ON THE USE OF WINDS IN FLIGHT PLANNING

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ABSTRACT

The first part of the paper reviews the theory of the single-heading flight of an airplane on a plane surface with unchanging geostrophic wind and shows that the simplicity of the formula for the heading of such a flight is lost if the surface is spherical or if the wind field is changing. It is also shown that the single-heading flight is neither necessarily faster nor necessarily slower than the straight-line flight.

In the second part, the problem of determining the quickest flight path between two given points for a given air speed is discussed. When the wind is not everywhere zero, time can frequently be saved by deviating from the great-circle route to take advantage of stronger tail winds or weaker headwinds. This is especially true on longer flights. To determine the paths having the least possible flight time for a given wind field (in general varying in both space and time) is a fairly complicated problem in the calculus of variations. It was first solved by Zermelo in 1930; his solution is applicable, however, only to flat surfaces. The solution has now been extended to cover the case of flight on the surface of a sphere, such as the earth. The solution takes the form of a differential equation which the airplane's heading is to satisfy. A similar equation for flight in three dimensions is discussed.

CONTENTS

The purpose of the present paper is to discuss certain principles relative to the use of observed or forecast winds in the planning of flights. Attention is paid chiefly to the fundamental mathematical problems involved, with the aim of presenting as complete and rigorous solutions as possible.

The principal importance of winds in flight planning is usually regarded as lying in the selection of the quickest route between two given points. In the absence of any wind, an airplane seeking to take the path of shortest time between two points would use the great-circle route, which is the route of shortest distance. However, in the presence of a wind field, the flight time may well be shortened by deviating from the great-circle route to take advantage of stronger tail winds or weaker head winds, the greater length of the route being more than compensated for by the increase in ground speed. In short flights, the deviations from the great-circle route cannot be very large, so that no great advantage over the great-circle route can be obtained, except in unusual situations. In longer flights (over 1000 miles), on the other hand, deviations from the great-circle route may well save considerable flight time. With the steady increase in the effective range of modern airplanes, the possibilities and importance of saving time by suitable choice of the route grow.

While the principles set forth in the foregoing paragraph are universally accepted, little or no attention has been paid, in the ordinary literature on flight planning, to the determination of the exact route which can be flown in the minimum length of time. The problem of choosing a curve so as to minimize some given quantity (in this instance, flight time) is referred to in mathematics as a problem of the calculus of variations. The exact mathematical solution is presented in part II of the present work, for the case where the air speed of the airplane is constant during the flight.

In the last few years, the development of the radio altimeter has stimulated discussion of the problem of determining the geostrophic wind from an airplane in flight. Unexpectedly, the solution of this problem has led to a method of determining a single heading such that an airplane starting from a given origin with that heading will, without change of heading, reach a desired destination. These results have been published. In part I these results are proved in a more concise mathematical form and some of the previously un-

[1] Based on "The minimization of flight-path time" presented by the author on 29 January 1946 at the Annual Meeting, A.M.S., New York. The author is now at the University of Chicago.

proved assertions concerning the properties of this single-heading path are examined.

The reader who is interested principally in the results and the underlying ideas, rather than the mathematical details, is advised to read sections 1 and 7–9.

I. The single-heading flight

1. Summary

Under certain circumstances, it is possible to find a single heading such that an airplane flying on that heading without change will go from a given origin to a given terminus. Assume that the airplane is flying on an isobaric surface (rather than the true spherical surface of the earth), that the wind is geostrophic and unchanging in time, and that the Coriolis parameter is the same everywhere. If g is the acceleration of gravity, Z_1 the height of the isobaric surface on which the flight takes place at the origin of the flight, Z_2 the height of this isobaric surface at the destination of the flight, c the air speed of the airplane, f the Coriolis parameter, D the straight-line distance from the origin to the destination and ω the angle measured clockwise from the straight line joining the origin and destination to the heading (or the angle measured counterclockwise from the heading direction to the straight line), then the heading defined by the equation

$$\sin \omega = g(Z_2 - Z_1)/(cfD) \qquad (0)$$

will suffice for a single-heading flight, provided that the maximum head wind (component of wind in a direction opposite to the heading) is less than the airplane's air speed. This formula was first derived by Bolton, Lambach, and Mansfield (1945).

A single-heading flight has obvious advantages from the point of view of the pilot. An additional advantage for navigation is that one coordinate of the airplane's position can always be determined easily if the airplane is flying a constant-heading course. Let the x-axis be chosen so as to pass through the origin of the flight and to have the direction of the heading. Then y is the perpendicular distance from the airplane's position to this line. Under the various assumptions which have been made,

$$y = (g/cf)(Z - Z_1),$$

where Z is the height of the isobaric surface at the position of the airplane.

A great deal of the usefulness of the single-heading procedure lies in the simplicity of (0), particularly in that ω depends only on the behavior of Z at the end points of the flight. (Otherwise, ω would be dependent on conditions along the route, which in turn depend on ω, so that in general the route would have to be found by trial and error.) It is shown below, by means of examples, that (0) is not valid if a spherical surface is considered instead of a plane, or indeed even if, on the plane surface, the wind is changing with time. In fact, in these cases the heading needed for a single-heading flight does not depend on the behavior of Z at the end points only. It is also shown, by means of examples, that the single-heading flight is not necessarily faster than the straight-line flight.

