntyre and Palmer 1983) of the midlatitude winter hemisphere. The surf zone has two well-defined edges: one at its poleward side, the edge of the polar vortex (McIntyre and Palmer 1983), and the other in the winter subtropics (McIntyre 1990), which is marked by strong gradients in tracer distributions at 10°–20° latitude (e.g., Murphy et al. 1993; Randel et al. 1993; Grant et al. 1996).

By reducing the angular momentum of the zonal flow, the wave drag drives air poleward, a process described by Holton et al. (1995) as the “extratropical pump.” If (for the purposes of illustration) the angular momentum surfaces can be assumed to be vertical, then (1) gives simply

$$\zeta \omega = -F,$$

where $\zeta = (a^2 \cos \phi)^{-1} \partial \ln \omega / \partial \phi$ is the absolute vorticity. The situation is thus as illustrated in Fig. 1a. It is assumed that the wave drag is entirely confined to the surf zone, denoted by the shading on the figure. Within this region, the mean flow is “pumped” poleward, according to (2). Continuity requires vertical motion at the edges of the surf zone; according to (1) the flow must (if steady) be along angular momentum contours outside the surf zone where $F = 0$, and connect downward, for reasons discussed by Haynes et al. (1991). Thus, if the surf zone terminates a finite distance away from the equator in a region of nonzero angular momentum gradient (finite absolute vorticity), the steady circulation must be as depicted in Fig. 1a, with the extratropical pump being fed by upwelling in the winter subtropics and in turn feeding downwelling near the edge of the polar vortex.

Note the difference here between the “vertical” (strictly, along $\mu$ contours) and “latitudinal” (across $\mu$ contours) structure of the response. In the vertical, the response to the imposed drag is nonlocal, since flow along $\mu$ contours makes no impact on the angular momentum budget, though of course diabatic relaxation is required to permit crossing of the potential temperature surfaces. In latitude, on the other hand, the response is under local control by the wave drag, since crossing of the $\mu$ contours is, by (2), permitted only through the agency of local drag. Thus, as long as $\zeta$ is nonzero, the steady-state circulation cannot extend outside the latitudes at which drag is applied.

The stratospheric circulation diagnosed from observations does not support this picture of upwelling maximizing in the winter subtropics. Figure 2 shows the residual vertical velocity at 68 hPa determined from temperature and constituent data acquired by the Microwave Limb Sounder (MLS) and Cryogenic Limb Array Etalon Spectrometer (CLAES) instruments on board the Upper Atmosphere Research Satellite. These velocities have been diagnosed from diabatic heating rates computed by means of a sophisticated radiative transfer code; the sources of uncertainties in the calculations have been extensively documented (Eluszkiewicz et al. 1996; Eluszkiewicz et al. 1997). Two determinations of CLAES-based velocities are shown, for cases in which the effects of aerosols from the Mount Pinatubo eruption in June 1991 are both excluded and included in the heating calculation. However, as shown in Fig. 13 of Eluszkiewicz et al. (1997), the aerosol effects are relatively minor at 68 hPa and cease to be important by the second half of 1992.

Several features in Fig. 2 are noteworthy. The lati-

![Fig. 1. Schematic of the steady meridional circulation produced by midlatitude wave drag (cf. Haynes et al. 1991). Wave drag is confined to the winter midlatitude surf zone, denoted by the shading. Light lines show contours of angular momentum, assumed here to be dominated by the planetary term. Arrows depict the circulation. See text for discussion of cases a and b.](Image)
Fig. 2. Vertical velocity at 68 hPa computed using (a) MLS data (Eluszkiewicz et al. 1996), and (b) and (c) CLAES data (Eluszkiewicz et al. 1997). In (c), the effects of aerosols are included in the calculation. The contour interval is 0.1 mm s\(^{-1}\); negative values are shaded. The solid line indicates the latitude of maximum upwelling.

The longitudinal band of upward velocities is shifted toward the summer hemisphere, a fact previously noted by Rosenlof (1995). Moreover, there is annual-mean upwelling at the equator (in fact, upwelling is found all year within about 20\(^\circ\) of the equator\(^1\)), and the latitude of maximum upwelling migrates toward the summer subtropics in the MLS-based circulation, except during the 1992/93 northern winter period, when it is located at the equator. These summer hemisphere excursions of the maximum upwelling are also noticeable in the CLAES-based Center (recently renamed the National Centers for Environmental Prediction) analyses (which are used in MLS version 3 temperature files at pressures greater than 22 hPa).

\(^1\) The negative values at the equator during July and August 1994 seen in the MLS fields are caused by large positive values of the temperature tendency term computed from National Meteorological
fields, with the exception of the 1991/92 northern winter period. Note that this was the period of large aerosol loading from the Pinatubo eruption, which, given the tropical peak in the aerosol concentration and in the associated heating (see Fig. 12 in Eluszkiewicz et al. 1997), contributes to the equatorial shift of maximum upwelling when aerosols are included in the heating calculations (the reasons for this shift are unclear).

Thus, the observed circulation does not resemble that depicted in Fig. 1a, but rather looks more like the schematic of Fig. 1b, with most of the upwelling occurring deep in the Tropics and into the summer hemisphere. According to (1), the circulation cannot extend equatorward of the surf zone if

1) the flow is steady,
2) the angular momentum gradient is nonzero equatorward of the subtropical edge (to be specific, if the angular momentum density at the subtropical edge differs from that in the near-equatorial region where the upwelling flow enters the stratosphere), and
3) \( F = 0 \) outside the surf zone.

In order to explain the observed circulation, one or more of these criteria must be violated. Of course, for most of the year, the circulation is not steady, as it undergoes its seasonal switching between hemispheres. Holton et al. (1995, see their Fig. 4b) show extension of the stress-driven circulation, as calculated by inviscid linear theory, extending into the Tropics during the time (early winter) when the wave drag is building up to its maximum intensity, but even then (as we shall see below), most of the upwelling is located in the winter subtropics, rather than near the equator. We can in fact see that nonsteadiness alone cannot resolve the difficulty by noting that, in the annual mean, tropical air is upwelling and that the annual mean of (1) is

\[
-fa \cos \varphi \overline{v} + \nabla \cdot (u a \cos \varphi) = (a \cos \varphi) \overline{F}, \tag{3}
\]

where the overbar denotes the annual mean; and we have neglected interannual variability so that \( \partial \overline{\text{time}} \overline{v} = 0 \). The second term, representing advection of relative angular momentum, is a nonlinear one, so we can conclude that \( \overline{v} = 0 \) wherever \( \overline{F} = 0 \) if the dynamics are linear. If the annual mean body force per unit mass \( \overline{F} = 0 \) over a finite band of the Tropics—between the most equatorward excursions of the edges of the northern and southern winter surf zones—then \( \overline{v} = 0 \) throughout that band, whence it follows from continuity that \( \overline{v} \) must also be zero throughout the tropical band, assuming that \( \rho \overline{v} \to 0 \) as \( z \to \infty \). It therefore follows that the existence of annual mean upwelling in the tropical stratosphere implies that either (i) the effective body force is nonzero there or (ii) the tropical circulation is fundamentally nonlinear.

In section 2, results are presented of some calculations with a nonlinear axisymmetric model of the circulation driven by a steady wave drag in the midlatitude Northern Hemisphere stratosphere. For this model, it is found that, if the wave drag is confined to latitudes greater than about 20° from the equator, there is little penetration of the meridional circulation equatorward of that latitude. If the region of wave drag is shifted farther equatorward, however, the circulation extends across the equator, and there is substantial upwelling at and around the equator. In the upper stratosphere of the model, the flow shows clear signs of nonlinearity. In the lower stratosphere, however, where most of the mass flux occurs, the response is very linear, and we will therefore focus on linear behavior in what follows.

As will be discussed in what follows, the reason for the extension of the model circulation into the Tropics is the presence in the model of viscosity, contributing to \( F \) and permitting flow across angular momentum contours. Thus, the achievement by the model of a qualitatively realistic circulation depends on what is probably its most unrealistic attribute, namely, the numerically necessary but otherwise arbitrary viscosity. While discussion of what the model results might imply for the real stratosphere will be deferred until section 5, we note here that the model viscosity is much weaker than thermal dissipation: the rate for damping the large-scale circulation by viscosity, \( \alpha_v \), is about \((1.5 \text{ yr}^{-1})\), some 50 times slower than the thermal relaxation rate \( \alpha_T \) in the model. Nevertheless, it is shown that viscosity is important within an internal tropical boundary layer that has width of order \( L_p P^{1/4} \) (about 8° latitude for the model; we shall in fact see that its actual width is about twice this scale), where \( L_p \) is an equatorial Rossby deformation radius and \( P = \alpha_v/\alpha_T = 0.02 \) may be regarded as a Prandtl number for this kind of mechanical and thermal dissipation. It will be shown that the tropical upwelling in the model is entirely insensitive to imposed wave drag poleward of about 20° latitude.

