

3D Radiative Transfer in Weakly Inhomogeneous Medium. Part II: Discrete Ordinate Method and Effective Algorithm for Its Inversion

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ABSTRACT

The forward discrete ordinate method (DOM) has been reformulated to include effects of a weak inhomogeneity of a medium. The modification is based on an expansion of the direct beam source term. This treatment of the source term is similar to the gradient correction (GC) method, presented in the first part of the paper for the diffusion approximation. The same requirement on the scales of variations applies to the current method: length of horizontal variations of optical properties of the medium should be large in comparison to the mean radiative transport length. The modification has another important advantage in that it permits obtaining particular solutions for both infrared and direct beam radiative transfer in one computational step.

An effective algorithm for solving inverse radiative transfer problems has been developed following the above reformulation. This algorithm modifies the DOM using Newton's iterative scheme in order to find a solution of the inverse problem. The algorithm convergence rate is very high. Typically, two to three iterations are enough in order to obtain a solution with sufficiently high accuracy. The algorithm can be applied to the plane-parallel radiative transfer as well as to the GC method.

The combination of the GC approach and the effective inverse algorithm creates an extremely useful and efficient tool for extraction of cloud fields from satellite imagery. It allows the algorithm to use radiance data with multiple views of the same pixel as an input and produce correct output even for large solar zenith or satellite view angles, when an independent pixel approximation fails.

1. Introduction

The importance of horizontal flux divergences in the process of radiative transfer in marine stratocumulus clouds (which are, without any doubt, the best candidates for the application of the plane-parallel model) has been pointed out recently by several authors (Marshak et al. 1995; Davis et al. 1997a,b; Loeb and Coakley 1998). These conclusions are based on the Monte Carlo simulations of the fractal model of the cloud field (Cahalan et al. 1994), as well as on the analyses of the observational data. However, due to a complexity of solution techniques of the radiative transfer equation for inhomogeneous medium, rather than relying on scientifically proven justifications, the two widespread methods used in global circulation models and in satellite imagery are based on the plane-parallel treatment of the radiative transfer in clouds, where the horizontal fluxes are neglected.

In a recent paper (Galinsky and Ramanathan 1998)

the simple analytical solution for the problem of radiative transfer in a weakly inhomogeneous medium, which takes into account the horizontal radiative fluxes, has been derived using the delta-Eddington approximation, so-called gradient correction (GC) method. The horizontal fluxes may result either from variations of the geometrical boundaries of the layer, that is, bumps on the top surface, or from variations of the internal optical properties of the medium, that is, dependence of the extinction coefficient on the horizontal coordinate.

Although the solution was supposed to be used for improvement of cloud retrieval from satellite imagery, it cannot be applied directly to remote sensing analyses. Indeed, it is based on the diffusion approximation and, hence, calculates an optical depth from a set of hemispheric fluxes, usually not being measured directly. Most satellite imagers record radiances or bidirectional reflectances and in an inhomogeneous case there is no simple transformation from radiances to hemispheric fluxes similar to an azimuthally averaged bidirectional reflection function (BDRF) method used in the satellite imagery based on the independent pixel approximation (IPA) method (Barker and Liu 1995; Loeb et al. 1997; Boer and Ramanathan 1997). And of course, normalization to the Lambertian reflectance and an isotropy

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assumption are even less applicable in an inhomogeneous case than the azimuthally averaged BDRF method. Therefore, in order to be able to use the GC method for an optical-depth retrieval from satellite imagery a modification of the method is required. The modification should allow one to start from a set of radiances and obtain a corresponding set of optical depths without an assumption that the radiances could be converted into hemispheric fluxes using any of the plane-parallel conversion algorithms, like the above-mentioned BDRF method. Because, the delta-Eddington approximation is a lowest-order version of the discrete ordinate method (DOM) (Chandrasekhar 1950), it seems very convenient to introduce this modification into N streams DOM algorithm.

Another problem, which complicates application of the GC method to remote sensing, is an absence of effective inverse algorithms. The discrete ordinate method is one of the most effective and commonly used methods in the plane-parallel radiative transfer, but currently it can be used only for solving direct or forward problems. All applications of the DOM to remote sensing are currently based on the precomputing of all possible combinations of parameters by the direct method and using obtained tables with a simple interpolation for looking the results up. There is no effective inversion of the DOM even in the simplest plane-parallel case. In an inhomogeneous case, the lookup tables method will become almost impossible to implement due to an increase in a number of parameters. Hence, an effective inverse method is required.

The paper is organized as follows. In section 2 general formulation of the forward DOM is presented with emphasis on four basic steps of a solution process (sections 2a–d). The important new form of the source term appropriate for use in a weakly inhomogeneous medium is presented in section 2b. The derivation of this new form for the direct beam source term in a weakly inhomogeneous media is provided in section 3. In section 4, a procedure for solving an inverse problem of radiative transfer using an iterative inversion of the discrete ordinate method is presented both for the standard one point-of-view radiance data (section 4a) and for the case of a weakly inhomogeneous medium, where the radiances were measured from two points of view (section 4b).

2. Discrete ordinate method

We will follow Stamnes’s (Stamnes and Swanson 1981; Stamnes and Dale 1981; Stamnes et al. 1988) formulation of the discrete ordinate method. The method was originally introduced by Chandrasekhar (1950) and is based on angular discretisation of the equation for the transfer of monochromatic radiation in a plane-parallel medium,

$$\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - J(\tau, \mu, \phi), \quad (1)$$

where $I(\tau, \mu, \phi)$ is the specific intensity of unit pencil of radiation at the level τ along the direction μ, ϕ (τ is the optical depth, μ is cosine of zenith angle, and ϕ is azimuth angle). Here $J(\tau, \mu, \phi)$ is the source function given by

$$J(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^1 p(\mu, \phi, \mu', \phi') I(\tau, \mu', \phi') d\mu' d\phi' + Q(\tau, \mu, \phi). \quad (2)$$

In order to obtain equations of the discrete ordinate method from (1) and (2), the phase function should be expanded in a series of Legendre polynomials, and the intensity and the source term, in a Fourier cosine series (Chandrasekhar 1950). The discrete ordinate approximation can be written as (Stamnes and Swanson 1981)

$$\mu_i \frac{dI_i^m}{d\tau} = I_i^m - \sum_{j=1}^n (C_{i-j}^m I_{-j}^m + C_{ij}^m I_j^m) - Q_i^m, \quad (3)$$

where

$$C_{ij}^m = \frac{\omega}{2} \sum_{l=m}^{2n-1} a_l (2l + 1) \frac{(l - m)!}{(l + m)!} g_l P_l^m(\mu_i) P_l^m(\mu_j). \quad (4)$$

In (4) I_i^m and Q_i^m are Fourier components of the intensity and the source function, respectively; a_j and μ_j are quadrature weights and points ($a_{-j} = a_j$ and $\mu_{-j} = -\mu_j$); g_l ’s are the coefficients of the Legendre polynomial; $P_l(\mu)$ is the expansion of the phase function; and $P_l^m(\mu)$ is the associated Legendre polynomial.