2. Equations of motion of an airplane

The motion of an airplane relative to the ground may be regarded as made up of two components: the motion of the airplane relative to the ambient air (the proper motion of the airplane) and the motion of the air relative to the ground (wind). Consider that a rectangular coordinate system has been established on the surface on which the airplane is flying, and let x, y be the coordinates. Let c be the air speed, which is the magnitude of the proper motion of the airplane, that is, the speed of the airplane relative to the air. Let θ be the heading of the airplane, that is, the angle made with the x-direction by the direction of the airplane's proper motion; θ is considered as increasing in a counterclockwise direction (this is the usual convention in mathematics, but in navigation the heading is taken as increasing in a clockwise direction). Then $c \cos \theta$ and $c \sin \theta$ are the x- and y-components of the airplane's proper motion, respectively. Let u and v be the x- and y-components of the wind, respectively. In general, u and v depend on position and time; let t stand for time (position is defined by x and y). Finally, let dx/dt and dy/dt be the x- and y-components of the movement of the airplane relative to the ground. (The differential symbol d will refer to rates of change as observed on the airplane.) The motion of the airplane must then satisfy the following equations, according to the previous remarks:

$$dx/dt = u + c \cos \theta,$$
$$dy/dt = v + c \sin \theta. \qquad (1)$$

3. Determination of wind from an airplane in flight

This problem will be discussed chiefly for its bearing on the theory of the single-heading flight. Bellamy (1945, pp. 53–54) derived a formula for determining the component of wind (assumed to be geostrophic) normal to an airplane's track. For the present purposes, it is more convenient to use the formula for the wind component normal to an airplane's heading.[2]

At any given point, the coordinate system mentioned in section 2 will be chosen so that the x-axis points in the direction of the airplane's heading. Then

[2] AAF Weather Service Weather Information Branch, Special Study No. 638, 1943, *Use of the Radio Altimeter in Determining Wind and Drift While in Flight;* also, AAF Weather Service Report No. 708 or 708 (Revised), 1944, *Determination of Absolute Height and Wind for Aircraft Operations*, pp. 51–58.

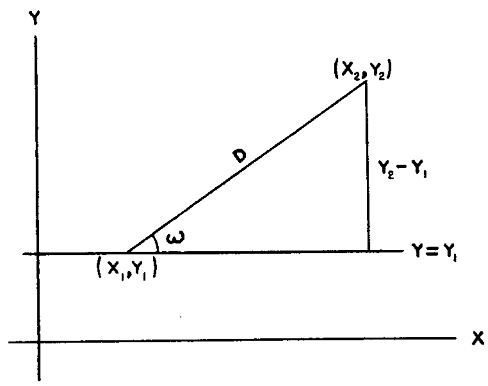

FIG. 1. Single-heading flight on a plane surface.

$\theta = 0$. From (1), the track of the airplane satisfies the equations

$$dx/dt = u + c, \quad dy/dt = v. \tag{2}$$

The wind is assumed to be geostrophic. Assume further that the airplane is flying on an isobaric surface (so that the pressure altimeter gives a constant reading). Then

$$u = -gf^{-1}\,\partial Z/\partial y, \quad v = gf^{-1}\,\partial Z/\partial x. \tag{3}$$

Since d denotes differentiation with respect to the moving airplane, dZ/dt is the rate of change of height above sea level (as measured, *e.g.*, by a radio altimeter) observed on the airplane as it flies along an isobaric surface. In general, Z depends on x, y, and t; however, it is here assumed not to change with time (*i.e.*, the contour lines of the isobaric surface do not move). Then

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt}. \tag{4}$$

Substitute from (3) into (2):

$$\begin{aligned} dx/dt &= -(g/f)\partial Z/\partial y + c, \\ dy/dt &= (g/f)\partial Z/\partial x. \end{aligned} \tag{5}$$

Substitute from (5) into (4):

$$\frac{dZ}{dt} = c\frac{\partial Z}{\partial x} = \frac{cf}{g}v. \tag{6}$$

By solving for v, the wind component normal to the airplane's heading is found. The wind component normal to a second heading can be found, and the wind is then determined from the two wind components.

To repeat, it has been assumed in this derivation that the wind is geostrophic and that the height of the isobaric surface does not change with time in the neighborhood of the point at which the observations are made. It has not been assumed that the surface has any particular form; since only differentials are involved, the arguments are equally valid for a plane, a sphere, or any other two-dimensional surface.

4. Determination of the single-heading flight on a plane surface

The problem here is to determine a single heading such that, if the airplane flies with this heading without change, starting from a given origin, it will arrive at a desired destination. Bolton, Lambach, and Mansfield (1945) derived a formula for this heading under the assumption that such a heading exists. Their treatment will be restated here, and some further results obtained.

Substitute for v from (2) into (6):

$$\frac{dy}{dt} = \frac{g}{cf}\frac{dZ}{dt}. \tag{7}$$

Suppose that the airplane flies from the point (x_1, y_1) at time t_1 to the point (x_2, y_2) at time t_2 with a constant heading. Equation (7) holds at each point of the flight if the x-direction is always in the direction of the heading. In order that this direction may be the same from point to point, the airplane is assumed to fly on a plane surface, and this assumption must be added to those previously made. (This point is not brought out clearly by Bolton, Lambach, and Mansfield.) It is also assumed that the Coriolis parameter is constant; in effect, the earth as a rotating sphere has been replaced by a rotating disc.

Integrate both sides of (7) over the path of the airplane, assuming the heading is such that the destination will be attained.

$$y_2 - y_1 = (g/cf)(Z_2 - Z_1). \tag{8}$$

From fig. 1, $y_2 - y_1$ is the perpendicular distance from (x_2, y_2) to the line $y = y_1$ (which represents the path which the airplane would follow with the given heading if there were no wind). Then,

$$\sin\omega = (y_2 - y_1)D^{-1} = g(Z_2 - Z_1)(cfD)^{-1}, \tag{9}$$

from equation (8), so that the necessary heading can easily be found. It is noteworthy that the heading depends only on the values of Z at the end points of the flight.