In section 3, a formal linear analysis is presented, showing explicitly the dependence of the induced tropical circulation driven by an imposed, steady wave drag on \( P \). For a surf-zone-like drag, which is zero equatorward of some specified latitude \( \varphi_0 \), the upwelling is concentrated near \( \varphi_0 \) (i.e., like Fig. 1a) unless \( \varphi_0 \) lies within the tropical boundary layer, when the upwelling spreads across the equator (as in Fig. 1b). A second consequence of the importance of weak viscosity in the angular momentum budget is the breakdown of the inviscid assertion that diabatic forcing alone is incapable of sustaining a steady circulation. When \( P \) is nonzero, some component of the circulation may be entirely independent of wave drag, driven thermally by latitudinal gradients in solar heating. Linear calculations in section 3 confirm that the steady thermally driven circulation is entirely confined to the Tropics, and driven by small latitudinal gradients of thermal forcing within the Tropics alone.
In section 4, we return to the nonlinear model and show the results of experiments with seasonally varying wave drag and thermal forcing, both separately and in combination. Using a simplified representation of mid-latitude wave drag of realistic strength that extends to sufficiently low latitudes, tropical upwelling is found to be of similar magnitude to observational results. Only by including thermal forcing as well as the drag can a qualitatively satisfactory pattern of tropical upwelling be produced, although the wave drag, overall, is the dominant effect. Despite predictions from the linear calculations of strong penetration of the thermally driven tropospheric Hadley circulation into the stratosphere, the nonlinear model shows penetration that is much weaker, though still perhaps sufficient to make a modest contribution to the lower-stratospheric upwelling rate. Implications of these results for our understanding of upwelling in the tropical stratosphere, including limitations on the concept of the extratropical pump as a controlling influence on the circulation and on troposphere–stratosphere exchange, will be discussed in section 5.

2. The circulation driven by steady, midlatitude wave drag

In this section results will be presented from experiments with a nonlinear, hydrostatic, zonally symmetric model, formulated in log-pressure coordinates $z = -\ln p$ (where $H$ is constant), based on the equations

$$
\frac{\partial u}{\partial t} + v \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) - f + \frac{w}{\varphi} = D + V(u),
$$

$$
\frac{\partial v}{\partial t} + f u = - \frac{1}{a} \frac{\partial \Phi}{\partial \varphi} + V(u),
$$

$$
\frac{1}{a \cos \varphi} \frac{\partial}{\partial z} (v \cos \varphi) + \frac{\partial}{\partial z} (\rho w) = 0,
$$

$$
\frac{\partial \Phi}{\partial z} = \frac{R}{H} T_e.
$$

Here $(u, v, w)$ are the eastward, northward, and upward components of velocity; $\Phi$ the geopotential; and $T$ the departure of temperature from a horizontally uniform reference profile $T_0(z)$, which has a corresponding static stability $S = dT_e/dz + \frac{g}{c_p}$. The corresponding buoyancy frequency $N = \sqrt{RS/H}$ is specified to be $2 \times 10^{-2}$ s$^{-1}$ above the tropopause at 15-km altitude, and $1.22 \times 10^{-2}$ s$^{-1}$ below. Here, $(\varphi, z)$ are latitude and height, $f$ is the Coriolis parameter, $a$ is the earth’s radius, and $R$ is the gas constant for air. The forcing and dissipation terms are as follows. The specified body force per unit mass is $D(\varphi, z)$, intended to represent Rossby wave drag acting on the mean flow; its form will be discussed below. The terms $V(\cdot)$ represent diffusion; in the vertical direction, it is second order, with a diffusivity of 0.25 m$^2$ s$^{-1}$; it is fourth order in the horizontal with a coefficient chosen empirically to dissipate a two-grid wave at the same rate as the vertical viscosity. Finally, $T_e(\varphi, z, t)$ is a specified equilibrium temperature distribution (the departure from the reference profile), toward which temperature is relaxed at a rate $\alpha_T$. For these experiments, the thermal relaxation time $\alpha_T^{-1} = 10$ days is spatially uniform.

The model is global, and extends to 60-km altitude, with $60 \times 50$ grid points regularly spaced in $\sin \varphi$ and in height. Within 5 km of the rigid lid, the viscosity and thermal relaxation are increased smoothly to a maximum of five times their interior values. In the bottom layer, a linear drag is also imposed, with rate coefficient $(1$ day$)^{-1}$. The model is semi-implicit, with all linear terms except the fourth-order horizontal diffusion being treated implicitly.

a. Response to steady wave drag

In the first series of experiments to be described, the model is run without any thermal forcing, in the sense that $T_e = 0$ everywhere, and with a specified “perpetual northern winter” wave drag that is steady in time. This drag has the form $D = D_0 B(\varphi, \varphi_0) Z(z)$, where

$$
B(\varphi, \varphi_0) = \begin{cases} 
0, & \varphi < \varphi_0 \\
\sin \varphi_0, & \varphi_0 < \varphi < \varphi_0 + 40^\circ \\
0, & \varphi > \varphi_0 + 40^\circ,
\end{cases}
$$

and

$$
Z(z) = \begin{cases} 
0, & z < 15 \text{ km} \\
\sin \left( \frac{\pi}{45} (z - 15 \text{ km}) \right), & z > 15 \text{ km}.
\end{cases}
$$

This form is chosen to give a simple representation of wave drag within a stratospheric surf zone lying between latitudes $\varphi_0$ and $\varphi_0 + 40^\circ$, peaking at 45°N and 37.5-km altitude. Except where otherwise stated, $D_0 = -2.0 \times 10^{-2}$ m s$^{-2}$. This value was chosen to be sufficient, given the highly simplified distribution, to generate upwelling of reasonable magnitude. The peak value is similar to that deduced by Rosenlof (1995, see Fig. 7 therein) but a factor of 2 smaller than the momentum residual of Eluszkiewicz et al. (1996) and Eluszkiewicz et al. (1997). What actually matters to the forced upwelling flux is the net integrated force (rather than force per unit mass; Haynes et al. 1991) and, for the vertical distribution (6), this is dominated by the wave drag in the lowest one- or two-scale heights of the stratosphere.

Time–latitude sections of the vertical velocity at 20-km altitude, for cases with $\varphi_0 = 30^\circ$ and 21°N, are
shown in Fig. 3. In the first case, after a brief initial adjustment, upwelling and downwelling are concentrated near the edges of the surf zone. In particular, there is very little extension of the upwelling deep into the Tropics. When the subtropical edge of the surf zone is moved to $\phi_0 = 21^\circ$N (Fig. 3b) tropical upwelling is increased, but only slightly, and collapses after a few months. Only when the edge is moved to $\phi_0 = 12^\circ$N (Fig. 4a) does the upwelling extend substantially away from the edge and deep into the Tropics. In this case

![Fig. 3](image1.png)

Fig. 3. Vertical velocity at 20 km for experiments with a steady wave drag imposed within a Northern Hemisphere surf zone whose lowest latitude is (a) $30^\circ$N, (b) $21^\circ$N. Contour interval: 0.05 mm s$^{-1}$. Positive contours solid, negative contours dashed, zero contour dotted.

![Fig. 4](image2.png)

Fig. 4. Vertical velocity at 20 km for experiments with a steady wave drag imposed within a Northern Hemisphere surf zone whose lowest latitude is $12^\circ$N: (a) $D_a = -2 \times 10^{-5}$ m s$^{-2}$, contour interval: 0.05 mm s$^{-1}$; (b) $D_a = -1 \times 10^{-4}$ m s$^{-2}$, contour interval: 0.0025 mm s$^{-1}$. Positive contours solid, negative contours dashed, zero contour dotted.
the upwelling, though still peaking near the edge of the forcing, spreads right across the Tropics; its maximum magnitude is 0.23 mm s\(^{-1}\), similar to what is observed.