A discrete ordinate solution of (3) may be constructed in four basic steps.

a. Finding eigenvalues and eigenvectors of the algebraic-homogeneous problem

The solution of (3) may be written as

$$\tilde{I}^m(\tau, \mu_i) = \tilde{I}_i^m = \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m g_j^m(\mu_i) e^{-k_j^m \tau}, \quad (5)$$

where k_j^m, g_j^m are eigenvalues and eigenvectors of the algebraic-homogeneous system obtained from (3),

$$\mu_i^{-1} \left[g_i^m - \sum_{j=1}^n (C_{i-j}^m g_{-j}^m + C_{ij}^m g_j^m) \right] = k_i^m g_i^m, \quad (6)$$

and L_j^m are constants of integration to be determined from the boundary conditions.

b. Finding particular solution for the source term

We will write the source term $Q^m(\tau, \mu_i)$ in the following form

$$Q^m(\tau, \mu_i) = \left[\sum_{r=0}^R X_r^m(\mu_i) \tau^r \right] e^{-\tau/\mu_0}, \quad (7)$$

$$X_r^m(\mu_i) = \frac{\omega I_r}{4\pi} (2 - \delta_{m0}) \sum_{l=m}^{2n-1} (-1)^{l+m} (2l + 1) \times \frac{(l - m)!}{(l + m)!} g_l P_l^m(\mu) P_l^m(\mu_0). \quad (8)$$

This particular form is useful because it permits obtaining the particular solutions for both infrared and direct beam radiative transfer in one step. But the most important feature of this form is that it includes effects of weak horizontal inhomogeneity of the medium (Galinsky and Ramanathan 1998). As we will show later, an appropriate choice of I_r will result in accurate treatment of “cloud-top bumps” by the discrete ordinate method.

The particular solution may then be written as

$$I_p^m(\tau, \mu_i) = \left[\sum_{r=0}^R Z_r^m(\mu_i) \tau^r \right] e^{-\tau/\mu_0}, \quad (9)$$

where $Z_r^m(\mu_i)$ will be determined from

$$\sum_{\substack{j=-n \\ j \neq 0}}^n \left[\left(1 + \frac{\mu_j}{\mu_0} \right) \delta_{ij} - C_{ij} \right] Z_r^m(\mu_j) = X_r^m(\mu_i), \quad r = R, \quad (10)$$

$$\sum_{\substack{j=-n \\ j \neq 0}}^n \left[\left(1 + \frac{\mu_j}{\mu_0} \right) \delta_{ij} - C_{ij} \right] Z_r^m(\mu_j) = X_r^m(\mu_i) + (r + 1) \mu_i Z_{r+1}^m(\mu_i), \quad r = R - 1, \dots, 0. \quad (11)$$

c. Imposing top and bottom boundary conditions

In order to find the integration constants L_j^m in (6), the appropriate boundary conditions should be imposed

$$\tilde{I}^m(0, -\mu_i) + I_p^m(0, -\mu_i) = I_i^\downarrow, \quad i = 1, \dots, n \quad (12)$$

$$\tilde{I}^m(\tau^*, \mu_i) + I_p^m(\tau^*, \mu_i) = I_i^\uparrow, \quad i = 1, \dots, n, \quad (13)$$

where I_i^\uparrow and I_i^\downarrow can be either known incident intensities at the boundaries or expressions taking into account reflecting properties of the bounding surfaces.

Using (6) and (9), we may write a system of linear equations for L_j^m :

$$\sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m g_j^m(\mu_i) = I_i^\downarrow - Z_0^m(\mu_i), \quad (14)$$

$$\sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m g_j^m(\mu_i) e^{-k_j^m \tau^*} = I_i^\uparrow - \left[\sum_{r=0}^R Z_r^m(\mu_i) \tau^{*r} \right] e^{-\tau^*/\mu_0}. \quad (15)$$

It should be noted that I_i^\downarrow and I_i^\uparrow may also contain linear combinations of L_j^m .

d. Interpolating directional intensity

Finally, we can obtain expressions for Fourier components of the specific intensity at any view angle using the formal solutions to (1) in integral form (similar to Stamnes and Swanson 1981):

$$I^m(\tau, +\mu) = \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m \frac{g_j^m(\mu)}{1 + k_j^m \mu} [e^{-k_j^m \tau} - e^{-(k_j^m \tau^* + (\tau^* - \tau)/\mu)}] + \sum_{r=0}^R Z_r^m(\mu) \frac{e^{\tau/\mu}}{\mu} \sum_{p=0}^r \frac{r!}{p!} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)^{r-p+1} \times [t^p e^{-t(1/\mu + 1/\mu_0)}]_{\tau^*}^{\tau}, \quad (16)$$

$$I^m(\tau, -\mu) = \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m \frac{g_j^m(-\mu)}{1 + k_j^m \mu} [e^{-k_j^m \tau} - e^{-\tau/\mu}] + \sum_{r=0}^R Z_r^m(-\mu) \frac{e^{-\tau/\mu}}{\mu} \sum_{p=0}^r \frac{r!}{p!} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)^{r-p+1} \times [t^p e^{-t(1/\mu_0 - 1/\mu)}]_0^{\tau}, \quad (17)$$

where

$$g_j^m(\mu) = \frac{\omega}{2} \sum_{l=0}^{2n-1} (2l + 1) \frac{(l - m)!}{(l + m)!} g_l P_l^m(\mu) \times \sum_{\substack{j=-n \\ j \neq 0}}^n a_j P_l^m(\mu_j) g_l^m(\mu_j), \quad (18)$$

$$Z_r^m(\mu) = \frac{\omega}{2} \sum_{l=0}^{2n-1} (2l + 1) \frac{(l - m)!}{(l + m)!} g_l P_l^m(\mu) \times \left[\sum_{\substack{j=-n \\ j \neq 0}}^n a_j P_l^m(\mu_j) Z_r^m(\mu_j) + \frac{I_r}{2\pi} (-1)^{l+m} (2 - \delta_{m0}) P_l(\mu_0) \right]. \quad (19)$$

The specific intensity at any view and azimuthal angle may be calculated from $I^m(\tau, \mu)$ using inverse cosine Fourier transform

$$I(\tau, \mu, \phi) = \sum_{m=0}^M I^m(\tau, \mu) \cos[m(\phi - \phi_0)], \quad (20)$$

and that concludes the process of solving (1) by the discrete ordinate method.

3. Direct beam in weak horizontally inhomogeneous medium

Although the discrete ordinate method has been developed for a plane-parallel medium, we will show in

this section how it can be modified to include effects of weak horizontal inhomogeneity. We can take into account effects of variations of the upper boundary of the layer (as in Galinsky and Ramanathan 1998) in the expression for the direct beam source term Q (see also Gabriel and Evans 1996).

Consider a parallel beam of light with intensity $I^\perp(\mathbf{\Omega}_0)$ incident on a layer,

$$I^\perp(\mathbf{\Omega}_0) = I_{\text{inc}} \delta(\mathbf{\Omega} - \mathbf{\Omega}_0), \quad (21)$$

where $\mathbf{\Omega}_0 = (n_x \nu_0, n_y \nu_0, \mu_0)$ is a unit vector in the direction of the beam, and $\mathbf{n} = (n_x, n_y) = (\cos\phi_0, \sin\phi_0)$ is a unit vector of a projection of $\mathbf{\Omega}_0$ into the horizontal plane.