This proof has been predicated on the assumption that there is a heading which will satisfy the desired conditions. It is not hard to prove, however, that the heading given by (9) will in fact cause the airplane to fly between the two desired points, if we add the further assumption that the maximum head wind attained, say w, is less than the air speed of the airplane; that is, $u \geqq -w > -c$ every where. Then, from (2),

$$dx/dt = u + c \geqq c - w > 0.$$

Integrate from t_1 to t:

$$x \geqq (c - w)(t - t_1) + x_1.$$

This inequality implies that x increases indefinitely with time; as $x = x_1$ when $t = t_1$; $x = x_2$ for some value of t. Let the airplane cross the line $x = x_2$ at (x_2, y_3), and suppose $y_2 \neq y_3$. Equation (7) is valid here; by integrating over the path from t_1 to the time when $x = x_2$, it is found that

$$y_3 - y_1 = (g/cf)(Z_3 - Z_1), \qquad (10)$$

where Z_3 is the height of the constant-pressure surface at (x_2, y_3). But the direction of the heading has been chosen so as to satisfy (9), and therefore (8) also holds. Subtract (8) from (10) and solve for c:

$$c = gf^{-1}(Z_3 - Z_2)(y_3 - y_2)^{-1}. \qquad (11)$$

But the term on the right is the average wind component against the heading (head wind) between the points (x_2, y_2) and (x_2, y_3). Equation (11) asserts that this average is c; but the head wind is nowhere greater than w, and $w < c$ by assumption, so that the average must be less than c. Therefore, $y_3 = y_2$, and the desired destination will be attained.

To sum up, it has been proved that for unchanging geostrophic wind on a rotating disc with maximum head wind less than the air speed of the airplane, (9) gives the one and only heading for a single-heading flight. The assumptions regarding the wind field may be stated more generally as the existence of an unchanging streamline flow on a plane, with maximum headwind less than the air speed of the airplane; formula (9) can be modified by replacing $f^{-1}gZ$ by the stream function. (The divergence of the flow is then always zero.)

Bolton, Lambach, and Mansfield are not entirely clear as to the properties asserted to hold for this single-heading path. There are advantages for navigational purposes, as indicated by them. Flying without change of heading is clearly a great convenience to the pilot. A single-heading flight also simplifies the work of the navigator, for (10) is valid at any point of the route, so that one coordinate of the airplane's position may be determined simply by reading the radio altimeter.

However, the authors also imply that the single-heading flight is the most advantageous from the viewpoint of time (Bolton, Lambach, and Mansfield, 1945, p. 20). The path requiring least time is derived in part II of the present paper, and the result will show that this path coincides with the single-heading flight only in particular cases. An example (suggested by Dr. George E. Forsythe) shows that the single-heading path is not necessarily quicker even than the straight-line path joining the origin and destination. Suppose the height of the constant-pressure surface has the form

$$Z = (f/g)[\tfrac{1}{2}(x^2 - y^2) - y] + \text{constant}.$$

Then, from the equations of the single-heading flight between the points $(-1, 0)$ and $(1, 0)$ and from the equations of the straight-line flight between the same two points, it can be deduced that the latter flight requires less time.

The wind field just discussed was one of pure deformation. It is also of some interest to consider a case of pure vorticity. (This example also is due to Dr. Forsythe.) Let

$$Z = (f/2g)(x^2 + y^2) + \text{constant},$$

and again consider a flight from $(-1, 0)$ to $(1, 0)$. In this case, the single-heading path is quicker than the straight-line path.

These two examples indicate that the single-heading path will probably be quicker than the straight-line path when vorticity, in some sense, outweighs deformation, and not otherwise.

5. The single-heading flight on a sphere

It was mentioned above that, apart from any question of speed, the single-heading path on the plane has certain advantages from the viewpoint of navigation. These are essentially of the nature of conveniences and must be weighted against the difficulty of determining the heading. Formula (9), valid on the plane surface, is extremely simple in view of the fact that it depends only on the values of Z at the end points. It can be shown, by means of an example, that on a sphere not only is (9) not valid but the heading needed for a single-heading flight is not determined by the values of Z at the end points alone.

Let ϕ and λ be latitude and longitude, respectively. The airplane is to fly from the point (ϕ_0, λ_0) to the point (ϕ_1, λ_0) on the same meridian. First, suppose Z has the distribution

$$Z = [(Z_1 - Z_0)\cos\phi - Z_1\cos\phi_0 + Z_0\cos\phi_1] \times (\cos\phi_1 - \cos\phi_0)^{-1}.$$

Let θ_0 be the heading of the corresponding single-heading flight measured from the meridian at each point of flight. Then consider a second distribution of Z,

$$\begin{aligned} Z = {} & [(Z_1 - Z_0)\cos\phi - Z_1\cos\phi_0 + Z_0\cos\phi_1] \\ & \times (\cos\phi_1 - \cos\phi_0)^{-1} - (2c\omega a g^{-1})\{\lambda\sin 2\phi \\ & + [(\lambda_0\sin 2\phi_0 - \lambda_0\sin 2\phi_1)\cos\phi \\ & + \lambda_0\sin 2\phi_1\cos\phi_0 - \lambda_0\sin 2\phi_0\cos\phi_1] \\ & \times (\cos\phi_1 - \cos\phi_0)^{-1}\}. \end{aligned}$$

At each end point of the flight, the values of Z are the same under the two distributions; yet it may be shown that the single-heading flight with heading θ_0 will not take an airplane from (ϕ_0, λ_0) to (ϕ_1, λ_0) if the wind field is given by the second distribution of Z. Hence, θ, the heading of a single-heading flight, cannot be given by a formula which depends only upon the values of Z at the end points.[3]