The meridional structure of the flow in this last case is shown in Fig. 5. The circulation throughout most of the stratosphere is in accord with the solutions discussed by Haynes et al. (1991), with the flow being poleward within the surf zone, connecting mostly downward into the troposphere (and, in fact, into the lowest model layer where the surface drag allows closure of the circulation). The reverse circulation at high levels is a manifestation of artificial “upward control” (Shepherd et al. 1996) whereby dissipation near the model lid allows the flow to return across the angular momentum contours. The distribution of angular momentum (m) contours shows the effects of advection near and above 30 km, where the contours are strongly distorted in the Tropics, leaving a region of weak gradient between the equator and the forcing region, within which nonlinear dynamics may be important in allowing the flow to move finite latitudinal distances without violating angular momentum conservation (cf. Held and Hou 1980; Dunkerton 1989). Through much of the tropical stratosphere, in fact, the streamlines are approximately parallel to the m contours. Nevertheless, the model response is found to be very close to linear below about 30 km. This is confirmed by comparing cases a and b of Fig. 4; the latter shows results from a case with \(\varphi_0 = 12^\circ\), but with reduced forcing magnitude \(D_0 = 1.0 \times 10^{-6} \text{ m s}\(^{-2}\)\), 20 times smaller than in case a (note that the contour interval is similarly reduced). The response at 20 km is simply reduced by almost exactly the same factor. (Near and above 30 km, however, this is not the case, suggesting that nonlinearity is important in the upper stratosphere.)

b. Response to steady thermal forcing

In the next series of experiments, no wave drag is applied, but a steady, latitudinally varying distribution of equilibrium temperature is imposed. Cross sections of radiative–convective equilibrium temperature calculated by Wehrbein and Leovy (1982) and Fels (1985) for the middle atmosphere show a strong gradient in winter middle latitudes, especially at the edge of polar night, a weak maximum at the equator, and weak gradients in the summer hemisphere. As a simple representation of this in perpetual northern winter, \(T_e\) is specified to be

\[
T_e(\varphi, z) = \begin{cases} 
0, & \varphi \leq \varphi_c, \\
T_{e0} Z(z) \sin \left[ \pi (\varphi - \varphi_c / 2) / \varphi_c \right], & \varphi > \varphi_c, 
\end{cases}
\]

(7)

where \(T_{e0} = -30 \text{ K}\) and \(Z(z)\) is given by (6). [The pole-to-equator equilibrium temperature contrast is weaker by a factor of about 2 from those shown by Wehrbein and Leovy (1982) and Fels (1985); however, as we shall see below, it is the tropical structure of \(T_e\) that is important here, and the value \(-30 \text{ K}\) was chosen to capture that.]

Vertical velocity at 20 km is shown in Fig. 6 for two cases with \(\varphi_c = 30^\circ\) (a) and (b) \(0^\circ\). In the first case, a thermally direct circulation spins up in northern middle latitudes, but collapses after about a month to leave a very weak steady circulation in middle latitudes. In the second case, the global circulation also shows signs of collapse after an early intense stage, but a weak though persistent tropical circulation remains, with a maximum upwelling velocity at 20 km of 0.016 mm s\(^{-1}\), considerably weaker than the stress-driven case of Fig. 4a. As in that case, experiments with reduced \(T_{e0}\) (not shown here) confirm the linearity of the circulation in the lower stratosphere.

Thus, the model results imply that (i) only wave pumping and thermal forcing gradients within the Tropics influence tropical upwelling and (ii) in the lower stratosphere, the model response is linear. In what follows, we shall exploit the latter finding to explain the former.

3. Linear solutions

Writing \(\mu = \sin \varphi\), the steady, linearized versions of (4) are

\[
-2\Omega \mu v = D + V(u),
\]

\[
2\Omega \mu u = -\frac{1}{a} \sqrt{1 - \mu^2} \frac{\partial \Phi}{\partial \mu},
\]

\[
\frac{1}{a} \frac{\partial}{\partial \mu} \left[ \nu \sqrt{1 - \mu^2} \right] + \frac{1}{\rho} \frac{\partial (\rho u)}{\partial z} = 0,
\]

\[
\frac{\partial \Phi}{\partial z} = \frac{R}{H} T, \quad \text{and}
\]

\[
w S = -\alpha_s (T - T_e). \quad (8)
\]

We have neglected the effects of viscosity in the northward momentum equation, since it competes with the large terms representing balance of the zonal flow, and diffusion of heat, which we assume to be much weaker than the relaxation term. Moreover, we replace the viscous term in the zonal momentum equation by a Rayleigh friction:

\[
V(u) = -\alpha_s u.
\]

This is done for mathematical convenience and has no justification other than an expectation that it is the rate at which angular momentum anomalies are dissipated, rather than the precise functional form of that dissipation, that is of greatest importance. With this modification, (8) can be reduced to a single equation for \(w\):
Fig. 5. Meridional structure of the response to wave drag reaching 12°N. (top) Meridional flow (largest arrow corresponds to a northward velocity of 0.112 m s\(^{-1}\)); (middle) mass streamfunction (units: 10\(^8\) kg s\(^{-1}\)); (bottom) angular momentum density (units: 10\(^9\) m\(^2\) s\(^{-1}\)).
Fig. 6. Vertical velocity at 20 km in response to steady thermal forcing of the form given by (7) for (a) \( \varphi_0 = 30^\circ \text{N} \), (b) \( \varphi_0 = 0^\circ \). Contour interval 0.01 mm s\(^{-1}\). Contouring convention as for Fig. 2.

\[
P \frac{\partial}{\partial \mu} \left[ \frac{(1 - \mu^2)}{\mu^2} \frac{\partial w}{\partial \mu} \right] + h^2 \left( \frac{N^2}{(N^2)} \frac{\partial}{\partial \zeta} \left[ \frac{1}{\rho} \frac{\partial (\rho w)}{\partial \zeta} \right] \right) \]

(9)

where \( \theta = \alpha_s RT_s/(N^2 H) \), \( h = 2\Omega a/N_s \) is a deformation height scale (46.25 km) based on the earth’s radius and the constant stratospheric buoyancy frequency \( N_s \), and \( \alpha_s \) is a Prandtl number for the problem. Equation (9) is in fact the steady limit of the more general relation given in Eq. (2) of Garcia (1987). Similar equations have been discussed for the undissipated, steady case (Plumb 1982) and for the seasonally varying case with thermal damping only (Haynes et al. 1991).

From the model viscosity, an appropriate value of \( \alpha_s \) is estimated to be about (500 days)\(^{-1}\), whence \( P \approx 0.02 \). This being so small, it is tempting to neglect the terms involving \( P \) in (9). To do so would, in fact, return us to the balance (2), with all that that implies for the latitudinal confinement of the circulation. However, inspection of (9) suggests that the first term on the left of the equation will in fact become significant for motions that are confined within an internal tropical boundary layer of sine-latitude width \( \mu_s \sim (Ph_0/h)_{\text{E}} \) about the equator, where \( h_0 \) is the smaller of the density-scale height and the vertical scale of the circulation. The fourth root makes this distance (about 0.15, or about 8.4\(^{\circ}\), for \( h_0 = 7 \text{ km} \) not so small, despite the smallness of \( P \). Note that, in terms of distance and for small \( \mu_s \), the width is \( L_s = a \mu_s \sim P \sqrt{L_s} \), where \( L_s = \sqrt{N_s h_0 a/2\Omega} \approx 2500 \text{ km} \) is the internal equatorial deformation radius. This scale is directly analogous to that discussed by Holton et al. (1995) in connection with seasonally varying forcing; their Eq. (9) is recovered if \( \alpha_s \) is replaced by frequency in the definition of \( P \).

The importance of near-equatorial forcing is illustrated in Fig. 7. In general, the solution to (9) can be expressed as

\[
w(\mu, z) = \frac{1}{\rho} \int_{-1}^{1} \int_{0}^{\alpha_{\text{max}}} G_D(\mu, z, \mu', z') D(\mu', z') \rho(z') d\mu' dz' + \int_{-1}^{1} \int_{0}^{\alpha_{\text{max}}} G_T(\mu, z, \mu', z') \frac{\partial}{\partial \mu} (\mu', z') d\mu' dz',
\]

(11)

where \( G_D \) and \( G_T \) are Green’s functions for forcing by imposed drag and equilibrium temperature gradients, respectively.\(^3\) These functions were calculated numerically.

