NOTES AND CORRESPONDENCE

How Do Friction and Pressure Torques Affect the Relative and Ω Angular Momenta of the Atmosphere?

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ABSTRACT

The axial Ω and relative angular momenta, M_Ω and M_r, depend on the meridional mass distribution and relative zonal velocity, respectively. According to the conventional formulation, the time rate of change of M_Ω is determined by the Coriolis conversion induced by meridional velocity, $\mathcal{C}$, whereas M_r is accelerated by $\mathcal{C}$ and the friction and pressure torques.

This note decomposes $\mathcal{C}$ into three components, according to different ways of distributing mass in meridional direction. It is shown that the first two components are identical to the pressure and friction torque, respectively, and the last one equals the conversion induced by the ageostrophic meridional flow, $\mathcal{C}_a$. The decomposition identifies $\mathcal{C}_a$ as the only forcing of M_r. The resulting budgets suggest that the torques change the angular momentum of a rotating fluid with the aid of mass transports, rather than by directly accelerating the rotation speed, as in the case of a rigid body.

1. Introduction

The absolute angular momentum about the earth's rotation axis can be decomposed into the Ω and relative angular momenta, which are functions of meridional distribution of surface pressure and relative zonal velocity, respectively. The equations of these two angular momenta per unit volume, $m_\Omega = \rho a^2 \cos^2\varphi\Omega$ and $m_r = \rho a \cos\varphi u$, are derived from the condition of conservation of mass and the equation of zonal momentum (Peixoto and Oort 1992). When using the traditional approximation (Phillips 1966, 1968), whereby assuming a constant earth's radius $r = a$, these equations read (neglecting furthermore extraterrestrial torques)

$$\frac{\partial m_\Omega}{\partial t} + \nabla \cdot (m_\Omega \mathbf{v}) = -a \cos\varphi 2\Omega \sin\varphi v\rho \quad \text{and} \tag{1}$$

$$\frac{\partial m_r}{\partial t} + \nabla \cdot (m_r \mathbf{v}) = a \cos\varphi 2\Omega \sin\varphi v\rho - \frac{\partial p}{\partial \lambda} + a \cos\varphi\rho F_\lambda. \tag{2}$$

The three-dimensional velocity $\mathbf{v}$ has zonal and meridional components u and v, Ω is the angular speed of the earth's rotation, φ latitude, λ longitude, and ρ the density of the air. The second terms on the left-hand sides of Eqs. (1) and (2) originate from the nonlinear advections and describe the net fluxes of Ω and relative angular momenta across the considered volume. The pressure torque is represented by $\partial p/\partial\lambda$, F_λ is the zonal component of the divergence of the stress tensor, and $a \cos\varphi\, \rho F_\lambda$ is the friction torque. The terms involving $2\Omega \sin\varphi$ can be considered as the torque due to the Coriolis force, $2\Omega \sin\varphi\, v\rho$, acting with moment arm $a \cos\varphi$. It converts, through the meridional flow, relative angular momentum to or from Ω angular momentum and is referred to as the Coriolis conversion. Equatorward (poleward) flow increases (decreases) Ω angular momentum and decreases (increases) relative angular momentum. Note that the Coriolis conversion is proportional to but not equal to the meridional mass transport, ρv.

Integrating Eqs. (1) and (2) over the entire volume of the atmosphere yields the budgets for the global Ω and relative angular momenta, $M_\Omega = \int_V m_\Omega\, dV$ and $M_r = \int_V m_r\, dV$:

$$\frac{dM_\Omega}{dt} = \mathcal{C} \tag{3}$$

$$\frac{dM_r}{dt} = -\mathcal{C} + \mathcal{F} + \mathcal{P}, \tag{4}$$

where the global friction and pressure torques, $\mathcal{F}$ and $\mathcal{P}$, are defined by

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$$\mathcal{F} = \int_S a \cos\varphi \tau_s \, dS \tag{5}$$

$$\mathcal{P} = -\int_V \frac{\partial p}{\partial \lambda} \, dV \tag{6}$$

and the global Coriolis conversion, $\mathcal{C}$, is defined by

$$\mathcal{C} = -\int_V a \cos\varphi 2\Omega \sin\varphi \upsilon \rho \, dV. \tag{7}$$

The zonal component of the surface stress is τ_s, $\int_S dS$ denotes the integral over the earth's surface; and $\int_V dV$ the integral over the volume of the atmosphere.