6. Variation of the wind field in time

Up to this point, it has been assumed that Z and therefore u and v are unchanging in time. We will now consider again the case of the plane surface and remove this restriction. It is easy to derive a formula for the heading of a single-heading path in this instance. Returning to the derivation in section 3, equation (4) is replaced by

$$\frac{dZ}{dt} - \frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt}.$$

The derivation then proceeds as before with $(dZ/dt - \partial Z/\partial t)$ replacing (dZ/dt). Equation (8) then becomes

$$y_2 - y_1 = \frac{g}{cf}\left[Z_2(t_2) - Z_1(t_1) - \int_{\mathrm{C}} \frac{\partial Z}{\partial t} dt \right],$$

where $Z_1(t)$ is the value of Z at the origin of the flight at time t and $Z_2(t)$ is the value of Z at the destination at time t and where the integration is carried out along the path C of the airplane. As

$$Z_2(t_2) = Z_2(t_1) + \int_{\mathrm{P}_2} \frac{\partial Z}{\partial t} dt,$$

where this last integral denotes the integral from t_1 to t_2 taken at the destination P_2, this equation may be written

$$y_2 - y_1 = \frac{g}{cf}\left[Z_2(t_1) - Z_1(t_1) + \left(\int_{\mathrm{P}_2} - \int_{\mathrm{C}} \right) \frac{\partial Z}{\partial t} dt \right]. \quad (13)$$

A formula corresponding to (9) can be derived. In the particular case where u and v are unchanging with time, $\partial Z/\partial t$ must be independent of position. The two integrals in (13) are then equal, and we return to (8) and (9), with Z_0 and Z_1 evaluated at the starting time. The results of section 4 are thus valid when the condition that Z be unchanging in time is replaced by the weaker condition that the wind field be unchanging in time.

The form of (13) suggests immediately that in the general case the heading cannot be determined exclusively by the values of Z at the end points. This conjecture can be verified by an example. If

$$Z = (f/2g)(x^2 + y^2 + 2ty) + \text{constant} \quad (14)$$

and $c > 2$, the values of Z at the points $(-1, 0)$ and $(1, 0)$ are the same as for (12); yet an airplane starting from $(-1, 0)$ with a suitable constant heading would arrive at $(1, 0)$ with wind field (12) and not arrive at $(1, 0)$ with the same heading with wind field (14).

[3] This result does not contradict those of Gringorten (1948), since his equation (14) for the single-heading flight on the sphere, where, indeed, the heading is dependent only on the heights of the isobaric surface at the endpoints, is derived on the basis of three approximations, those involved in passing from (10) to (10a), from (13) to (13a), and from (13a) to (14). (The numbers refer to formulae in Gringorten's paper.)

II. The flight path of least time

7. Introduction

In this part, the problem is that of determining which, among the various steerable routes available to an airplane going from a given origin to a given terminus, will require the least amount of time. This route will be referred to as the minimal path. It will first be assumed that the airplane is restricted to traveling at a fixed altitude above the surface of the earth (so that its path must lie on a two-dimensional spherical surface) and at a fixed air speed. This problem has received a certain amount of attention in the literature of the calculus of variations, especially since a very similar problem appears in connection with optics. According to Fermat's principle, light always travels in such a way as to require the least time. As far back as 1918, Frank pointed out the close mathematical relation between the problem of finding the minimal path of an airplane and that of finding the path of a light ray in a moving medium. The determination of the minimal flight path (under the conditions specified above) was first accomplished by Zermelo in a lecture at Prague (1930). A fuller exposition was given by Zermelo in a later paper (1931). Zermelo's solution is an extremely elegant one and uses highly original methods. Subsequently, Levi-Cività (1931) derived Zermelo's result along more orthodox lines, very similar to those which had been used in optical problems.

Zermelo's results, however, apply only to plane surfaces. It is necessary to generalize his results to the two-dimensional spherical surface to make them applicable to the navigation of aircraft over the surface of the earth.

An informal discussion of the problem and a presentation of the solution are given first; they are followed by the complete mathematical derivation. For clarity of exposition, the informal discussion takes up Zermelo's solution for the plane before proceeding to the case of the sphere.

8. Informal discussion of the plane case

On the plane surface on which the flight is to take place, consider that a rectangular coordinate system x, y has been imposed. As in section 2, the motion of the airplane satisfies the equations (1). Each route is defined by a function which states the heading θ of the airplane at any time t and by the conditions that the airplane start from a given point P_1 at time t_1 and pass at some time through a given point P_2. In the present case, we consider u and v, the x- and y-components of the wind respectively, to be arbitrary functions of x, y, and t; that is, we do not require that they be steady with time or space. It is only required that u and v possess continuous derivatives with respect to x, y, and t. The air speed c is assumed to be constant. Then Zermelo's solution takes the form of an equation which must be satisfied by the function $\theta(t)$ in order that the path be minimal. To give this solution its clearest form, consider the rectangular coordinate system to be so chosen that the x-axis is in the direction of the heading of the airplane at a given point of the minimal path. The speeds u and v are then the tail wind and cross wind, respectively. Zermelo's result is that

$$d\theta/dt = -\,\partial u/\partial y \tag{15}$$

at that point of the minimal path.

Equations (1) and (15), together with the boundary conditions, determine the minimal path. However, they may not determine it uniquely, because there may be several paths starting from P_1 at time t_1 and passing through P_2 which satisfy (1) and (15). One of these is the minimal path; which one it is can be found by an actual calculation of the time required on each. If, instead of the airplane being required to pass through a given destination, the initial heading is fixed, the solution is unique. In practice, an initial heading is estimated and the corresponding solution is found by numerical integration. If the resulting path does not pass through the desired destination, a second guess is made.

The significance of (15) may be clarified by considering the special case where the streamlines are all parallel to the line joining the origin and destination but are more crowded on one side. Clearly, the minimal path is achieved by starting out with a heading into the side with the more favorable tail winds or weaker head winds. But, once started, the heading must be continually altered so that the airplane will eventually pass through the destination. That is, along the flight path, the airplane must change its heading toward the side with the stronger head winds or weaker tail winds. This is precisely what is required by (15).

9. Informal discussion of the spherical case

In effect, the spherical case is made to reduce to the plane case by mapping the sphere onto the plane. As is well known, every such mapping involves distortion; to compensate, a correction term must be added to (15).