In order to find the direct beam intensity inside the medium, we have to integrate a three-dimensional radiative transfer equation, neglecting the multiple-scattering term J ,

$$\mathbf{\Omega} \cdot \nabla I(\mathbf{r}, \mathbf{\Omega}) = -kI(\mathbf{r}, \mathbf{\Omega}). \quad (22)$$

The solution is simple and represents an exponential attenuation of the intensity along the path of the beam,

$$\begin{aligned} I(\mathbf{r}, \mathbf{\Omega}) &= I_{\text{inc}} \exp\left(-\int_{r_0}^r k dr\right) \delta(\mathbf{\Omega} - \mathbf{\Omega}_0) \\ &= I_{\text{inc}} \exp(-\tau_d) \delta(\mathbf{\Omega} - \mathbf{\Omega}_0). \end{aligned} \quad (23)$$

For a plane-parallel layer, the direct beam optical depth τ_d and the vertical optical depth τ are simply geometrically related as $\tau_d = \tau/\mu_0$. This relationship is not valid in the general three-dimensional case. The direct beam optical depth τ_d becomes function not only of τ , but will also depend on the position where the beam enters the medium. For the weak horizontally inhomogeneous layer, we may expand $I(\mathbf{r}, \mathbf{\Omega})$ in a series

$$I(x, y, \tau) = I_{\text{inc}} e^{-\tau/\mu_0} \left(1 + \sum_{r=1}^R \tau^r h_r(x, y)\right). \quad (24)$$

After substitution of (24) into (22), we will obtain the following recurrence relationship for determining h_r :

$$\begin{aligned} h_{r+1}(x, y) &= -\frac{1}{r+1} \frac{1}{k} \frac{\nu_0}{\mu_0} (\mathbf{n} \cdot \nabla) h_r(x, y) \\ r &= 1, \dots, (R-1). \end{aligned} \quad (25)$$

Until this point, we did not make any assumptions about type of inhomogeneity present in the layer (it may be created either by horizontal variations of extinction in a layer with constant height or by height variations of a layer with constant extinction). Therefore, the first coefficient h_1 may be found either from geometrical or from optical properties of the medium. For the layer with varying top surface $H_0(x, y)$, it will be [see (24) in Galinsky and Ramanathan 1998]

$$h_1(x, y) = -\frac{\nu_0}{\mu_0^2} (\mathbf{n} \cdot \nabla) H_0(x, y). \quad (26)$$

We will not proceed with the second case of inhomogeneous extinction, because it was shown that for the same scales of inhomogeneity it is less important for calculations of radiative properties than variations in geometry (Loeb and Davies 1996; Galinsky and Ramanathan 1998).

Taking into account that the source term Q and the direct beam I related as

$$Q(x, y, \tau, \mu, \phi) = \frac{\omega}{4\pi} p(\mu, \phi, -\mu_0, -\phi_0) I(x, y, \tau), \quad (27)$$

we may easily see that the expressions (24), (25), and (26) have the same form as the expressions (7) and (8) for the source term Q (where $I_r = I_{\text{inc}} h_r$, $r = 0, \dots, R$, $h_0 = 1$), and can be used for finding solution of radiative transfer in a weak horizontally inhomogeneous medium.

Examples of discrete ordinate solutions using (24)–(26) as a source term are shown in Figs. 1a,b. In Fig. 1a the mean optical depth $\bar{\tau}$ equals to 2, and in Fig. 1b it equals to 8. The solutions plotted for three cases, using 0, 1, and 2 terms in the source term expansion (24), R equals to 0, 1, and 2, respectively. The case $R = 0$ corresponds to the independent pixel approximation solution. The results of a numerical solution of radiative transfer equation using three-dimensional spherical harmonics discrete ordinate method (SHDOM) (Evans 1998) are also shown in the figure. In all cases, Henyey–Greenstein phase function with asymmetry parameter $g = 0.85$ was used and angular resolution $N_\mu \times N_\phi$ was varied from 16×32 to 32×64 . As we can see, an increase in a number of the expansion terms improves the solution, and for $R = 2$ agreement between SHDOM and our solution is very good.

Another numerical test has been performed for a more “realistic” inhomogeneous cloud field, shown in the next figure (Fig. 2a). The cloud field information has been obtained as a result of numerical cloud model–large eddy simulations of stratocumulus performed by Moeng (Moeng et al. 1996; Evans 1998). In Figs. 2c,d, reflectances of this cloud field calculated by discrete ordinate method (using 0 and 1 additional term in source term expansion) are again compared with three-dimensional SHDOM solutions. The cosine of solar zenith angle is 0.3 in both cases, and relative azimuthal angle is equal to 0 and π , respectively. The angular resolution in all cases was $N_\mu \times N_\phi = 16 \times 32$ and the spatial resolution was $N_x \times N_z = 64 \times 30$.

The agreement between three-dimensional SHDOM solutions and corrected DOM solutions is pretty decent, taking into account that the three-dimensional reflectances were calculated using a fully inhomogeneous cloud field, while the corrected DOM results were obtained from column-averaged values. The phase function in the corrected DOM calculations was also replaced by Henyey–Greenstein phase function with effective asymmetry parameter. The variations of the op-