In the past, interest in the axial angular momentum of the atmosphere has stemmed, to a large extent, from studies of the global budget of the earth system with emphasis on the relationship between changes in the axial component of the atmospheric angular momentum and that of the solid earth (for a review see Rosen 1993). Within this context, only the budget of the global absolute angular momentum, $M_a = M_\Omega + M_r$, is relevant and there is no need to study the separate budgets (3) and (4).

The situation changes for the most pronounced planetary-scale modes of the atmosphere. Using an integration with the coupled atmosphere–ocean general circulation model, it is found that one of these modes has essentially only relative angular momentum but no Ω angular momentum, while two other modes have much larger Ω than relative angular momentum (von Storch 1999a, 2000). The mode with large relative angular momentum has large zonal wind anomalies in the tropical troposphere and is also described by Kang and Lau (1994) using observational data. The two modes with large Ω angular momentum operate in the mid- and high latitudes. They are the modeled Antarctic and Arctic oscillations, which were identified as the most pronounced extratropical modes by Thompson and Wallace (2000). The observation that the dominant planetary-scale modes have, to the first order, either relative or Ω angular momentum, was the initial motivation for studying the separate budgets Eqs. (3) and (4).

How do these budgets work? Consider an eastward stress that results in a positive $\mathcal{F}$. An intuitive suggestion is that $\mathcal{F}$ will accelerate the relative angular momentum. The suggestion is certainly true, if the atmosphere were a rigid body. In this case, an eastward stress will change the relative rotation speed of the body. Since no mass movement is possible within the body, the global Ω angular momentum would remain constant. The budgets (3)–(4) reduce to

$$\frac{dM_\Omega}{dt} = 0 \tag{8}$$

$$\frac{dM_r}{dt} = \mathcal{F} + \mathcal{P}. \tag{9}$$

For the atmosphere as a rotating fluid, the situation becomes complicated. As long as the above considered eastward stress is not located at the equator, it accelerates M_r and, at the same time, induces an equatorward Ekman transport, resulting in a positive $\mathcal{C}$. This, according to Eq. (4), decelerates M_r and cancels, to a yet unknown extent, the direct acceleration by the stress. Thus, the friction torque exerted on a rotating fluid cannot freely change the rotation speed as in the case of a rigid body. Moreover, the torque also affects M_Ω, which is impossible for a rigid body.

This note aims to understand how the friction and pressure torques change the global Ω and relative angular momenta. For this purpose, Eqs. (3) and (4) are reformulated by decomposing the total Coriolis conversion into parts induced by the geostrophic, the Ekman, and the residual ageostrophic meridional velocities, respectively. The conversion induced by the residual ageostrophic meridional velocity is further quantified in section 3.

2. An alternative formulation

The meridional velocity υ in Eq. (7) can be written as

$$\upsilon = \upsilon_g + \upsilon_e + \upsilon_a, \tag{10}$$

where υ_g is the geostrophic velocity defined by

$$\upsilon_g \equiv \frac{1}{\rho 2\Omega \sin\varphi a \cos\varphi} \frac{\partial p}{\partial \lambda}, \tag{11}$$

for $\varphi \in (0, \pi/2)$ or $\varphi \in (-\pi/2, 0)$, and υ_e is the ageostrophic Ekman velocity defined by

$$\upsilon_e \equiv \frac{1}{\rho 2\Omega \sin\varphi} \frac{\partial \tau_{zx}}{\partial z}, \tag{12}$$

for $\varphi \in (0, \pi/2]$ or $\varphi \in [-\pi/2, 0)$. In Eq. (12) τ_{zx} is the vertical eddy flux of zonal momentum near the lower boundary of the atmosphere; υ_a represents the residual ageostrophic flow $\upsilon - \upsilon_g - \upsilon_e$. The definition of υ_g and υ_e can be found in, for example, Peixoto and Oort (1992).

Since υ_g is defined for any φ with $\varphi \in (0, \pi/2)$ or $\varphi \in (0, -\pi/2)$, the Coriolis conversion induced by geostrophic flow υ_g, $\mathcal{C}_g$, is given by

$$\mathcal{C}_g \equiv -\int_V \rho a \cos\varphi 2\Omega \sin\varphi \upsilon_g \, dV = -\lim_{\delta\varphi \to 0} \left(\int_z \int_\lambda \int_{-\pi/2+\delta\varphi}^{0-\delta\varphi} (\rho a \cos\varphi 2\Omega \sin\varphi \upsilon_g) a^2 \cos\varphi \, d\varphi \, d\lambda \, dz \right.$$