\(^3\) One could of course express the response in terms of a single Green’s function, relating \( w \) at the base point to the combined forcing as expressed on the right-hand side of (9). However, we shall find it useful later to separate them in this way. Note also that we have included a density factor in the definition of \( G_D \), but not of \( G_T \); this is somewhat arbitrary, but it makes \( G_D \) an expression of the sensitivity to the imposed force per unit volume, rather than per unit mass.
we compare two solutions to the steady linear problem forced thermally, according to (7) with $T_\infty = -30$ K and $\varphi_r = 0$, in (top) an unbounded domain and (bottom) in one with a rigid lid at $z = 60$ km. While there are clear differences between the two within two scale heights (14 km) or so of the lid, the solutions are almost identical below 40 km. (Results, not shown, with forcing by wave drag show the same characteristic.)

Given this result then, provided we restrict attention to the region below 40 km, we may seek solutions in a domain with a rigid lid at 60 km. This opens up an alternative avenue of attack, namely, to expand the solution first in vertical modes of the bounded domain, and then to determine the horizontal structure of these modes. Since our region of interest is the Tropics, it proves an expedient simplification to make the equatorial beta-plane approximation, replacing $1 - \mu^2$ in the first term of (9) by unity and by replacing the boundary conditions at the poles by ones of boundedness as $\mu \to \pm \infty$. Thus, we have replaced the laterally bounded, vertically unbounded atmosphere with one that is vertically bounded but horizontally infinite. This approach—and, indeed, much of the mathematics to follow—is reminiscent of Gill’s investigations (see Gill 1982, chap. 11, and references therein) of equatorial wave dynamics. As we shall see, it allows us to capture the characteristics of the tropical circulation in a very concise way.

### a. Vertical modes of the bounded problem

We begin by expanding the forcing and solution to (9) in the form

$$w(\mu, z) = \sum_n w_n(\mu)Z_n(z),$$

$$\theta(\mu, z) = \sum_n \theta_n(\mu)Z_n(z),$$

$$V_z(\mu, z) = \sum_n V_{zn}(\mu)V_n(z),$$

with

$$V_n(z) = \frac{a}{\rho} \frac{d}{dz} \left[ \rho Z_n(z) \right],$$

where $Z_n(z)$ are the eigenfunctions and $\nu_n$ the corresponding eigenvalues of the vertical operator in (9):

$$h^2 \frac{N^2}{\Delta z} \left[ \frac{d}{dz} \rho \frac{d}{dz} Z_n \right] = -\nu_n^2 Z_n,$$

subject to boundary conditions of no vertical motion at the lid ($Z_n = 0$ on $z = z_{\text{max}}$) and $d(\rho Z_n)/dz = 0$ on $z = 0$. The latter condition is appropriate at the top of the surface drag layer, in the limit of large drag coefficient. The eigenfunctions are normalized according to
The first three functions are plotted in Fig. 9.

From (12) and (13), the horizontal structure problem becomes, from (9) and making the equatorial beta-plane approximation,

\[
\frac{1}{H\rho(0)} \int_{0}^{\text{max}} \rho \frac{N^2}{N_0^2} Z(z) Z_v(z) \, dz = \delta_{\text{np}}. \tag{14}
\]

As we shall be considering different latitudinal structures of thermal and mechanical forcing, we shall consider these two cases separately.

b. Forcing by subtropical wave drag

With no thermal forcing, (16) becomes

\[
P \frac{d^2 v_n}{d\mu^2} - v_n^2 \mu^2 v_n = R \frac{d^2 \theta_n}{d\mu^2} + \mu \Delta_n. \tag{17}
\]

If wave drag is nonzero only poleward of a sharp edge
at $\mu = \mu_0$, the steep gradients of the Green’s function for this problem (see the appendix) ensures that only the drag close to the edge will influence tropical upwelling. That being the case, the tropical circulation will be insensitive to the precise latitudinal dependence of wave drag within the surf zone. A simple case with $2Gm/m > m_0$, $d \Delta_\gamma(\mu) = \left\{ \begin{array}{ll} -\Gamma \mu, & \mu > \mu_0 \\ 0, & \mu < \mu_0 \end{array} \right.$ (18) will therefore suffice to illustrate the solutions. Given (18), then $\psi$ is uniform within the surf zone:$^4$

$$v_\mu = \frac{\Gamma}{\psi_\mu}, \quad \mu > \mu_0.$$ 

In $\mu < \mu_0$, the appropriate solution to (17), bounded as $\mu \rightarrow -\infty$, and matched at $\mu_0$, is

$$v_\mu = \frac{\Gamma}{\psi_\mu} \frac{D_{-1/2}(-\gamma_\mu)}{D_{-1/2}(-\gamma_0 \mu_0)} \equiv \frac{\Gamma}{\psi_\mu} \Psi(\gamma_\mu, \gamma_0 \mu_0).$$

The vertical velocity component is then $w_\mu(\mu) = 0$ in $\mu > \mu_0$, and

$$w_\mu(\mu) = \frac{d}{d\mu} \frac{\psi_\mu}{\psi_\mu} \frac{D_{1/2}(\gamma_\mu)}{D_{1/2}(\gamma_0 \mu_0)} = \frac{\gamma_\mu}{\psi_\mu} \Psi(\gamma_\mu, \gamma_0 \mu_0).$$

\footnote{$^4$ This is, in fact, simply the steady, inviscid balance (2) within the surf zone.
}

in $\mu < \mu_0$. The functions $\Psi(x, x_0)$ and $W(x, x_0)$ are shown in Fig. 10 for $x_0 = \gamma_0 \mu_0 = 0.5, 1, 2, 3$, illustrating once again the sensitivity of the tropical upwelling to the location of the subtropical surf zone edge. For $x_0 \geq 3$, the upwelling is mostly confined to $x > x_0 - 1$, with little at the equator. It begins to spread across the equator only if the edge of the surf zone is close enough to the equator that $x_0 < 2$. Even then, the upwelling peaks quite strongly near the edge of the forcing region. For $x_0 = 1$, the upwelling is more uniform across the equator, within $2 < x < x_0$.

\subsection*{c. Thermal forcing}

With no body force, (16) becomes

$$\frac{d^2 \psi_n}{d\mu^2} - \frac{\psi_\mu^2}{\psi_\mu} \psi_n = -\frac{a}{H \, d\mu} \frac{d\theta}{d\mu}.$$ (21)

In general, solutions to (21) can be obtained by expanding both forcing and response in $D_\mu(\gamma \mu)$, the eigenfunctions of the operator on the left-hand side. These functions have corresponding eigenvalues $\kappa_\mu = -(m + \frac{1}{2})$ and are normalized such that

$$\int_{-\infty}^\infty D_\mu(\eta)D_\mu(\eta) \, d\eta = n! \sqrt{2\pi} \delta_{n \mu}.$$ (22)

(e.g., Abramowitz and Stegun 1972, 775). Since, from (7), $\partial \theta / \partial \mu = -\Delta \mu N_0^2 Z(z) / N_0^2(z)$, where

$$\Lambda = -\frac{2aR}{N_0^2 H} \frac{d\theta}{d\mu} = 6.15 \times 10^{-3} \, \text{m s}^{-1},$$

$\Lambda \ll \mu_0$. The functions $\Psi(x, x_0)$ and $W(x, x_0)$ are shown in Fig. 10 for $x_0 = \gamma_0 \mu_0 = 0.5, 1, 2, 3$, illustrating once again the sensitivity of the tropical upwelling to the location of the subtropical surf zone edge. For $x_0 \geq 3$, the upwelling is mostly confined to $x > x_0 - 1$, with little at the equator. It begins to spread across the equator only if the edge of the surf zone is close enough to the equator that $x_0 < 2$. Even then, the upwelling peaks quite strongly near the edge of the forcing region. For $x_0 = 1$, the upwelling is more uniform across the equator, within $2 < x < x_0$.