The mapping must be conformal. In such a mapping, the scale is a function of position only, and angles are preserved. Of course, most weather maps are on conformal projections.

Suppose the sphere to have been mapped conformally onto a plane surface. Establish now on the plane map a system of rectangular coordinates x, y. The inverse of the conformal mapping is a conformal mapping of the plane onto the sphere; the coordinate lines on the plane are mapped as coordinate curves on the sphere. There are two such curves passing through any point on the sphere, a curve x = const. and a curve y = const. The direction of the first may be regarded as the y-direction, that of the second, the x-direction, exactly as on the plane. The two directions are perpendicular on the sphere, since they are so on the plane and the mapping is conformal. Let $u(x, y, t)$ and $v(x, y, t)$ be the components of the wind in the x- and y-directions, respectively, at time t at the point on the sphere corresponding to the point (x, y) on the plane. Let c be the airplane's air speed, assumed constant, and let θ be the heading of the airplane at any point measured counterclockwise from the x-direction. Let $S(x, y)$ be the scale of the mapping of the sphere onto the plane at the point defined by coordinates x, y on the plane. Because of the conformality of the mapping, S is uniquely defined at each point.

As in the plane case, in order to obtain the simplest form of the equation which must be satisfied by $\theta(t)$, choose the coordinate system just described so that the x-direction through the position of the airplane is in the direction of the airplane's heading. Let n be distance on the sphere measured along the curve x = const. through the airplane's position. Then the equation is

$$d\theta/dt = -\,\partial u/\partial n - (u + c)S^{-1}\,\partial S/\partial n. \tag{16}$$

The first term is the same as for the plane case. The meaning of the second term can best be seen by considering the case where the wind is everywhere zero. Then

$$d\theta/dt = -\,cS^{-1}\,\partial S/\partial n.$$

But, if the wind is everywhere zero, the minimal path is simply a great circle. The second term thus represents, in effect, the amount of turning necessary to keep the airplane on the great circle.

10. Derivation of minimal path formula for the sphere

The derivation here will follow the general principles of Levi-Cività's proof for Euclidean spaces. Since the only property of the sphere that will be used is the possibility of mapping it conformally onto a plane, the result will actually be applicable to any regular two-dimensional surface, since any such surface can be mapped conformally onto a plane.

Consider a two-dimensional surface mapped onto a plane. If dx_1, dx_2 represent a small displacement on the map, the corresponding small displacement on the sphere has the components

$$(dx_1/S), \quad (dx_2/S). \tag{17}$$

Similarly, let w_1, w_2 be the magnitude on the map of the wind (*i.e.*, the map distance representing wind travel per unit time), so that

$$w_1/S = u, \quad w_2/S = v; \tag{18}$$

and let c_1, c_2 be the magnitude on the map of the proper motion of the airplane, so that

$$c_1/S = c \cos\theta, \quad c_2/S = c \sin\theta. \tag{19}$$

Equations (1) become

$$S^{-1}\, dx_i/dt = S^{-1}(w_i + c_i), \quad (i = 1, 2).$$

Multiply both sides by S and transpose:

$$dx_i/dt - w_i = c_i. \tag{20}$$

By (19), $c_1^2 + c_2^2 = S^2c^2$, so that

$$\sum_i \left(\frac{dx_i}{dt} - w_i\right)^2 = S^2c^2. \tag{21}$$

Let[4]

$$p_d = S^{-2} \sum_i w_i\, dx_i, \tag{22}$$

$$w^2 = S^{-2} \sum_i w_i^2, \tag{23}$$

$$(ds)^2 = S^{-2} \sum_i (dx_i)^2. \tag{24}$$

Clearly, w is the actual wind speed and ds/dt the ground speed of the airplane. Substitute (22–24) into (21).

$$(ds/dt)^2 - 2p_d/dt + w^2 = c^2,$$
$$(c^2 - w^2)(dt)^2 + 2p_d\, dt - (ds)^2 = 0. \tag{25}$$

It is now assumed that $q^2 = c^2 - w^2 > 0$ everywhere, *i.e.*, the air speed of the airplane is greater than the wind speed. In that case, (25) has exactly one positive root for dt.

$$dt = q^{-2}\{-p_d + [p_d^2 + q^2(ds)^2]^{\frac{1}{2}}\} = L(x, dx, t). \tag{26}$$

By (22–24), p_d and ds are homogeneous functions of the first degree in dx_1, dx_2, while c and w do not involve the dx's. Hence, $L(x, dx, t)$ is a homogeneous function of the first degree in dx_1, dx_2. Divide through in equation (26) by dt; it follows that, along the route,

$$L(x, \dot{x}, t) = 1, \tag{27}$$

where the dot denotes differentiation with respect to t.

[4] Throughout section 10 the index will be understood to run from 1 to 2 in all summations.

Also, from (25),

$$q^2L^2 + 2pL - v^2 = 0, \tag{28}$$

where

$$p = p_d/dt = S^{-2} \sum_i w_i\dot{x}_i, \tag{29}$$

$$v^2 = (ds/dt)^2 = S^{-2} \sum_i (\dot{x}_i)^2. \tag{30}$$

Let T be the length of time necessary to travel from P_1 to P_2 along a given route. By (26),

$$T = \int_{P_1}^{P_2} L(x, dx, t). \tag{31}$$

The time T is to be minimized. However, the variables of integration are related by (27), which may be expressed as

$$dt - L(x, dx, t) = 0. \tag{32}$$

Equations (31) and (32) are the same as Levi-Cività's equations (10) and (7′), respectively (see Levi-Cività, 1931, p. 316; the notation varies slightly). He then shows that the minimization of T, subject to condition (32), entails the differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial t}\frac{\partial L}{\partial \dot{x}_i}. \tag{33}$$

(Levi-Cività, 1931, p. 317, I). The minimal path must then satisfy (33) and the condition (27).