$$+ \int_z \int_\lambda \int_{0+\delta\varphi}^{\pi/2-\delta\varphi} (\rho a \cos\varphi 2\Omega \sin\varphi v_g) a^2 \cos\varphi \, d\varphi \, d\lambda \, dz \Bigg)$$

$$= -\lim_{\delta\varphi \to 0} \left(\int_z \int_\lambda \int_{-\pi/2+\delta\varphi}^{0-\delta\varphi} \left(\frac{\partial p}{\partial \lambda}\right) a^2 \cos\varphi \, d\varphi \, d\lambda \, dz + \int_z \int_\lambda \int_{0+\delta\varphi}^{\pi/2-\delta\varphi} \left(\frac{\partial p}{\partial \lambda}\right) a^2 \cos\varphi \, d\varphi \, d\lambda \, dz \right) = -\int_V \frac{\partial p}{\partial \lambda} \, dV$$

$$= \mathcal{P}. \tag{13}$$

Note that even though the *local* expression (11) has singularities at the equator and poles, the *global* expression (13) is well defined.

If the earth's surface would be flat, the Coriolis conversion induced by v_g would be zero, reflecting the known fact that the zonally averaged v_g exists only between longitudinal barriers. Thus, as $\mathcal{P}$, $\mathcal{C}_g$ is different from zero only in the presence of mountains.

Similarly, one finds that the Coriolis conversion induced by the Ekman velocity v_e, $\mathcal{C}_e$, is given by

$$\mathcal{C}_e \equiv -\int_V a \cos\varphi \rho 2\Omega \sin\varphi v_e \, dV = \int_S a \cos\varphi \tau_s \, dS$$

$$= \mathcal{F}. \tag{14}$$

Again, the global expression (14) is well defined, despite the singularity of Eq. (12) at the equator. The term $\mathcal{C}_e$ represents the Coriolis conversion induced by the total Ekman transport.

Equations (13) and (14) are particularly useful for estimating the torques exerted on the ocean. Neglect the residual ageostrophic meridional velocity v_a. The pressure torque exerted on the oceanic interior can be obtained by replacing v with v_g in Eq. (13) (since $v_e \sim 0$ in the oceanic interior and v_a is neglected) and performing the vertical integral from the sea floor to the bottom of the surface Ekman layer. This approximation was used by Ponte and Rosen (1994). The torque exerted by the wind stress at the sea surface, on the other hand, can be obtained by replacing v_e with v in Eq. (14) (when neglecting v_a and $\int_\lambda v_g \, d\lambda$ within the Ekman layer) and performing the vertical integral from the sea surface to the bottom of the Ekman layer. Equations (13) and (14) are less useful for estimating $\mathcal{F}$ and $\mathcal{P}$ exerted on the atmosphere. This is because both v_e and v_g with nonzero zonal integral are located in the lower part of the atmosphere so that a separation of the two is difficult.

Substituting

$$\mathcal{C} = \mathcal{C}_g + \mathcal{C}_e + \mathcal{C}_a \tag{15}$$

with

$$\mathcal{C}_g = \mathcal{P}, \qquad \mathcal{C}_e = \mathcal{F},$$

$$\mathcal{C}_a \equiv -\int_V a \cos\varphi \rho 2\Omega \sin\varphi v_a \, dV \tag{16}$$

in Eqs. (3) and (4), one obtains an alternative formulation of the budgets of M_r and M_Ω:

$$\frac{dM_\Omega}{dt} = \mathcal{F} + \mathcal{P} + \mathcal{C}_a \tag{17}$$

$$\frac{dM_r}{dt} = -\mathcal{C}_a. \tag{18}$$

The new formulations (17) and (18) do not involve additional approximations, but rely only on Eq. (15), which decomposes the total Coriolis conversion $\mathcal{C}$ according to different ways of distributing mass in meridional direction. First, mass can be distributed through meridional geostrophic flows through pressure differences set up between longitudinal topographic barriers. The Coriolis conversion related to this mass transport is identical to pressure torque. Second, mass can be moved in form of Ekman transport induced by the zonal wind stress. The associated Coriolis conversion is identical to friction torque.