\subsection*{c. Thermal forcing}

With no body force, (16) becomes

$$\frac{d^2 \psi_n}{d\mu^2} - \frac{\psi_\mu^2}{\psi_\mu} \psi_n = -\frac{a}{H \, d\mu} \frac{d\theta}{d\mu}.$$ (21)

In general, solutions to (21) can be obtained by expanding both forcing and response in $D_\mu(\gamma \mu)$, the eigenfunctions of the operator on the left-hand side. These functions have corresponding eigenvalues $\kappa_\mu = -(m + \frac{1}{2})$ and are normalized such that

$$\int_{-\infty}^\infty D_\mu(\eta)D_\mu(\eta) \, d\eta = n! \sqrt{2\pi} \delta_{n \mu}.$$ (22)

(e.g., Abramowitz and Stegun 1972, 775). Since, from (7), $\partial \theta / \partial \mu = -\Lambda \mu N_0^2 Z(z) / N_0^2(z)$, where

$$\Lambda = -\frac{2aR}{N_0^2 H} \frac{d\theta}{d\mu} = 6.15 \times 10^{-3} \, \text{m s}^{-1},$$

$\Lambda \ll \mu_0$. The functions $\Psi(x, x_0)$ and $W(x, x_0)$ are shown in Fig. 10 for $x_0 = \gamma_0 \mu_0 = 0.5, 1, 2, 3$, illustrating once again the sensitivity of the tropical upwelling to the location of the subtropical surf zone edge. For $x_0 \geq 3$, the upwelling is mostly confined to $x > x_0 - 1$, with little at the equator. It begins to spread across the equator only if the edge of the surf zone is close enough to the equator that $x_0 < 2$. Even then, the upwelling peaks quite strongly near the edge of the forcing region. For $x_0 = 1$, the upwelling is more uniform across the equator, within $2 < x < x_0$.
it follows from (12) and (14) that \( \frac{d\theta}{d\mu} = -\Lambda \epsilon_n \mu \), where

\[
\epsilon_n = \frac{1}{H \rho(0)} \int_0^{\tau_{\text{max}}} \rho \bar{Z}(z) Z_n(z) \, dz.
\]

With the vertical profile \( Z(z) \) given by (6), the expansion coefficients for the first three vertical modes (\( \epsilon_1, \epsilon_2, \epsilon_3 \)) are (0.158, 0.013, -0.049). Note that the projection is greatest onto the first vertical mode, which is likely to dominate the solution.\(^5\)

Now, since we anticipate that it is the forcing near the equator that is important here, and \( D_1(x) = x \exp(-\frac{1}{2}x^2) \), it is a reasonable simplification to write

\[
\frac{1}{\gamma_n} \frac{d\theta}{d\mu} \equiv -\frac{\Lambda \epsilon_n}{\gamma_n} D_1(\gamma_n \mu).
\]

Then the solution to (21) is similarly confined to the \( m = 1 \) mode and is

\[
v_n(\mu) = -\frac{2a \Lambda \epsilon_n}{3 \gamma_n^2} D_1(\gamma_n \mu)
\]

and

\[
w_n = \frac{2 \Lambda \epsilon_n}{3 \gamma_n^2} D_1'(\gamma_n \mu), \tag{23}
\]

where \( D_1'(x) = D_1(x)/dx = (1 - \frac{1}{2}x^2) \exp(-\frac{1}{2}x^2) \).

Among other things, \( w \) changes sign at \( \gamma_n \mu = \pi \) (or, with our numbers, \( \mu \approx 0.185 \), i.e., latitude 10.6°, for the first vertical mode).

The solution thus reconstructed, for the first vertical mode only, is shown in Fig. 11. This figure should be compared with Fig. 8b. The upwelling is a little weaker and narrower here but, as the comparison makes clear, this single mode solution does indeed capture the full solution rather well.

4. Seasonally varying circulation

a. Northern winter wave drag

As we saw from (3), the steady-state constraints apply to the annual-mean circulation even when the forcing varies seasonally. It is not clear, however, to what extent the seasonally varying component of the circulation reflects the same constraints. Figure 12 shows the vertical velocities at 20-km altitude in response to forcing applied in half the year (meant to represent northern winter) only. There is no spatial or temporal variation in \( T_e \), but, like the cases of Fig. 3, the circulation is driven by an imposed drag \( D = D_0 \bar{Z}(z)B(\phi, \varphi_0)\tau_\alpha(t) \), where \( \bar{Z} \) and \( B \) are given by (5) and (6), \( D_0 = -2.0 \times 10^{-3} \) m s\(^{-2} \), as in the steady cases, but

\[
\tau_\alpha(t) = \begin{cases} 
\sin \left( \frac{2\pi t}{t_e} \right), & \sin \left( \frac{2\pi t}{t_e} \right) > 0 \\
0, & \sin \left( \frac{2\pi t}{t_e} \right) < 0,
\end{cases}
\]

where \( t_e = 1 \) yr. Thus, defining \( t = 0 \) as northern autumnal equinox, the drag—which is located only in the Northern Hemisphere—maximizes at winter solstice, decreases to zero at vernal equinox, and is zero throughout northern summer. Though a crude representation of Northern Hemisphere midlatitude wave drag, this is of approximately the right magnitude and annual variation, and will suffice to illustrate the characteristics of the response.

As in the cases shown in Fig. 3, for the three cases of Fig. 12 the equatorward edge of the drag is shifted progressively closer to the equator. In each case, the
degree of equatorial penetration of the circulation is very much like that of the steady solutions; indeed, despite the unsteadiness of the forcing, the response at any time is quite similar to the steady response. The first 1.5 yr of each integration is shown, with little change between the first and second “winter,” after which the pattern repeats with a regular annual cycle. With the drag terminating at 30°N (Fig. 12a), most upwelling and downwelling occurs almost simultaneously with the imposed drag, with the upwelling near its subtropical edge and the downwelling near and poleward of its poleward edge; some circulation does linger through the summer, mainly in low latitudes. During the development of the drag during fall, there is some penetration of the upwelling to and across the equator (though this is too weak to be visible in the figure), as shown in Fig. 4b of Holton et al. (1995) and in the transient phase of the steady forcing cases shown above, but it is weak and, more to the point in the context of (3), is followed by a corresponding downwelling as the drag declines during spring. Thus, the annual-mean upwelling at the equator is close to zero (in fact, less than 1% of the maximum upwelling near 30°N); from (3), this is further confirmation of the linearity of the circulation. As the equatorward edge of the imposed drag moves farther equatorward, however, the upwelling moves across the equator, just as in the steady case.

With the equatorward edge at 12°N, the upwelling region straddles the equator, without any compensating subsidence at the equator during northern summer. Even in this case, however, the maximum upwelling is still located on the winter side of the equator, just as in the linear equatorial \( \beta \)-plane solutions (cf. Fig. 10).

b. Wave drag in each winter hemisphere

Figure 13 shows the responses to annually varying wave drag imposed in each winter hemisphere. In each case, the imposed stress in the Northern Hemisphere is the same as that shown in Fig. 12c. Imposed wave drag—the mirror image about the equator, lagged by 6 months—is added in southern winter, in case a with one-half the amplitude of northern winter, and in case b, the same as in northern winter.

The results are what one would expect from a linear superposition of the two hemispheric components of the imposed stress. In case b, the response becomes antisymmetric between northern and southern winters, and in each winter looks very similar to the northern winter of Fig. 12c. There is an exception, in the winter subtropics, where the lingering upwelling through early summer (cf. Fig. 12c) renders the upwelling more symmetric about the equator in Fig. 13b, though now with two maxima straddling the equator, that on the winter side of the equator being marginally the more intense. This feature is less apparent in northern winter in case a, simply because the circulation lingering from southern winter is weaker, but in southern winter the upwelling (weak though it is) actually maximizes on the northern side of the equator.

The maximum upwelling in these experiments (max-
c. Wave drag and stratospheric thermal forcing

Figure 13c shows results from a calculation with imposed drag as Fig. 13a, but with stratospheric thermal forcing, through a latitudinal gradient of $T_e$, also included. The equilibrium temperature (the departure from the reference temperature) is specified to be

$$T_e(\varphi, z, t) = \begin{cases} 0, & \varphi < 0 \\ T_{e0} Z(z) \sin\left(\frac{2\pi t}{t_y}\right) \sin\varphi, & \varphi > 0, \end{cases}$$

where $t_y = 1$ yr, $T_{e0} = -30$ K, and $Z(z)$ is the profile given in (6), during northern winter [$\sin(2\pi t/t_y) > 0$]. In southern winter, the mirror image distribution is used. As is evident from Fig. 13c, the impact of the thermal forcing on the circulation magnitude is modest, though by no means negligible. In fact, the response to sea-
sonally varying thermal forcing is considerably larger than the steady response shown in Fig. 6; as is evident from that figure, the timescale for decay of the response to steady forcing is comparable with the seasonal timescale. Thermal forcing produces a small increase in the maximum upwelling in northern winter to about 0.18 mm s\(^{-1}\); the maximum high-latitude downwelling is hardly affected at all. It makes a bigger impact on the overall mass flux; including the thermal forcing (the difference between Figs. 13a and c) increases the upwelling flux through 20 km by 17% in northern winter (\(t = 1.25\) yr), by 36% in southern winter (\(t = 0.75\) yr), and by 21% on the annual average. Perhaps the most significant change, however, is in the structure of the tropical upwelling; there is now a modest but clear bias of the upwelling toward the summer side of the equator.