It remains to evaluate the derivatives of L. For the moment, let z represent indifferently any of x, $\dot{x}$, or t. Differentiate equation (28) with respect to z:

$$(q^2L + p)\frac{\partial L}{\partial z} = -\frac{L^2}{2}\frac{\partial w^2}{\partial z} - L\frac{\partial p}{\partial z} + \frac{1}{2}\frac{\partial v^2}{\partial z}. \tag{34}$$

Substitute from equation (27):

$$(q^2 + p)\frac{\partial L}{\partial z} = -\frac{1}{2}\frac{\partial w^2}{\partial z} - \frac{\partial p}{\partial z} + \frac{1}{2}\frac{\partial v^2}{\partial z}, \tag{35}$$

where w^2, p, and v^2 are given by (23), (29), and (30). First, let $z = \dot{x}_i$:

$$\frac{\partial w^2}{\partial \dot{x}_i} = 0, \quad \frac{\partial p}{\partial \dot{x}_i} = \frac{w_i}{S^2}, \quad \frac{\partial v^2}{\partial \dot{x}_i} = \frac{2\dot{x}_i}{S^2}.$$

Substitute into (35), and recall (20):

$$(q^2 + p)\, \partial L/\partial \dot{x}_i = S^{-2}(\dot{x}_i - w_i) = S^{-2}c_i.$$

Let

$$a_i = c_i/c, \tag{36}$$
$$\rho = c[S^2(q^2 + p)]^{-1}. \tag{37}$$

Then

$$\frac{\partial L}{\partial \dot{x}_i} = \rho a_i. \tag{38}$$

Now restrict z to x_i or t.

$$\frac{\partial w^2}{\partial z} = \frac{2}{S^2}\sum_j w_j \frac{\partial w_j}{\partial z} - \frac{2}{S^3}\frac{\partial S}{\partial z}\sum_j w_j^2,$$
$$\frac{\partial p}{\partial z} = \frac{1}{S^2}\sum_j \frac{\partial w_j}{\partial z}\dot{x}_j - \frac{2}{S^3}\frac{\partial S}{\partial z}\sum_j w_j\dot{x}_j, \qquad (39)$$
$$\frac{\partial v^2}{\partial z} = -\frac{2}{S^3}\frac{\partial S}{\partial z}\sum_j \dot{x}_j^2.$$

Also, by (20) and (36),

$$\dot{x}_i = w_i + ca_i. \qquad (40)$$

Substitution of (39) and (40) into (35) gives

$$(q^2+p)\frac{\partial L}{\partial z} = -\frac{c}{S^2}\sum_j a_j\frac{\partial w_j}{\partial z} + \frac{2c}{S^3}\frac{\partial S}{\partial z}\sum_j w_ja_j - \frac{2c}{S^3}\frac{\partial S}{\partial z}\sum_j w_ja_j - \frac{c^2}{S^3}\frac{\partial S}{\partial z}\sum_j a_j^2, \qquad (41)$$

where several terms have been cancelled. Let

$$W = S^{-2}\sum_j w_ja_j. \qquad (42)$$

Then,

$$\frac{\partial W}{\partial z} = \frac{1}{S^2}\sum_j a_j\frac{\partial w_j}{\partial z} - \frac{2}{S^3}\frac{\partial S}{\partial z}\sum_j w_ja_j, \qquad (43)$$

so that

$$-\frac{c}{S^2}\sum_j a_j\frac{\partial w_j}{\partial z} + \frac{2c}{S^3}\frac{\partial S}{\partial z}\sum_j w_ja_j = -c\frac{\partial W}{\partial z}. \qquad (44)$$

Also, by (21) and (36),

$$\sum_j a_j^2 = S^2. \qquad (45)$$

Substitute (42, 44, 45) into (41):

$$(q^2+p)\partial L/\partial z = -c\,\partial W/\partial z - S^{-1}c(c+2W)\partial S/\partial z.$$

Divide through by (q^2+p), recalling (37):

$$\partial L/\partial z = -S^2\rho\,\partial W/\partial z - S\rho(c+2W)\,\partial S/\partial z.$$

Now let $z = x_i$, t respectively; $(\partial S/\partial t = 0.)$

$$\partial L/\partial x_i = -S^2\rho\,\partial W/\partial x_i - \partial S/\partial x_i\,S\rho(c+2W), \qquad (46)$$
$$\partial L/\partial t = -S^2\rho\,\partial W/\partial t. \qquad (47)$$

Substitute (38, 45, 47) into (33).

$$d(\rho a_i)/dt = -S^2\rho\,\partial W/\partial x_i - \partial S/\partial x_i\,S\rho(c+2W).$$

But $d(\rho a_i)/dt = \dot{\rho}a_i + \rho\dot{a}_i$; substitute, transpose, and divide through by ρ. We find

$$\dot{a}_i = -S^2\,\partial W/\partial x_i - S(c+2W)\partial S/\partial x_i - [(\dot{\rho}/\rho) + S^2\rho\,\partial W/\partial t]a_i. \qquad (48)$$

Equations (48) are the general differential equations for the direction of the airplane's motion under the conditions of the problem, as the a_i are essentially direction numbers.