Finally, mass can also be transported through an ageostrophic meridional flow v_a. Since the total mass is conserved, v_a must transport mass in the direction opposite to the transport induced by $\mathcal{F} + \mathcal{P}$. For $\mathcal{F} + \mathcal{P}$ as external forcing, v_a can be understood as the adjustment of the atmosphere to changes in the moment of inertia induced by mass movements associated with $\mathcal{F}$ and $\mathcal{P}$. On the other hand, angular momentum variations can also be induced through internal momentum transports. When substituting Eq. (10) into Eq. (2) (whereby neglecting the horizontal eddy fluxes of zonal momentum, τ_{xx} and τ_{yx}), one finds

$$\frac{\partial m_r}{\partial t} + \nabla \cdot (m_r \mathbf{v}) = a \cos\varphi 2\Omega \sin\varphi v_a \rho, \tag{19}$$

which is the local version of the global budget (18). Equations (19) and (18) suggest that the internal momentum transports [which is manifested in $\nabla \cdot (m_r \mathbf{v})$] can only change the global relative angular momentum by initiating a nonzero ageostrophic conversion. This is because the global integral of momentum transport (as the global integral of mass transport) vanishes, but the global integral of Coriolis conversion, which represents a latitudinally weighted mass transport, is generally not zero. The adjustments to the mass transport associated with $\mathcal{C}_a$ produce, within the mass conversation constraint, changes in surface torques. In this case, the atmosphere exerts rather than receives a torque.

The fact that the new formulations (17) and (18) contain only Coriolis conversions as forcing terms suggests that the angular momentum of a rotating fluid cannot be changed through a direct acceleration of rotation speed by the torques, as in the case of a rigid body. Instead, it is changed by Coriolis conversion with the aid of mass movements.

3. The ageostrophic Coriolis conversion $\mathcal{C}_a$

According to Eqs. (17) and (18), the relative strength of $\mathcal{C}_a$ controls whether variations of relative or those of Ω angular momentum are largest. If $|\mathcal{C}_a|$ is comparable to $|\mathcal{P} + \mathcal{F}|$, one would have a nearly complete compensation with $\mathcal{C}_a \sim -(\mathcal{F} + \mathcal{P})$. Equations (17) and (18) would reduce to Eqs. (8) and (9). Variations of M_r would be much larger than those of M_Ω. On the other hand, if $|\mathcal{C}_a|$ is much smaller than $|\mathcal{P} + \mathcal{F}|$, there would be no term that can significantly counteract $\mathcal{F} + \mathcal{P}$ in Eq. (17). Equations (17) and (18) could be replaced by

$$\frac{dM_\Omega}{dt} = \mathcal{P} + \mathcal{F} \quad (20)$$

$$\frac{dM_r}{dt} = 0. \quad (21)$$

Variations of M_Ω would be much larger than those of M_r.

But what determines the strength of $\mathcal{C}_a$? For a small volume element, the answer to the question is suggested by Eq. (2), which is the sum of Eq. (19) and the balance

$$a \cos\varphi 2\Omega \sin\varphi(v_g + v_e)\rho - \frac{\partial p}{\partial \lambda} + a \cos\varphi\rho F_\lambda = 0. \quad (22)$$

It is readily shown that Eq. (22) can be used to approximate the momentum budget, or the ageostrophic conversion in Eq. (19) can be neglected, when the considered flow has a small Rossby number. For the global atmosphere, one has to taken into account that the zonal integral of the pressure torque vanishes in the absence of mountains. Integrating Eq. (22) over a shell of the atmosphere that is located well above the mountains, one finds that all terms in the resulting equation are essentially zero. The integrated Eq. (22) is degenerated and cannot be used to approximate the momentum budget. To the contrary, Eq. (22) integrated over a shell located in the lower part of the atmosphere (where the integrals of friction and pressure torques are not zero) represents a true momentum balance. In this case, the ageostrophic conversion is neglected.

The suggestion that one might neglect the ageostrophic Coriolis conversion in the lower atmosphere is quantified using an integration with the ECHAM1/LSG model (von Storch et al. 1997). In Eqs. (17) and (18) $\mathcal{C}_a$ is replaced with the ageostrophic conversion in the upper atmosphere, $\mathcal{C}_a^u$. The relation between $M_r(M_\Omega)$ and the corresponding forcing $-\mathcal{C}_a^u(\mathcal{C}_a^u + \mathcal{P} + \mathcal{F})$ is studied in terms of phase and squared coherence spectra (Figs. 1 and 2). The lower boundary of the upper layer is chosen to be at 500 hPa, which is roughly the elevation of the summit of Himalaya in the ECHAM model with T21 resolution. The ageostrophic conversion, $\mathcal{C}_a^u$, can be calculated using the full meridional velocity, v^u, since both the Ekman velocity v_e^u and the zonal integral of the geostrophic velocity v_g^u are essentially zero above 500 hPa.