5. Discussion

a. The importance of weak viscosity

Perhaps the most surprising aspect of the results presented here is the importance of viscosity to the dynamics of the tropical circulation, even when the viscous timescale is greater than a year. The key parameter, in fact, is the ratio, \(P\), of the dissipation rates of angular momentum and thermal anomalies; the equatorial \(\beta\)-plane analysis predicts that viscosity allows flow across the angular momentum contours within a tropical boundary layer whose width is about \(2L_c\), where \(L_c = L_e P^{1/4}\), \(L_e\) being the equatorial Rossby radius based on the height scale of the vertical mode in question (of the order of a density scale height for deep modes). This width is substantial (about 17° of latitude) even for \(P = 0.02\). This phenomenon is one further illustration—of which the quasi-biennial oscillation (QBO) is the best known—of the vulnerability of the tropical atmosphere to weak forces. In middle latitudes, by contrast, strong coupling of the heat and momentum budgets through the meridional circulation ensures that viscosity is negligible whenever \(P\) is small.\(^7\)

In the full meridional circulation model used here, as in other models, some viscosity is numerically necessary. To the extent that effective body forces in the tropical stratosphere can be thought of as dissipative of angular momentum anomalies (we shall return to this below), the rate of this dissipation in the model may be exaggerated. The existence of the QBO implies that the timescale for such dissipation cannot be much less than a year or so (Plumb 1984). The timescale of 1.5 yr estimated for the nonlinear model and used in the linear calculations must therefore be close to an upper limit for the true dissipation rate. Correspondingly, the estimate of the width of the tropical upwelling regime, reaching in these numerical calculations about 15° from the equator (consistent with the linear scaling), must also be an upper limit, given a 10-day thermal timescale. In the lower stratosphere, however, where the thermal dissipation time \(\alpha_T^2\) is longer, the width of the tropical boundary layer will be greater than for a 10-day timescale.\(^8\) Note that the fourth-root dependence of \(L_e\) on \(P\) makes the sensitivity, both to mechanical and thermal dissipation, weak.

Since the model tropical circulation depends so fundamentally on viscosity, is it possible that the whole nature of the circulation is controlled by the model viscosity? The qualitative similarity of the model circulation to that diagnosed in the stratosphere (given that the model representation of momentum dissipation is unlikely to be realistic) suggests otherwise, but the theoretical indications are mixed. Suppose that the wave drag terminates at some fixed subtropical latitude; how would the induced circulation change as viscosity is reduced? The results of the linear calculations of section 3c indicate that the width of the tropical boundary layer would decrease and the upwelling would move off the equator to become concentrated near the subtropical edge of the drag region (i.e., the reverse of the sequence depicted in Fig. 10). The behavior of the nonlinear model suggests another possibility. In Fig. 5 we noted the nonlinearity of the middle- and upper-stratospheric circulation, manifested as the distortion of near-equatorial angular momentum contours at higher altitudes. Numerical constraints preclude us from reducing viscosity further. However, in response to an increase in amplitude of the imposed drag, the nonlinear region of distorted angular momentum contours shifts downward, with no dramatic change in the circulation other than an increase in amplitude. It is conceivable, therefore, that the model viscosity, rather than permitting the tropical circulation to exist at all, is to a first approximation merely constraining the angular momentum contours and keeping the dynamics linear. Note, however, that nonlinearity of the lower-stratospheric circulation appears weak unless the imposed wave drag, and the induced tropical upwelling, become unrealistically large.

b. Is the real lower-stratospheric circulation linear?

These considerations lead us naturally to ask what observations tell us about the linearity of the circulation in the tropical stratosphere. Given our poor quantitative understanding of tropical stratospheric dynamics, it is

---

\(^7\) The length scale at which viscous effects are important in extratropical latitudes is \(-L_e P^{1/2}\); where \(L_e\) is the midlatitude deformation radius based on the local value of the Coriolis parameter. The difference from the tropical value, dominated by the square-root rather than fourth-root dependence on \(P\), makes this scale much smaller than in the Tropics.

\(^8\) In fact, repeating the earlier experiments with \(\alpha_T^2 = 50\) days makes the case of Fig. 3b, with the subtropical edge of the imposed stress at 21°, look like that of Fig. 4a, with upwelling across the Tropics.
difficult to address this directly through analysis of the angular momentum budget. (Analysis of output from general circulation models is unlikely to be definitive, as they suffer the same problem as the nonlinear model used here, namely, inevitable model viscosity.) We can, however, look for the signature of nonlinearity in the angular momentum distribution. If the horizontal gradient of angular momentum is nonzero in the Tropics, then any mean meridional flow must cross angular momentum contours, as in the linear regime of interest here. A nonzero gradient does not preclude upwelling, but since diagnosed mass fluxes decrease with altitude there must be substantial meridional flow out of the tropical stratosphere. A vanishing gradient of angular momentum between the equator and 15° latitude requires the zonal wind at 15° to exceed that at the equator by about 30 m s⁻¹. There seems little doubt that, on a climatological average, this is not the case except very close to the tropopause near the tropospheric subtropical jet. Short-term fluctuations of angular momentum contours are of little significance to this argument, as what matters are the gradients on the timescale taken by a zonal tube of air to travel out to the edge of the tropical boundary layer (at 0.1 m s⁻¹, it takes about 6 months to move from the equator to 15° latitude). Only during the easterly phase of the QBO does it appear at all plausible that the gradient could be weak.

Overall, then, it seems likely that the lower-stratospheric circulation is not nonlinear and, therefore, that the extension of the meridional circulation deep into the Tropics implies some role for an in situ contribution to the angular momentum budget analogous to that played by viscosity in the present model. One should beware, however, of dismissing entirely the possible significance of the QBO modulation of the tropical angular momentum structure. There are equatorial easterlies at some lower-stratospheric height during the greater part of the QBO cycle [see, e.g., the time–height sections of Naujokat (1986)]. Tropical tracer observations (Trepte and Hitchmann 1992) reveal a QBO influence on the tropical meridional circulation; while the deduced circulation anomaly is consistent with that predicted from the internal dynamics of the tropical QBO (Plumb and Bell 1982), one cannot dismiss entirely the possibility that the QBO easterlies provide a tropical conduit for an angular momentum conserving circulation extending deep into the Tropics from the subtropical edge of the winter surf zone, thus modulating the Rossby wave-driven component of the circulation. Further investigation of this is, however, beyond the scope of this paper.

c. What, if anything, does the model viscosity represent?

The results of this analysis show that the model's success in producing a qualitatively reasonable tropical circulation is directly attributable to the weak model viscosity. Perhaps the important conclusion to be drawn, with the real stratosphere in mind, is not so much the dependence of the circulation on an arbitrary and artificial model viscosity as the importance of weak contributions to the angular momentum budget within the Tropics. If the circulation of the tropical lower stratosphere is indeed linear, what real processes might play the role of the model viscosity? Since molecular viscosity is far too weak, eddy processes must be responsible for local violation of angular momentum conservation.

It should first be noted that eddy momentum transport has been represented, to some extent arbitrarily, in two separate ways in these calculations. The effects of Rossby wave breaking have been represented as a fixed, externally imposed drag on the flow. This may have some justification, to the extent that Rossby wave drag is, for the most part, sign definite (easterly, corresponding to a convergence of the Eliassen–Palm flux), though it will in reality be sensitive to the distribution of mean wind [see Garcia (1991) for a procedure by which this dependence might be parameterized]. The model viscosity affects the circulation in a different way. It does not act alone, but nevertheless allows a circulation in the presence of other agencies: for example, diabatic forcing generates zonal wind anomalies, on which viscosity acts, the consequent local contribution to the angular momentum budget then pumping the meridional circulation.