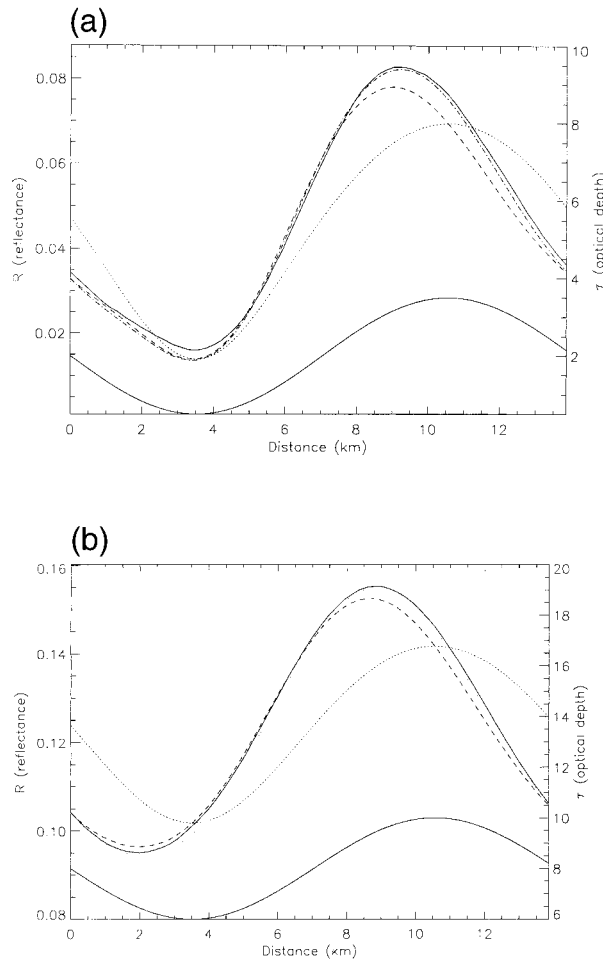


FIG. 1. (a) Plot of nadir reflectances for the gradient correction approximation with 0 (IPA), 1, and 2 terms in expansion (dotted, dashed, and dash-dot curve, respectively) and for 3D spherical harmonics discrete ordinate method (upper solid curve), where $\bar{\tau} = 2$, $\tau(x) = kH_0(x) = k\bar{Z}[1 - \delta \sin(2\pi x/L_x)]$, $k = 7 \text{ km}^{-1}$, $\delta = 0.75$, $\bar{Z} = 0.5 \text{ km}$, $g = 0.85$, $L_x = 14.08 \text{ km}$, $\mu_0 = 0.3$. Optical depth profile shown by bottom curve. (b) The same as (a), but with the following parameters: $\bar{\tau} = 8$, $\tau(x) = kH_0(x) = k\bar{Z}[1 - \delta \sin(2\pi x/L_x)]$, $k = 10 \text{ km}^{-1}$, $\delta = 0.25$, $\bar{Z} = 1.0 \text{ km}$, $g = 0.85$, $L_x = 14.08 \text{ km}$, $\mu_0 = 0.5$.

tical depth τ (shown in Fig. 2b) are resulted from both cloud-top height variations and pixel-by-pixel variations of column-averaged extinction. The gradient correction field $h_1(x)$ (also shown in Fig. 2b by dashed line) has been obtained using (described in the next section) inverse method from two additional sets of reflectances calculated with the cosine of solar zenith angle $\mu_0 = 0.5$ and relative azimuthal angles $\phi_0 = 0$ and $\phi_0 = \pi$. This more realistic cloud field violates the weak inhomogeneity assumption, but correction methods are often useful beyond the range where they are strictly valid.

We also qualitatively compared a behavior of the solution under different viewing conditions with the conclusions made by Loeb and Davies (1997). Their results are based on the analysis of Earth Radiation Budget

Satellite (ERBS) data. Figures 3a,b show reflectances in forward and backward directions, respectively, with all other parameters being the same in both cases. The reflectances in forward directions usually show higher deviations from independent pixel (plane parallel) computations than those in backward directions. Of course, in the results of Loeb and Davies (1997) both solar zenith angle and azimuthal information have been averaged out, but qualitatively this angular behavior of reflectance is in agreement with their findings.

4. Inverse discrete ordinate method

The classic formulation of the discrete ordinate method (section 2) gives us a solution of a forward or direct problem of radiative transfer. In this case all optical (phase function, single-scattering albedo, extinction, and so on) and geometrical (height of the layer) properties of the medium are known, and we can calculate reflected and/or transmitted radiances $I_d(\tau, \mu, \phi)$ at any observational angle and for any optical depth.

There are a wide range of different problems, where radiances at some angle $\tilde{I}(\mu, \phi)$ are known and optical and/or geometrical properties of the medium require determination. This inverse radiative transfer problem is a standard problem in remote sensing. The solution process of inverse problem is usually more complicated than that of direct problem. In addition, the inverse problem is often ill posed, so the solution may be not unique.

In its simplest formulation, the inverse problem requires determination of an optical depth of a plane-parallel layer from a set of radiances measured on top of the layer. The radiances depend not only on view angle μ, ϕ , but also on many other functional parameters, that is, the frequency ν ; the optical depth of the layer $\tau^* = \tau^*(x, y)$; the solar angle $\mu_0 = \mu_0(x, y)$; $\phi_0 = \phi_0(x, y)$; the phase function $p = p(\mu, \phi, \mu', \phi')$; the extinction $k = k(x, y, z)$; the single-scattering albedo $\omega = \omega(x, y, z)$; and so on

$$\tilde{I}(\mu, \phi) = \tilde{I}(\mu, \phi; x, y, \tau^*, \nu, \mu_0, \phi_0, \omega, k, p, \dots). \tag{28}$$

In order to provide a unique solution, some properties of the medium, like the phase function, the single-scattering albedo, or the extinction, are often assumed to be known from other sources. The common approach consists of creating large lookup tables with the radiances precomputed by the direct method for the whole range of every important parameter (Loeb and Davies 1996; Loeb et al. 1997). Even an oversimplified approach to an optical depth retrieval from satellite imagery will require at least a four-dimensional table with sufficiently high resolution [$I = I(\mu, \mu_0, \phi - \phi_0, \tau^*)$]. And in some cases, like for example, in the discussed later two points-of-view retrieval in a weakly inhomogeneous medium, creation of lookup tables may become almost impossible.