By (19) and (36),

$$a_1 = S\cos\theta, \quad a_2 = S\sin\theta,$$

so that

$$\dot{a}_2 = (S\cos\theta)\dot{\theta} + \dot{S}\sin\theta.$$

In particular, choose the coordinate system so that the x_1-axis coincides with the direction of the airplane's heading at a particular point. Then, at that point, $\theta = 0$, and

$$a_1 = S, \quad a_2 = 0, \quad \dot{a}_2 = S\dot{\theta}. \qquad (49)$$

Substitute in (48), with $i = 2$.

$$S\dot{\theta} = -S^2\,\partial W/\partial x_2 - S(c+2W)\,\partial S/\partial x_2. \qquad (50)$$

For $\partial W/\partial x_2$, substitute from (49) into (43), letting $z = x_2$.

$$\partial W/\partial x_2 = S^{-1}\,\partial w_1/\partial x_2 - 2w_1S^{-2}\,\partial S/\partial x_2. \qquad (51)$$

Substitute from (49) into (42).

$$W = w_1/S. \qquad (52)$$

Substitute (51, 52) into (50),

$$\dot{\theta} = -\partial w_1/\partial x_2 - c\,\partial S/\partial x_2. \qquad (53)$$

Let n be measured on the sphere along the curve $x_1 =$ const. (*i.e.*, normal to the heading). Then by (17) $S\,dn = dx_2$. Then, with the aid of (18) and (53), it follows that

$$\dot{\theta} = -S^{-1}\,\partial(Su)/\partial n - cS^{-1}\,\partial S/\partial n = -\partial u/\partial n - S^{-1}(u+c)\,\partial S/\partial n, \qquad (16)$$

as given before.

In the case of the plane surface, the mapping may be regarded as that of the surface onto itself, so that S is identically equal to 1, Equation (16) then becomes $\dot{\theta} = -\partial u/\partial n$, which is the same as (15).

11. The minimal path in three dimensions

Mathematically, the consideration of flight in three dimensions is simpler than in two since the space in which flights take place may be regarded as Euclidean (neglecting any considerations of relativity theory), even though the space is bounded by a spherical surface. Therefore, Levi-Cività's results may be taken as they stand. However, the results can be simplified by a change to a rectangular coordinate system x_1, x_2, x_3. This system is regarded as fixed to the earth; the earth's rotation does not enter into the present problem, which involves only velocities, not accelerations. As before, the airplane's proper speed c is regarded as a constant. Let $c_i (i = 1, 2, 3)$ be the components of the airplane's proper motion in the three directions, let $w_i (i = 1, 2, 3)$ be the three components of wind,

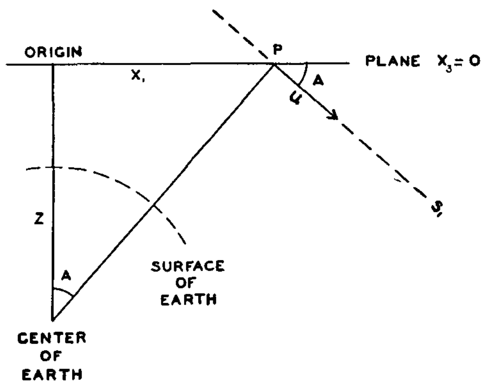

FIG. 2. Orientation of coordinate system for describing the minimal path in three dimensions.

and let

$$a_i = c_i/c \quad (i = 1, 2, 3), \tag{54}$$

$$W = \sum_{j=1}^{3} a_j w_j. \tag{55}$$

Then the equations for the minimal path are those given by Levi-Cività (1931, p. 322, eq. IV).

$$\dot{a}_i = -\,\partial W/\partial x_i + A a_i \quad (i = 1, 2, 3). \tag{56}$$

From (55)

$$\partial W/\partial x_i = \sum_{j=1}^{3} a_j\, \partial w_j/\partial x_i. \tag{57}$$

Let the heading of the airplane (*i.e.*, of the projection of its motion onto a surface $x_3 =$ const.) be θ and the angle of lift (considering x_3 as the vertical dimension) be α. Then

$$c_1 = c \cos\theta \cos\alpha,$$
$$c_2 = c \sin\theta \cos\alpha,$$
$$c_3 = c \sin\alpha.$$

By (54)

$$\begin{aligned} a_1 &= \cos\theta\cos\alpha,\\ a_2 &= \sin\theta\cos\alpha,\\ a_3 &= \sin\alpha. \end{aligned} \tag{58}$$

Differentiating with respect to time yields the expressions,

$$\begin{aligned} \dot{a}_1 &= -\,\dot{\alpha}\cos\theta\sin\alpha - \dot{\theta}\sin\theta\cos\alpha,\\ \dot{a}_2 &= -\,\dot{\alpha}\sin\theta\sin\alpha + \dot{\theta}\cos\theta\cos\alpha,\\ \dot{a}_3 &= \dot{\alpha}\cos\alpha. \end{aligned} \tag{59}$$

In particular, choose the coordinate system (fig. 2) so that the origin is at the position of the airplane, the plane $x_3 = 0$ is parallel to the plane tangent to the earth's surface at the point nearest the position of the airplane, and the x_1-axis points in the direction of the heading. Equations (58, 59) become, respectively,

$$a_1 = \cos\alpha, \quad a_2 = 0, \quad a_3 = \sin\alpha, \tag{60}$$

$$\dot{a}_1 = -\,\dot{\alpha}\sin\alpha, \quad \dot{a}_2 = \dot{\theta}\cos\alpha, \quad \dot{a}_3 = \dot{\alpha}\cos\alpha. \tag{61}$$

It is necessary to evaluate $\partial w_1/\partial x_i$, $\partial w_2/\partial x_i$ for this coordinate system. At any point in space, the wind is assumed to lie in a plane parallel to a plane tangent to the earth's surface at the nearest point (*i.e.*, vertical motion of air with respect to the earth is neglected). In fig. 2 let S_1 be this plane through any given point P on the surface $x_3 = 0$; let S_2 be the plane $x_2 =$ const. through that point. The plane of fig. 2 is the particular S_2 defined by the equation $x_2 = 0$. Then, u is defined as the component of wind along the intersection of S_1 and S_2; v is the component normal to u.