Given our inadequate knowledge of motions in the tropical stratosphere (especially those weak motions that act on timescales as long as a year), it is difficult to make any definitive statements about what kind of eddies may be important. Perhaps the most obvious candidates are vertically propagating equatorial waves or gravity waves, whose likely importance to the momentum budget of the QBO is well known. For gravity waves in particular, despite recent progress, current knowledge of the climatological wave spectrum in the tropical lower stratosphere is inadequate to define their importance to the mean tropical angular momentum budget. Moreover, while the effects of small-scale eddies are sometimes modeled as a viscosity, this is rarely
even qualitatively justifiable. If the tropical atmosphere is subjected to the effects of a broad spectrum of internal gravity waves or equatorial waves, there may be no preferred sign of the wave drag (and their effects are sometimes represented as Rayleigh friction), but the direct effect of such waves may act to amplify, rather than to dissipate, angular momentum anomalies [cf. the theory of the QBO; see Plumb (1984) and references therein].

Recent analyses of tropical tracers put some limits on possible timescales for tracer diffusion. Hall and Waugh (1997) estimate the vertical diffusivity of tracers to be no greater than $0.01 \text{ m}^2 \text{s}^{-1}$ and, therefore, much smaller than the vertical momentum diffusivity used here. While the transport of momentum could be very different from that of tracers, Hall and Waugh’s (1997) analysis implies that lateral mixing is more important in reality for tracers, and this may also be true for potential vorticity transport (which is equivalent to wave drag). Analysis of tropical tracer budgets (Avalleone and Prather 1996; Minschwaner et al. 1996; Mote et al. 1996; Volk et al. 1996; Hall and Waugh 1997) suggest that extratropical air is ingested into the lower-stratospheric “tropical pipe” at such a rate as to dilute tropical air on a timescale of $0.5–1 $ yr. While these studies do not address the mechanism by which this transport takes place, it seems likely that it is effected by large-scale motions. In terms of the distinction made at the beginning of this section, this may correspond to a leakage of surf zone eddies into the Tropics (cf. Polvani et al. 1995) rather than being a manifestation per se of small-scale motions.

**d. Upward influence in the Tropics?**

If small-scale, or otherwise “hidden,” contributions to the angular momentum budget are indeed important in the Tropics, then the concept of “downward influence” (Haynes et al. 1991) needs some qualification. In the strict sense, in the inviscid steady state, the absence of any circulation above the altitude of the imposed drag stems from the impossibility of maintaining steady, inviscid flow across angular momentum contours, and from the absence of any equivalent of the surface boundary layer. As Haynes et al. note, even for unsteady flow, density stratification biases any forced mass circulation to altitudes below the forcing. In the Tropics, however, viscous effects remove the former constraint, in principle. To some extent, the question is a semantic one, since all flow across angular momentum contours must be accompanied by a local drag, whether “imposed” or “viscous,” and so there cannot be flow above the altitude where the drag is felt. However, if we persist with the distinction between the large-scale drag and what is here included as viscosity, imprecise though that distinction may be (and the distinction is important when thermal forcing is considered), then in principle the strict condition of no upward influence must be abandoned in the Tropics.

The Green’s function for the drag-driven circulation was shown in Fig. 7a. Although there is some effect on $w(0^\circ, 20 \text{ km})$ from forcing at lower levels, this effect is weak and decays rapidly with height. Thus, drag-driven equatorial upwelling is likely to be dominated by subtropical wave drag at higher altitudes, and “downward control” remains approximately, if not perfectly, valid. The same statement cannot be made, however, of the response to thermal forcing. This should not be surprising: under steady conditions for which the downward control principle is strictly valid, the circulation driven by thermal forcing vanishes [cf. (9) as $P \rightarrow 0$]. Thus, the existence of a thermally driven circulation and the breakdown of downward control go together. The Green’s function for the response to gradients in $T_e$ (Fig. 7b) shows a somewhat more far-reaching sensitivity to forcing at levels below the equatorial base point than above.

**e. The impact of tropospheric thermal forcing on the stratospheric circulation**

The breakdown of downward control naturally leads one to ask whether and to what extent the tropospheric Hadley circulation can leak upward into the stratosphere. Since the tropospheric circulation is much stronger than that in the tropical stratosphere (tropospheric upwelling velocities are as much as $5 \text{ mm s}^{-1}$), only very weak leakage is needed for it to be a significant component of the stratospheric upwelling. In fact, linear theory predicts extensive upward penetration of a tropospheric, thermally driven circulation. Figure 14 shows the steady circulation calculated by the linear model in the absence of any imposed drag, but with thermal forcing confined to the troposphere; the latitudinally varying equilibrium temperature distribution is specified to be

$$T_e(\varphi, z) = T_{e0}Z_e(z) \sin^2 \varphi,$$

(25)

where

$$Z_e(z) = \begin{cases} \sin \left( \frac{\pi z}{z_T} \right), & z < z_t, \\ 0, & z > z_t, \end{cases}$$

and where $z_T = 15 \text{ km}$ is the nominal tropopause height. In (25), $T_{e0} = -30 K$, a value like that used by Held and Hou (1980) and which (as is evident from Fig. 14) produces a reasonable maximum upwelling (of about $7 \text{ mm s}^{-1}$) in the equatorial middle troposphere.

Immediately obvious from Fig. 14 is the strong penetration of the linear circulation all the way to the upper stratosphere. The upwelling velocity at $20 \text{ km}$ on the equator is about $3 \text{ mm s}^{-1}$, very much greater than the values we obtained with stratospheric forcing and, more significantly, than real stratospheric values. Other experiments (not shown here) with other vertical and latitudinal distributions of tropospheric $T_e$ (but similar tropical gradients) all show the same characteristic.
Unlike the cases with stratospheric forcing, however, the results of these tropospherically forced, linear calculations are very different from the results of nonlinear calculations. [Since the generally accepted paradigm for the tropospheric Hadley circulation is the nonlinear, inviscid theory of Held and Hou (1980), this should not be surprising.] The nonlinear circulation after 1.5 yr in response to the forcing (25) is shown in Fig. 15. The tropospheric circulation (maximum upwelling of about 2 mm s\(^{-1}\)) is weaker than in the linear case, but the most dramatic difference from the linear results is the sharp decrease in upwelling velocity across the tropopause. At 20 km altitude on the equator, the nonlinear model produces upwelling of 0.04 mm s\(^{-1}\), much less than the linear model result but larger than the values produced by stratospheric thermal forcing. The importance of nonlinearity in this case is illustrated by the strong splaying out from the equator of the angular momentum contours, in and above the upper troposphere.

In fact, theoretical arguments suggest that, in the inviscid, nonlinear limit, the stratospheric circulation forced by tropospheric forcing would vanish entirely. Consider an atmosphere with neither drag nor viscosity—thus conserving angular momentum—except perhaps in a surface boundary layer, and in which there are no horizontal gradients of equilibrium temperature above the tropopause. Assume, for simplicity, that the circulation is symmetric about the equator and is as depicted in Fig. 16. Extrema of angular momentum can only exist at the lower boundary (Schneider 1977) so we assume that all streamlines pass through the boundary layer.\(^{13}\) Consider streamlines A and B, along each of which angular momentum density \((M_A, M_B)\) is constant. Since the zonal flow must be weak in the boundary layer, the angular momentum leaving the boundary layer will be dominated by the planetary component; since streamline B is further from the equator than streamline A, it follows that \(M_A > M_B\).

Between points \(\gamma \) and \(\delta\), at the same latitude at the top of the cell, the vertical shear of the zonal flow must therefore be westerly. Now, above the tropopause \(T_e\) is horizontally uniform; the temperature at point \(\beta\), where air is descending, must therefore exceed that at point \(\alpha\) (in steady state, \(T = T_e - wN^2H/\alpha\gamma\)): temperature between \(\alpha\) and \(\beta\) must increase poleward. But, through thermal wind balance, this is inconsistent with the existence of a westerly vertical wind shear. Therefore, the circulation cannot exist above the altitude (here assumed to be the tropopause) at which the gradient of \(T_e\) vanishes.