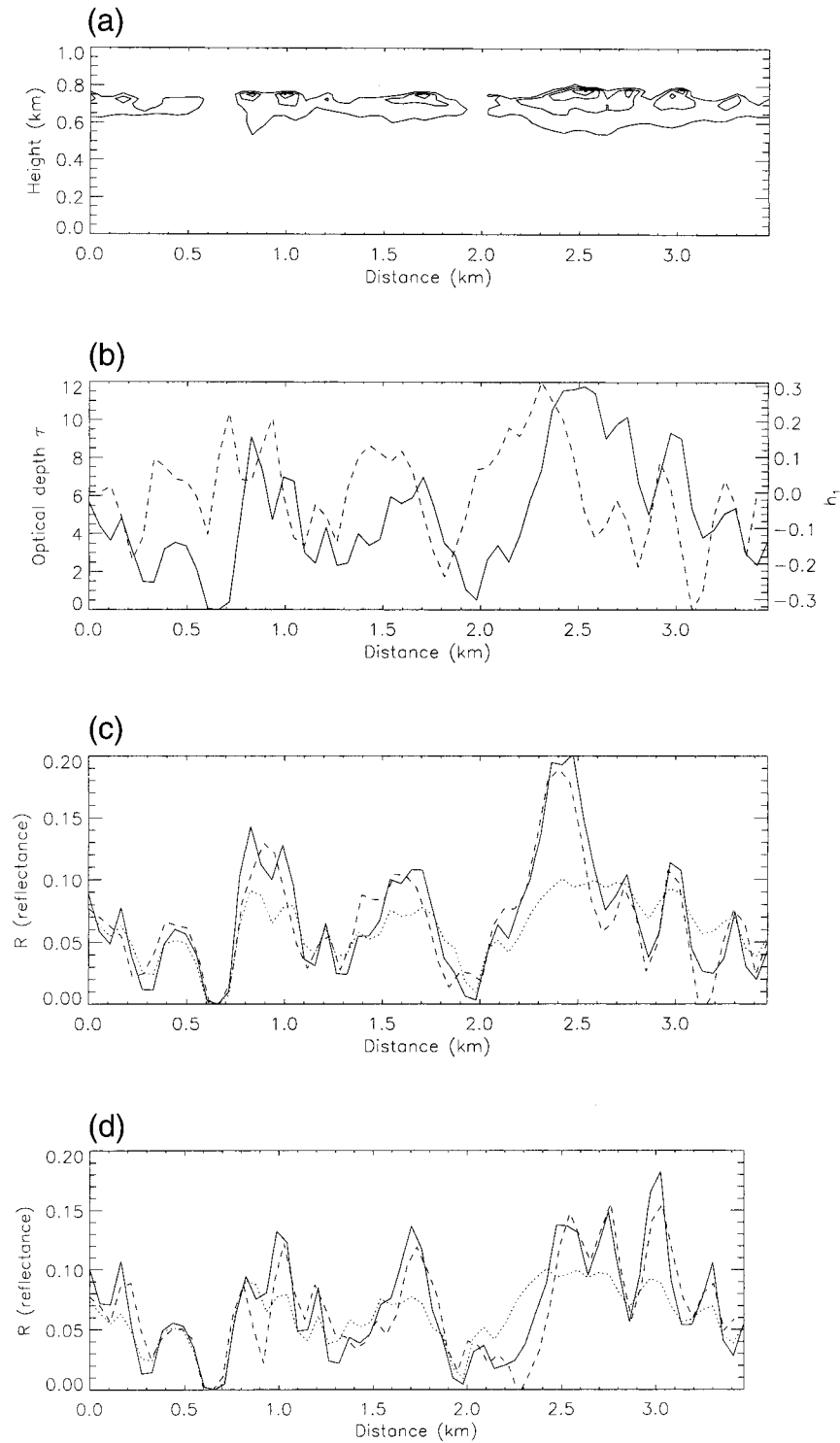


FIG. 2. (a) Contour lines of 2D large-eddy simulation of clouds (Moeng et al. 1996) used as a more "realistic" input cloud field. (b) Plot of the optical depth τ and the gradient correction parameter $h_1(x)$ (dashed line) for the above 2D cloud field. (c) Plot of nadir reflectances for the gradient correction approximation with 0 (IPA) and 1 term in expansion (dotted and dashed curve respectively) and for 3D spherical harmonics discrete ordinate method (solid curve) calculated for the above 2D cloud field with $\mu_0 = 0.3$ and $\phi_0 = 0$. (d) The same as (b), but using different illumination angle $\mu_0 = 0.3$ and $\phi_0 = \pi$.

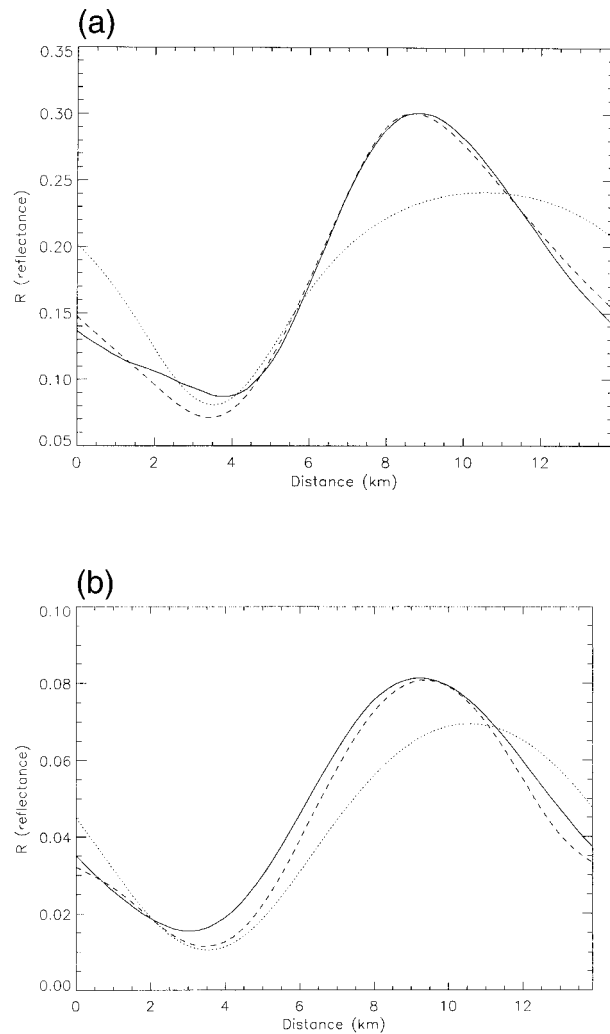


FIG. 3. (a) Plot of forward reflectances for the gradient correction method (dashed), the IPA (dotted), and for 3D spherical harmonics discrete ordinate method (solid), where $\mu_1 = 0.7$, $\phi_0 - \phi_1 = 0$ and all other parameters the same as in Fig. 1a. (b) The same as in (a), but for backward reflectance ($\mu_1 = 0.7$, $\phi_0 - \phi_1 = \pi$).

We will now show how the properties of the forward discrete ordinate method can be used to create an effective inverse algorithm. The method can be applied not only to the simplest plane-parallel case, but to a weakly inhomogeneous medium as well.

a. Weakly inhomogeneous layer, one point-of-view data

Let us consider first the simplest inverse problem. Suppose that we have to find an optical depth of a layer from the measurements of radiance $\tilde{I}(\mu, \phi)$, reflected from the top of the layer in the view direction μ, ϕ . The layer is illuminated by a parallel beam incident at the angle μ_0, ϕ_0 . We will take a phase function of the layer to be known from other sources, as well as an

extinction and a single-scattering albedo. We can create a simple iterative scheme based on Newton's method to find an optical depth of the layer

$$\tau^{*(n+1)} = \tau^{*n} + \frac{\tilde{I}(\mu, \phi) - I_d(\tau^{*n})}{\partial I_d / \partial \tau^* |_{\tau^{*n}}}, \quad (29)$$

where $I_d(\tau^{*n})$ is the radiance calculated by the forward discrete ordinate method (20), using the value of the optical depth of the layer from the previous iteration τ^{*n} .

In order to be able to use this iterative scheme, we have to find out an efficient way to calculate the derivative $\partial I_d / \partial \tau^* |_{\tau^{*n}}$. First of all, following a process of a solution of the forward discrete ordinate method (section 2), one can see that the solution (20) depends on τ^* only in two ways. First is a direct dependence through the Fourier components of the radiances $I^m(\tau, +\mu)$ [Eq. (16)] and $I^m(\tau, -\mu)$ [(17)]. Second is an indirect dependence through the integration constants L_j^m in the boundary conditions Eqs. (14) and (15).

The eigenvalues k_j^m and eigenvectors g_j^m [Eq. (6)], as well as the coefficients in the particular solution for the direct beam $Z_r^m(\mu_i)$ [Eq. (9)], are independent of τ^* . This fact means that the most computationally extensive parts of the discrete ordinate method (see sections 2a,b) do not need re-evaluation on each iteration. Both the particular solution and the solution of the homogeneous problem may be determined only once for the entire iterative process.

Following this discussion we can write linear equations for calculating the derivative in (29). We will move all the equations both for plane-parallel and for the weakly inhomogeneous layer in the appendix and will provide here only the summary of the method.