The low coherence at high frequencies (Figs. 1 and 2) may be induced by the sampling errors. The problem leads to low coherence at extremely low frequencies, which was discussed in von Storch (1999b). Apart from these problems, which are not the concern of this note, a highly significant 90° phase relation is found at intermediate frequencies for M_Ω and $\mathcal{F} + \mathcal{P} + \mathcal{C}_a^u$, as

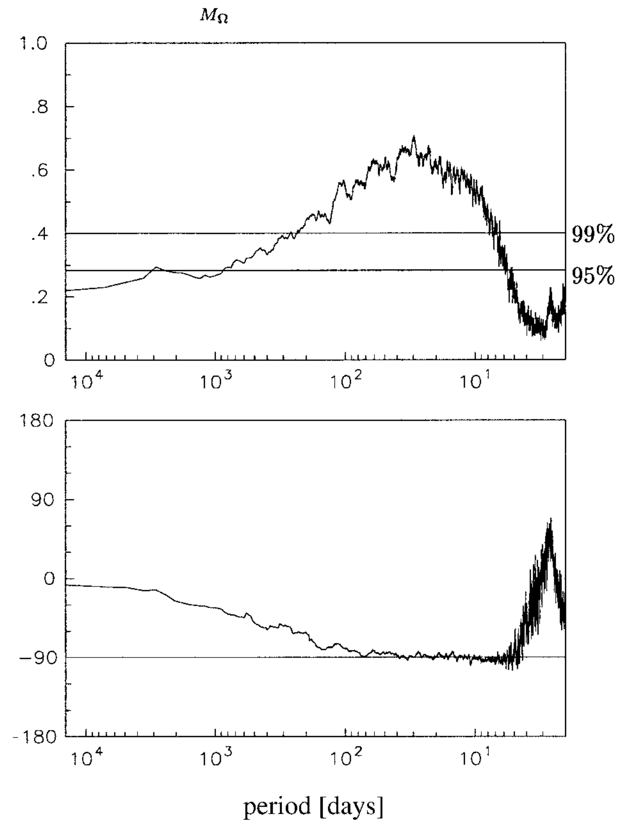

FIG. 1. Squared coherence (top) and phase (bottom) spectra between M_Ω and its approximated forcing $\mathcal{F} + \mathcal{P} + \mathcal{C}_a^u$. Here 200 yr of daily data obtained from an ECHAM1/LSG integration are used. The spectra are first derived from 10 consecutive chunks with the length of a chunk being 7200 and then further smoothed using running average with the average interval being 31. The corresponding 99%, 95% confidence levels are shown as the horizontal lines. The term $\mathcal{C}_a^u$ is derived from the full meridional velocity at 500, 400, 300, 200, 100, 50 hPa. The calculations of M_Ω, $\mathcal{P}$, and $\mathcal{F}$ are described in von Storch (1999a).

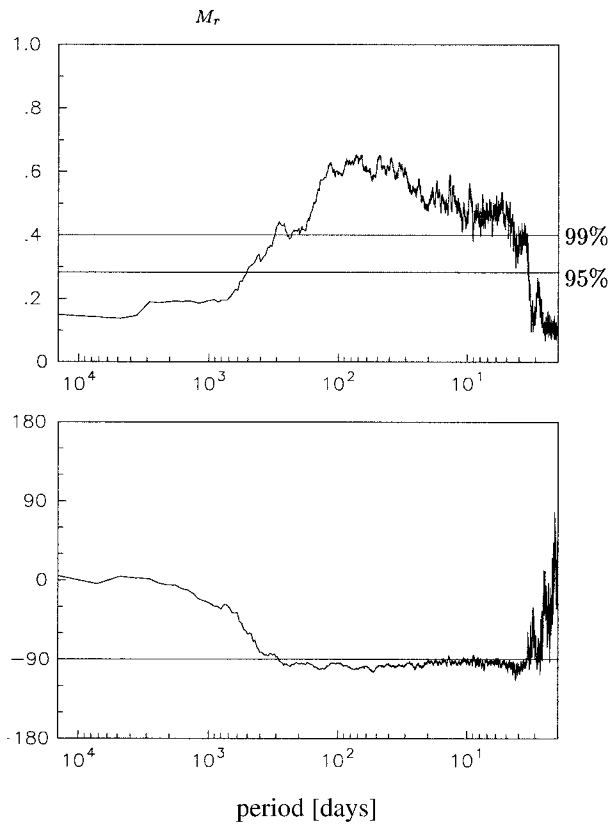

FIG. 2. As Fig. 1, but for squared coherence and phase spectra between M_r and its approximated forcing $-\mathcal{C}_a^u$. The term $\mathcal{C}_a^u$ is calculated as in Fig. 1. The calculation of M_r is given in von Storch (1999a).

well as for M_r and $-\mathcal{C}_a^u$. Thus, at least for the ECHAM atmosphere, the strength of $\mathcal{C}_a$ is essentially controlled by the ageostrophic meridional velocity in the upper atmosphere.