In this case, it will be noted that the line defined by the intersection of S_1 and the plane $x_3 = 0$ is perpendicular to the intersection of S_1 and S_2, so that the wind component v lies in the plane $x_3 = 0$. Let A be the angle at the center of the earth between the lines to the airplane's position and point P. Then, clearly,

$$w_1 = u\cos A, \quad w_2 = v, \quad w_3 = -\,u\sin A,$$

if $x_2 = x_3 = 0$ at P. Thus, these relations may be used to get the partial derivatives of w_1 and w_3 with respect to x_1 for $x_2 = x_3 = 0$.

$$\tan A = x_1/z,$$

so that

$$w_1 = uzh, \quad w_3 = -\,ux_1h,$$

where

$$h = (z^2 + x_1^2)^{-\frac{1}{2}}.$$

Since h has its maximum when $x_1 = 0$, at the origin, $h = 1/z$, $\partial h/\partial x_1 = 0$. Therefore, at the origin,

$$\begin{aligned} &\partial w_1/\partial x_1 = \partial u/\partial x_1, \quad \partial w_2/\partial x_1 = \partial v/\partial x_1,\\ &\partial w_3/\partial x_1 = -\,u/z. \end{aligned} \tag{62}$$

Similarly,

$$\begin{aligned} &\partial w_1/\partial x_2 = \partial u/\partial x_2, \quad \partial w_2/\partial x_2 = \partial v/\partial x_2,\\ &\partial w_3/\partial x_2 = -\,v/z. \end{aligned} \tag{63}$$

For $x_1 = x_2 = 0$, $w_1 = u$, $w_2 = v$, $w_3 = 0$, so that at the origin

$$\begin{aligned} &\partial w_1/\partial x_3 = \partial u/\partial x_3, \quad \partial w_2/\partial x_3 = \partial v/\partial x_3,\\ &\partial w_3/\partial x_3 = 0. \end{aligned} \tag{64}$$

We will now shift to another system of coordinates. As in fig. 2 let z denote distance from the center of the earth; and let dx denote a small displacement along a sphere, $z =$ const. in the direction of the airplane's heading. Let dy denote a similar displacement in a direction normal to and to the left of dx. Then, as the plane $x_3 = 0$ is tangent to a sphere of the form $z =$ const., differentiation with respect to x_1, x_2 at the origin may be replaced by differentiation with respect to x, y. As z differs only by a constant from x_3 (for $x_1 = x_2 = 0$), differentiation with respect to x_3 may be replaced by differentiation with respect to z. Then, from (62–64)

$$\partial w_1/\partial x_1 = \partial u/\partial x, \; \partial w_3/\partial x_1 = -\,u/z, \; \partial w_1/\partial x_2 = \partial u/\partial y,$$
$$\partial w_3/\partial x_2 = -\,v/z, \quad \partial w_1/\partial x_3 = \partial u/\partial z,$$
$$\partial w_3/\partial x_3 = 0.$$

Substituting in (57) and also (60) yields the relations,

$$\begin{aligned}\partial W/\partial x_1 &= (\partial u/\partial x)\cos\alpha - (u/z)\sin\alpha,\\ \partial W/\partial x_2 &= (\partial u/\partial y)\cos\alpha - (v/z)\sin\alpha,\\ \partial W/\partial x_3 &= (\partial u/\partial z)\cos\alpha. \qquad (65)\end{aligned}$$

In (56), set $i = 2$, and substitute from (61, 60, 65),

$$\dot{\theta} = -\,\partial u/\partial y + (v/z)\tan\alpha. \qquad (66)$$

Set $i = 1, 3$ in (56) and substitute from (61, 60, 65).

$$\begin{aligned}(-\sin\alpha)\dot{\alpha} &= -(\partial u/\partial x)\cos\alpha\\ &\qquad + (u/z)\sin\alpha + A\cos\alpha,\\ (\cos\alpha)\dot{\alpha} &= -(\partial u/\partial z)\cos\alpha + A\sin\alpha.\end{aligned}$$

Multiply the first equation by $-\sin\alpha$ and the second by $\cos\alpha$ and add.

$$\begin{aligned}\dot{\alpha} = (\partial u/\partial x)\cos\alpha\sin\alpha - (u/z)\sin^2\alpha\\ -(\partial u/\partial z)\cos^2\alpha. \qquad (67)\end{aligned}$$

Equations (66, 67) together form the differential equations for the direction of the minimal path in three dimensions. In (67), since α is generally small and $\partial u/\partial x$ and u/z are small compared with $\partial u/\partial z$, an excellent approximation is obtained by setting

$$\dot{\alpha} = -(\partial u/\partial z)\cos^2\alpha. \qquad (68)$$

12. Bibliography

Besides the papers of Frank (1918), Zermelo (1930; 1931), and Levi-Cività (1931) already cited, there are a few others dealing with the general topic of minimal flight paths. To the present writer's knowledge, the only paper which has discussed the problem of minimal flight on the sphere is Zita's dissertation (1931). He discusses the problem both for the sphere and the plane and gives examples, but the wind field is assumed to be stationary and to depend only on distance from the origin of the coordinate system. (Thus, for the sphere, he considers the case where the wind depends only on latitude.)

Von Mises (1931) considers the special case where the wind field is constant except for a discontinuity (front). In this case, the minimal flight obeys a formula analogous to Snell's law for refraction. The analogy between optical and minimal-flight problems was also pointed out by Frank. In addition to his earlier paper (1918) he has shown (1933) that the minimal flight problem is precisely analogous to light transmission in a stationary medium in which the velocity of light varies from place to place and also varies with direction (as in certain crystals).

De Mira Fernandes (1932) has generalized the results of Zermelo and Levi-Cività for Euclidean spaces to the case where the air speed is a preassigned function of position and time. The assumption that the air speed varies in a preassigned way with horizontal coordinates is unrealistic, but, if an airplane flies with constant gasoline consumption, the air speed will certainly vary with height (because of change in air density) and with time (because of change in load). De Mira Fernandes shows that the change in c with time has no effect on the formula while that with space has the same effect as a corresponding change in the winds.

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