The inverse DOM solution process may be summarized as follows:

- Preliminary step: finding eigenvalues k_j^m , eigenvectors $g_j^m(\mu_i)$, and coefficients of particular solution $Z_r^m(\mu_i)$ of the forward problem [Eqs. (6), (10), and (12)] and choosing initial τ^* ;
- Iteration step: finding from the boundary conditions both constants of integration L_j^m (14)–(15) and their derivatives $\partial L_j^m / \partial \tau^*$ (A3)–(A4) [or from (A6)–(A7) in an inhomogeneous case]; interpolating Fourier components of directional intensity I^m (16) and their derivatives $\partial I^m / \partial \tau^*$ (A2) [(A5) in an inhomogeneous case]; calculating radiance (20) and its derivative (A1); calculating new τ^* (29) and repeating iteration step if needed.

This iterative scheme is extremely effective. It may be explained by two reasons. First of all, Newton's scheme has a very high convergence rate. Second, a dependence of reflected radiance on optical depth is always a convex, monotonous, and nondecreasing function, hence the radius of convergence is also very large. Therefore, in most cases two or three iterations are

enough to obtain an answer with sufficiently high accuracy.

*b. Weakly inhomogeneous medium,
two points-of-view data*

The iterative scheme and the expressions for derivatives of radiance components and integration constants are only slightly more difficult in the weakly inhomogeneous case, than in the simple plane-parallel case, as can be seen from the appendix. But the inverse problem in the inhomogeneous medium, as it was formulated in the previous section, is not very interesting from a practical point of view. We must specify the parameters of inhomogeneity [I_r in (8) or h_r in (24)] in order to find the solution, and these parameters are usually unknown themselves.

Hence, we will consider here a little bit different inverse problem. We will suppose that a layer with geometrical variations of the top surface (cloud-top bumps) is illuminated by a parallel beam incident at the angle μ_0, ϕ_0 . The radiance reflected from the top of the layer is measured at two view angles (\tilde{I}_1 at μ_1, ϕ_1 and \tilde{I}_2 at μ_2, ϕ_2).

For the case of plane-parallel layer \tilde{I}_1 and \tilde{I}_2 are dependent. We can always express one through the other using the view angles information. In an inhomogeneous case, a relationship between \tilde{I}_1 and \tilde{I}_2 is not so simple and will include some characteristics of inhomogeneity. Using expressions for the direct beam source term in an inhomogeneous medium we can find a solution of this inverse problem that will satisfy both observed radiances. Once again we will for simplicity assume that all parameters except τ^* and h_1 are known from other sources (although we can easily relax this assumption, if we reformulate the problem as a multivariate minimization problem). An iteration scheme in this case will include two variables, τ^* and h_1 . Once again we will move the iteration scheme and the interpolation formulas and all sets of linear equations required to find the derivatives $\partial I_d / \partial h_1$ in the appendix.

To illustrate this two points-of-view retrieval we conducted a numerical experiment. The constant extinction, Gaussian-shaped cloud was illuminated by an oblique beam with $\mu_0 = 0.4$ and $\phi_0 = 0$. The spatial distributions of upwelling radiances in two directions ($\mu_1 = 1$ and $\mu_2 = 0.707$) were calculated once again using three-dimensional SHDOM method. The resulting reflectances are shown in Fig. 4a. These reflectances were afterward used for optical depth retrieval. Figure 4b shows that optical depth profiles independently extracted from each of these reflectance fields using independent pixel approximation (dashed line from $\mu_2 = 0.707$ view and dash-dotted line from nadir view) exhibit strong deviation from the original distribution (dotted line). The maximum optical depth extracted from $\mu_2 = 0.707$ reflectance almost 10 times larger than the original maximum. At the same time, when these two sets of

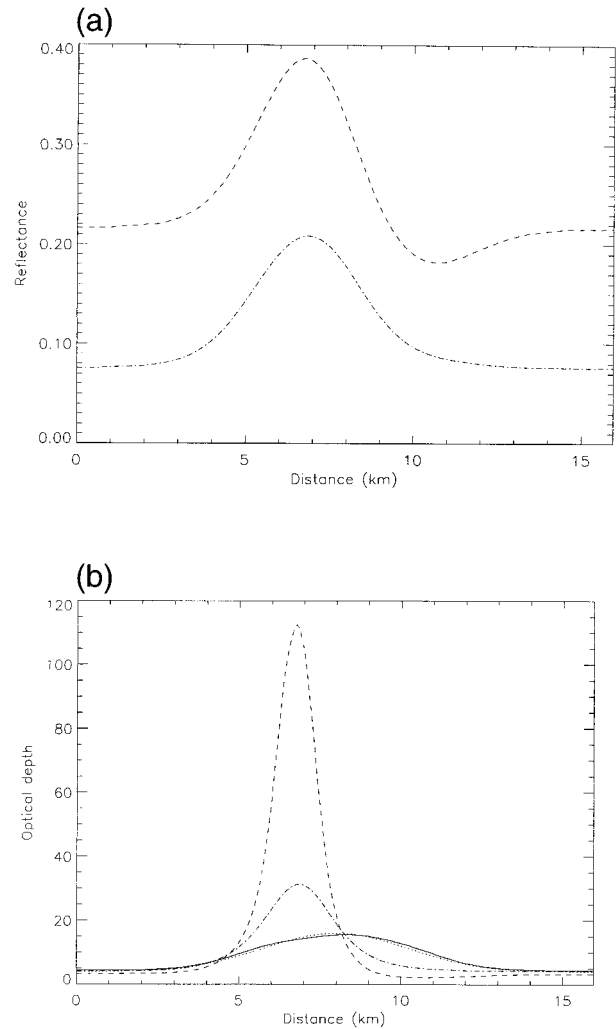


FIG. 4. An example of retrieval of optical depth of Gaussian-shaped cloud using two point-of-view data $\{\tau(x) = kH_0(x) = kZ_{\min} \{ [1 + \delta \exp[-(x - L_x/2)^2/\sigma^2]] \}$, $k = 32 \text{ km}^{-1}$, $\delta = 3$, $\sigma = 1/6 \text{ km}$, $Z_{\min} = 0.125 \text{ km}$, $g = 0.85$, $L_x = 16 \text{ km}$. (a) Spatial distribution of reflectances at $\mu_2 = 0.707$ view (dashed line) and at nadir $\mu_1 = 1.0$ view (dash-dot line) used as input for the retrieval. The cloud was illuminated by a beam with $\mu_0 = 0.4$. Relative azimuthal angle $\phi_0 - \phi$ is 0 in both cases. (b) Optical depths retrieved from the above reflectances using IPA method (dashed and dash-dot lines) and simultaneously from both sets using two point-of-view algorithm (solid line). The original optical depth shown by dotted line.

reflectances are used together in the two point-of-view retrieval algorithm they allow one to extract the original optical depth with much higher accuracy.

Of course, we should emphasize that this example is extremely artificial. There is no such thing as a Gaussian cloud in real life. In order to be able to apply this method to real data more research is needed. In particular, the questions of accuracy of the method and its stability to noise and various errors (e.g., collocation errors or errors due to differences in calibration of two instruments) need to be addressed.