4. Concluding remarks

The budgets of global relative and Ω angular momenta, M_Ω and M_r, are reformulated. The new formulation emphasizes that the angular momentum of a rotating fluid cannot be changed through a direct acceleration of relative rotation speed by the torques. Instead, it is changed by the Coriolis conversion with the aid of mass transports.

According to the new formulation, the relative strength of $\mathcal{C}_a$ controls the budgets of M_r and M_Ω. If the amplitude of $\mathcal{C}_a$ is comparable to that of $\mathcal{F} + \mathcal{P}$, the mass transport induced by $\mathcal{F} + \mathcal{P}$ would be significantly compensated by that related to $\mathcal{C}_a$. Variations of M_r would be larger that those of M_Ω. On the other hand, if $\mathcal{C}_a$ can be neglected relative to $\mathcal{P} + \mathcal{F}$, the torques would produce essentially only changes in Ω angular momentum. For the global (ECHAM) atmosphere, it is found that one can neglect $\mathcal{C}_a$ induced by the ageostrophic velocity in the lower part of the atmosphere. The main conversion that determines the time rate of change of M_r and counteracts the effect of $\mathcal{F} + \mathcal{P}$ originates from the ageostrophic velocity in the upper atmosphere.

When applying the result of this note to the angular momentum budgets of the ocean, one should be aware of the fact that the ocean exchanges angular momentum with the atmosphere at the sea surface and with the earth at the bottom and lateral boundaries. If the inputted angular momentum from the atmosphere is immediately transferred to the earth, as suggested by a model integration (Ponte and Rosen 1994), one would have $\mathcal{F} + \mathcal{P} \sim 0$. In this case, $\mathcal{C}_a$ remains the only forcing in Eq. (17) and cannot be neglected. The variations of M_r would be essentially generated by $\mathcal{C}_a$, and those of M_Ω by $-\mathcal{C}_a$. Amplitudes of variations of M_r would be comparable to those of variations of M_Ω. This issue remains to be further studied using improved ocean models.

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REFERENCES

Kang, I.-K., and K.-M. Lau, 1994: Principal modes of atmospheric circulation anomalies associated with global angular momentum fluctuations. *J. Atmos. Sci.,* **51,** 1194–1205.

Peixoto, J. P., and A. H. Oort, 1992: *Physics of Climate.* American Institute of Physics, 520 pp.

Phillips, N. A., 1966: The equations of motion for a shallow rotating atmosphere and the ''traditional approximation.'' *J. Atmos. Sci.,* **23,** 626–628.

——, 1968: Reply. *J. Atmos. Sci.,* **25,** 1155–1157.

Ponte, R. M., and R. D. Rosen, 1994: Oceanic angular momentum and torques in a general circulation model. *J. Phys. Oceanogr.,* **24,** 1966–1977.

Rosen, R. D., 1993: The axial momentum balance of the earth and its fluid envelope. *Surv. Geophys.,* **14,** 1–29.

Thompson, D. W. J., and J. M. Wallace, 2000: Annular modes in the extratropical circulation. Part I: Month-to-month variability. *J. Climate,* **13,** 1000–1016.

von Storch, J.-S., 1999a: The reddest atmospheric modes and the forcings of the spectra of these modes. *J. Atmos. Sci.,* **56,** 1614–1626.

——, 1999b: What determines the spectrum of a climate variable at zero frequency? *J. Climate,* **12,** 2124–2127.

——, 2000: Angular momenta of the Antarctic and the Arctic Oscillations. *J. Climate,* **13,** 681–685.

——, V. V. Kharin, U. Cubasch, G. C. Hegerl, D. Schriever, H. von Storch, and E. Zorita, 1997: A description of a 1260-year control integration with the coupled ECHAM1/LSG general circulation model. *J. Climate,* **10,** 1525–1543.