Like stratospheric thermal forcing, then, it appears that tropospheric thermal forcing will make only a modest (though, again, perhaps not negligible) contribution to the lower-stratospheric upwelling. There are, however, several reasons to defer judgment on how important it might be in practice. For one thing, these calculations are highly simplified and have several obvious shortcomings when compared with the atmosphere. The

---

\(^{13}\) The other possibility—that angular momentum density is uniform throughout the circulation cell—can be shown, with a small modification to the argument, to lead to the same result.
Angular momentum gradient, shown in Fig. 15, is flatter across the tropical upper troposphere than appears to be the case in reality, manifested as the subtropical jets (not shown here, but their positions are evident as the angular momentum maxima near 10° latitude in the upper troposphere) being too far equatorward. Momentum transport by large-scale eddies in the upper troposphere plays a significant role in the vorticity budget (e.g., Sardeshmukh and Hoskins 1988) and inhibits angular momentum conservation by the overturning circulation. Moreover, the modeled tropical angular momentum gradient remains weak throughout the depth of the stratosphere. This also appears at odds with reality, since the subtropical jets are observed to decay rapidly above the tropopause and so, as argued above, the angular

---

14 This reflects the nonlinear, inviscid prediction that the structure above the tropopause must be barotropic, whence the angular momentum contours must be vertical.
momentum gradient in the tropical lower stratosphere must be stronger than this nonlinear calculation predicts. Since it is in the nonlinear, inviscid limit (as opposed to the linear, viscous limit) that upward extension of the circulation is prevented, it is possible that the stratospheric upwelling would be stronger in a more realistic model.

On the other hand, there may be numerical shortcomings in the nonlinear calculation associated with the rapid decay of vertical velocity with height across the tropopause. In the nonlinear model, $w$ decreases by almost two orders of magnitude across a few vertical grid points; in the nonlinear, inviscid limit, it would decrease to zero. Some sensitivity tests suggest that these results are not unduly sensitive to resolution, but the tests are far from definitive.

f. The extratropical pump

The foregoing results suggest the need for some modification to the concept of the extratropical pump in the context of the global circulation. For the drag-driven component of the circulation (which, as we have seen, appears to be dominant), it remains true that it is “wave pumping,” the poleward drift of air consequent on zonal drag, that is the causal agent in driving the circulation. Within the extratropical surf zone, the link between the wave drag and the circulation is indeed as local and as direct as implied by the arguments of Haynes et al. (1991) and Holton et al. (1995). For tropical upwelling, however, it is [as noted by Holton et al. (1995) for the seasonally varying component of the circulation] the wave drag in the subtropics—at the very edge of the surf zone, which in turn will be related to the net entrainment of tropical air across the edge—that is the dominant factor, rather than the extratropical drag as a whole. Within the Tropics, the link between the drag and the circulation is more subtle, the details (including such important details as where the upwelling occurs) depending on very weak factors in the angular momentum budget. Even there, of course, the relationship (1) between angular momentum advection and the total body force $F$ (in the model, the sum of imposed drag and the viscous contribution) is direct in a diagnostic sense—a point to which we shall return below—but the issue is, to what extent can $F$ be sensibly regarded as being independent of the circulation itself? In the model Tropics, and perhaps in the real stratosphere, $F$ within the Tropics is very much a part of the response to the imposed extratropical wave drag, rather than something that can be thought of as being externally imposed. In one sense this is analogous to the role of diabatic heating in the extratropics, which must be regarded the same way (Haynes et al. 1991; Holton et al. 1995). Unlike extratropical diabatic heating, however, weak tropical contributions to $F$ do have a direct impact on the meridional circulation.

The other fundamental departure from the extratropical pump picture is the contribution of thermal forcing as a driving agent. In principle, thermal forcing is on an equal footing with wave drag in the Tropics (and, of course, it is regarded as the dominant process in the tropospheric Hadley circulation). In practice, on the basis of the highly simplified calculations of this paper, it appears to be a modest contributor, but the effects of both stratospheric and tropospheric thermal forcing may have a significant impact on the tropical upwelling, especially with regard to its latitudinal structure and during southern winter, when midlatitude wave drag is relatively weak.

g. Use of the “downward control principle” to diagnose the stratospheric circulation

What do these results, by identifying factors other than midlatitude, large-scale wave drag as important agents controlling the structure and intensity of the stratospheric circulation, imply for the technique of using “downward control” to calculate the circulation and the net stratospheric mass flux? The technique (Holton 1990; Rosenlof and Holton 1993) centers on using (1) to calculate the flow across angular momentum contours, given the body force $F$, together with the application of mass continuity (and a condition of no mass flux at infinite height, whence downward control).

It has already been noted above that the angular momentum budget linking the circulation and $F$ remains valid as a diagnostic statement, the possible causal contribution of diabatic heating notwithstanding. Being able to determine the circulation accurately in this way thus depends on an accurate determination of $F$ and of the angular momentum gradient $\nabla m$, both of which can be affected in principle by either mechanical (drag or viscous) or thermal forcing. (Note, however, that in the linear regime of interest here, changes in $\nabla m$ are negligible.) Quite apart from the issues discussed in this paper, difficulties arise in the calculation within the Tropics, because of limitations in analyzed winds, leading to errors in $F$ and in $\nabla m$, and of difficulties in integrating the mass budget along possibly convoluted $m$ contours. For the latter reason, Rosenlof and Holton (1993) limited their calculation to regions poleward of 15°.

Any contribution to $F$ from small-scale motions (and viscosity, in the model) is likely to be invisible to a calculation based on global stratospheric analyses (and even in one based on general circulation model data, unless care is taken to include them). In the context of the numerical results discussed here, Rosenlof and Holton’s approach would be quite accurate in the drag-driven cases, not because the viscous contribution to $F$ is negligible (we have seen otherwise) but because it is negligible outside the Tropics, and thus the impact of viscosity is to redistribute the upwelling within the Tropics (where the calculation is not made), rather than to make any substantial change to the magnitude of the
circulation. The same is not true, however, of the experiments with thermal forcing included. For example, for the two experiments shown in Figs. 13a,c, a downward control calculation in which \( F \) constitutes the imposed wave drag alone would give the same result in both cases and, thus, make an error of 21% in the annually averaged mass \( \bar{F} \) for the case with thermal forcing, most of the error being incurred in southern winter, when the wave drag is weak and the viscous term makes at least a 10% contribution to \( F \) equatorward of about 25\(^\circ\). Consideration of how important this is to the real stratosphere brings us back to the questions of what (if anything) the model viscosity corresponds to in reality, and what impact a more complete and realistic diabatic forcing would have on the circulation.

Acknowledgments. We thank Tim Hall, Peter Haynes, Adam Sobel, and Darryn Waugh for discussions, and the reviewers for suggested improvements. This work was supported by NASA, through Grants NAGW-1727 and NAG5-2812.

APPENDIX

The Green’s Function

The homogeneous solutions to (15) are in the form of derivatives of the parabolic cylinder functions \( D_{-1/2}(\pm \gamma_n \mu) \), where \( \gamma_n = (4\nu/\mu)^{1/4} \); these functions are discussed and tabulated by Abramowitz and Stegun (1972). Expressed in terms of a Green’s function, the solution can be written [cf. (11)]

\[
W(x, x') = \begin{cases}
  g(x')D_{-1/2}(x')D_{-1/2}(-x), & x < x' \\
  -g(x')D_{-1/2}(-x')D_{-1/2}(x), & x > x',
\end{cases}
\]

(A1)

and where

\[
g(x) = x \left( D_{-1/2}(x)D_{-1/2}(-x) + D_{-1/2}(-x)D_{-1/2}(x) \right)^{-1},
\]

and where \( D' \) denotes the derivative of \( D \). The Green’s function for an equatorial base point, \( W(0, x) \), is plotted in Fig. A1. The function peaks at \( \gamma_n \mu = 1.17 \) and falls off rapidly beyond \( \gamma_n \mu = 2 \). Thus, \( \gamma_n^{-1} = (P/4\nu^2)^{1/4} \) is the (sine–latitude) length scale for the range of influence into the Tropics of forcing by wave drag or by equilibrium temperature gradients. This scale is 0.131, 0.108, and 0.092 for the first, second, and third modes, respectively, corresponding to latitudes of 7.5\(^\circ\), 6.2\(^\circ\), and 5.3\(^\circ\), in agreement with the scales evident in Fig. 7. Note that since \( \nu^2 \sim n, \gamma_n^{-1} \sim n^{-1/2} \) for large \( n \).

REFERENCES


Brewer, A. W., 1949: Evidence for a world circulation provided by

![Fig. A1. The latitudinal Green’s function \( W_n(0, x) \) for an equatorial base point.](image)