Another important point is that in general it is not a good idea to apply root-finding techniques to multidimensional problems. In order to solve this type of problem correctly, one should reformulate it as a minimization problem, and apply some multidimensional minimization technique. For the above two-dimensional two points-of-view retrieval the conjugate gradient method can be used. At the same time, the nature of this particular problem allows us to apply root-finding technique successfully. Indeed, because we know in advance that reflectance depends monotonically on both the optical depth τ^* and the gradient h_1 , we can be sure that the algorithm will provide a correct solution, if the solution exists, and also will be more effective. But in order to use a modification of this method for determining τ^* , h_1 and unknown characteristics of phase function (e.g., the asymmetry parameter) simultaneously from some additional set of collocated measurements, one will have to use minimization techniques, because of complicated dependence of reflectance on phase function properties. Fortunately, it is easy to make these adjustments.

5. Summary and conclusions

The discrete ordinate method has been reformulated to include effects of weak inhomogeneity of a medium. The modified DOM allows the calculation of both angular and spatial distributions of radiances in an inhomogeneous medium as well as spatial distribution of fluxes. A more simple analytical solution for fluxes based on the diffusion approximation has been reported in the first part of the paper. The DOM's modification is based on the new form for the direct beam source term, which implicitly include horizontal fluxes. These horizontal fluxes may result from variations of the geometrical boundaries of the layer, that is, bumps on top surface.

We will repeat once again the requirements for the validity of this new method, which are the same as for the analytical solution presented in the first part of the paper:

- the characteristic scale of horizontal variations should be large in comparison to the mean radiative transport length, thus the medium is weakly inhomogeneous; and
- the horizontal extent of the medium should be larger than the vertical size, thus the horizontal boundary effects may be neglected.

Note, that the second requirement is not specific for the present method, but it is the requirement for the validity of the plane-parallel-independent pixel approximations as well.

The numerical solution using the inhomogeneous (with gradient corrections) DOM method has been compared to the solution of the three-dimensional radiative transfer equation using the spherical harmonics discrete

ordinate method (Evans 1998), and a good agreement has been found.

The solution accurately describes both the spatial and the angular distributions of the radiative field in a weakly inhomogeneous medium, showing deviations of the angular distribution (dependence of bidirectional reflectance on view angle) from the plane-parallel results similar to those of Loeb and Davies (1997).

Another important result presented here is the creation of an effective way for finding the inverse radiative transfer solution using the discrete ordinate method. The algorithm presented uses an iterative approach and solves the inverse problem as a subset of the forward discrete ordinate problem for derivatives of boundary conditions. It has a very high convergence rate as well as a high radius of convergence. That does not require solving the most computationally extensive part of the DOM, namely, the eigenvalues-eigenvectors problem on every iteration contributes to its efficiency.

In spite of its efficiency this inverse method does require solution of direct discrete ordinate problems, and of course, it cannot be made as fast as a simple search in a precomputed lookup table, especially for high angular resolutions. Therefore, the method is not supposed to replace lookup table calculations used in the independent pixel approximation (under conditions when the IPA already produces both right and accurate results) and is not supposed to be used in operational remote sensing systems, at least without further improvements.

Several acceleration techniques may be easily implemented to further improve efficiency of the method. For example, a modification based on multigrid methods (finding the solution using several angular resolutions simultaneously, both coarse and fine) may be easily applied.

The implementation of this effective inverse algorithm in an inhomogeneous medium creates an extremely useful and efficient tool for the extraction of cloud fields from satellite imagery. One of the most promising applications is optical depth retrieval using two satellites—two fields of view data. The independent pixel calculations will not give any new results in this case, because in the plane-parallel case the entire angular distribution of radiation can be determined from known phase function and reflectance at one view angle. Using the inhomogeneous inverse algorithm, we can verify the plane-parallel assumption and produce correct output optical depth even for large solar zenith or satellite view angles, when the independent pixel approximation fails in an inhomogeneous medium. An extension of the inversion technique to a single viewing angle, but using multiple pixels to perform the gradient correction may also be possible.

An application of this method to the real cloud fields, as well as its validation using either cloud models or simulated fractal clouds will be a subject of future research.

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APPENDIX

Inverse Discrete Ordinate Method

a. Inverse method in plane-parallel case

From (20) the derivative of I needed for iterative improvement of τ^* in (29) is simply

$$\frac{\partial I(0, +\mu, \phi; \tau^*)}{\partial \tau^*} = \sum_{m=0}^M \frac{\partial I^m(0, +\mu; \tau^*)}{\partial \tau^*} \cos[m(\phi - \phi_0)]. \quad (A1)$$

Now from (16) we can easily find required derivatives of the Fourier coefficients I^m (taking into account that in the plane-parallel case all Z_r , except Z_0 equal to 0)

$$\begin{aligned} \frac{\partial I^m(0, +\mu; \tau^*)}{\partial \tau^*} &= \sum_{\substack{j=-n \\ j \neq 0}}^n \frac{\partial L_j^m}{\partial \tau^*} \frac{g_j^m(\mu)}{1 + k_j^m \mu} [1 - e^{-(k_j^m \tau^* + \tau^*/\mu)}] \\ &+ \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m \frac{g_j^m(\mu)}{\mu} e^{-(k_j^m \tau^* + \tau^*/\mu)} \\ &+ \frac{Z_0^m(\mu)}{\mu} e^{-\tau^*(1/\mu + 1/\mu_0)}. \end{aligned} \quad (A2)$$

And finally from (14) and (15) we can obtain a set of linear algebraic equations for unknown derivatives of the constants of integration L_j^m :

$$\sum_{\substack{j=-n \\ j \neq 0}}^n \frac{\partial L_j^m}{\partial \tau^*} g_j^m(\mu_i) = 0, \quad (A3)$$

$$\begin{aligned} \sum_{\substack{j=-n \\ j \neq 0}}^n \frac{\partial L_j^m}{\partial \tau^*} g_j^m(\mu_i) e^{-k_j^m \tau^*} &= \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m k_j^m g_j^m(\mu_i) e^{-k_j^m \tau^*} \\ &+ \frac{1}{\mu_0} Z_0^m(\mu) e^{-\tau^*/\mu_0}. \end{aligned} \quad (A4)$$

Expressions (A1)–(A4) correspond to the last two steps of the forward discrete ordinate method (sections 2c,d, respectively).

b. Inverse method in weakly inhomogeneous medium

Interpolation formula for derivatives of Fourier components of radiance in weak horizontally inhomogeneous medium can be obtained from (16), if we let $\tau = 0$ and differentiate over τ^*

$$\begin{aligned} &\frac{\partial I^m(0, +\mu; \tau^*)}{\partial \tau^*} \\ &= \sum_{\substack{j=-n \\ j \neq 0}}^n \frac{\partial L_j^m}{\partial \tau^*} \frac{g_j^m(\mu)}{1 + k_j^m \mu} [1 - e^{-(k_j^m \tau^* + \tau^*/\mu)}] \\ &+ \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m \frac{g_j^m(\mu)}{\mu} e^{-(k_j^m \tau^* + \tau^*/\mu)} \\ &+ \sum_{r=0}^R Z_r^m(\mu) \frac{1}{\mu} \sum_{p=1}^r \frac{r!}{p!} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)^{r-p} \tau^{*p} e^{-\tau^*(1/\mu + 1/\mu_0)} \\ &- \sum_{r=0}^R Z_r^m(\mu) \frac{1}{\mu} \sum_{p=1}^r \frac{r!}{(p-1)!} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)^{r-p+1} \tau^{*(p-1)} \\ &\times e^{-\tau^*(1/\mu + 1/\mu_0)} + \sum_{r=0}^R Z_r^m(\mu) \frac{r!}{\mu} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)^r e^{-\tau^*(1/\mu + 1/\mu_0)}. \end{aligned} \quad (A5)$$

A set of equations for finding derivatives of the integration constants L_j^m are obtained from boundary conditions (14) and (15) differentiating over τ^*

$$\sum_{\substack{j=-n \\ j \neq 0}}^n \frac{\partial L_j^m}{\partial \tau^*} g_j^m(\mu_i) = 0, \quad (A6)$$

$$\begin{aligned} &\sum_{\substack{j=-n \\ j \neq 0}}^n \frac{\partial L_j^m}{\partial \tau^*} g_j^m(\mu_i) e^{-k_j^m \tau^*} \\ &= \sum_{\substack{j=-n \\ j \neq 0}}^n L_j^m k_j^m g_j^m(\mu_i) e^{-k_j^m \tau^*} + \frac{1}{\mu_0} Z_0^m(\mu_i) e^{-\tau^*/\mu_0} \\ &+ \left[\sum_{r=1}^R Z_r^m(\mu) \left(\frac{\tau^*}{\mu_0} - r \right) \tau^{*(r-1)} \right] e^{-\tau^*/\mu_0}. \end{aligned} \quad (A7)$$

c. Inverse method and two points of view

An iteration scheme for determining an optical depth from two points-of-view data can be obtained from expansions of \tilde{I}_1 and \tilde{I}_2

$$\begin{aligned} \tilde{I}_1 &= I_d(\mu_1, \phi_1) + \frac{\partial I_d(\mu_1, \phi_1)}{\partial \tau^*} \Delta \tau^* \\ &+ \frac{\partial I_d(\mu_1, \phi_1)}{\partial h_1} \Delta h_1, \end{aligned} \quad (A8)$$

$$\begin{aligned} \tilde{I}_2 &= I_d(\mu_2, \phi_2) + \frac{\partial I_d(\mu_2, \phi_2)}{\partial \tau^*} \Delta \tau^* \\ &+ \frac{\partial I_d(\mu_2, \phi_2)}{\partial h_1} \Delta h_1, \end{aligned} \quad (A9)$$

where $\Delta \tau^* = \tau^{*(n+1)} - \tau^{*n}$ and $\Delta h_1 = h_1^{(n+1)} - h_1^n$ and hence new $\tau^{*(n+1)}$ and $h_1^{(n+1)}$ can be found easily as

$$\tau^{*(n+1)} = \tau^{*n} + \left[\Delta I_1 \frac{\partial I_d(\mu_2, \phi_2)}{\partial h_1} - \Delta I_2 \frac{\partial I_d(\mu_1, \phi_1)}{\partial \tau^*} \right] / D, \quad (\text{A10})$$

$$h_1^{(n+1)} = h_1^n - \left[\Delta I_1 \frac{\partial I_d(\mu_2, \phi_2)}{\partial \tau^*} - \Delta I_2 \frac{\partial I_d(\mu_1, \phi_1)}{\partial h_1} \right] / D, \quad (\text{A11})$$

where

$$\Delta I_1 = \tilde{I}_1 - I_d(\mu_1, \phi_1), \quad \Delta I_2 = \tilde{I}_2 - I_d(\mu_2, \phi_2) \quad \text{and} \\ D = \frac{\partial I_d(\mu_1, \phi_1)}{\partial \tau^*} \frac{\partial I_d(\mu_2, \phi_2)}{\partial h_1} - \frac{\partial I_d(\mu_2, \phi_2)}{\partial \tau^*} \frac{\partial I_d(\mu_1, \phi_1)}{\partial h_1}. \quad (\text{A12})$$

Interpolation formulas for derivatives of Fourier components of intensity over the inhomogeneity parameter h_1 can be easily obtained from (16)

$$\frac{\partial I^m(0, +\mu)}{\partial h_1} = \sum_{j=-n}^n \frac{\partial L_j^m}{\partial h_1} \frac{g_j^m(\mu)}{1 + k_j^m \mu} [1 - e^{-(k_j^m \tau^* + \tau^*/\mu)}] \\ + \sum_{r=0}^1 \frac{\partial Z_r^m(\mu)}{\partial h_1} \frac{1}{\mu} \sum_{p=0}^r \frac{r!}{p!} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)^{r-p+1} \\ \times [t^p e^{-t(\mu+1/\mu_0)}]_{\tau^*}^0, \quad (\text{A13})$$

where

$$\frac{\partial Z_r^m(\mu)}{\partial h_1} = \frac{\omega}{2} \sum_{l=0}^{2n-1} (2l+1) \frac{(l-m)!}{(l+m)!} g_l P_l^m(\mu) \\ \times \left[\sum_{j=-n}^n a_j P_l^m(\mu_j) \frac{\partial Z_r^m(\mu_j)}{\partial h_1} \right. \\ \left. + \delta_{r1} \frac{I_{\text{inc}}}{2\pi} (-1)^{l+m} (2 - \delta_{m0}) P_l(\mu_0) \right]. \quad (\text{A14})$$

A set of equations for finding derivatives of the integration constants L_j^m are obtained from boundary conditions (14) and (15) differentiating over h_1

$$\sum_{j=-n}^n \frac{\partial L_j^m}{\partial h_1} g_j^m(\mu_i) = - \frac{\partial Z_0^m(\mu_i)}{\partial h_1}, \quad (\text{A15})$$

$$\sum_{j=-n}^n \frac{\partial L_j^m}{\partial h_1} g_j^m(\mu_i) e^{-k_j^m \tau^*} = - \left[\sum_{r=0}^R \frac{\partial Z_r^m(\mu_i)}{\partial h_1} \tau^{*r} \right] e^{-\tau^*/\mu_0}. \quad (\text{A16})$$

Because the direct beam source term depends on h_1 linearly, the following equations for the derivatives $\partial Z_{\{01\}}^m(\mu_j)/\partial h_1$ can be solved only once (similar to the equations for Z_j^m themselves)

$$\sum_{j=-n}^n \left[\left(1 + \frac{\mu_j}{\mu_0} \right) \delta_{ij} - C_{ij} \right] \frac{\partial Z_1^m(\mu_j)}{\partial h_1} = \frac{\partial X_1^m(\mu_i)}{\partial h_1}, \quad (\text{A17})$$

$$\sum_{j=-n}^n \left[\left(1 + \frac{\mu_j}{\mu_0} \right) \delta_{ij} - C_{ij} \right] \frac{\partial Z_0^m(\mu_j)}{\partial h_1} = \mu_i \frac{\partial Z_1^m(\mu_i)}{\partial h_1}. \quad (\text{A18})$$

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