The Roles of the Horizontal Component of the Earth's Angular Velocity in Nonhydrostatic Linear Models

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ABSTRACT

Roles of the horizontal component of the earth's rotation, which is neglected traditionally in atmospheric and oceanographic models, are studied through the normal mode analysis of a compressible and stratified model on a tangent plane in the domain that is periodic in the zonal and meridional directions but bounded at the top and bottom. As expected, there exist two distinct kinds of acoustic and buoyancy oscillations that are modified by the earth's rotation. When the cos(latitude) Coriolis terms are included, there exists another kind of wave oscillation whose frequencies are very close to the inertial frequency, 2Ω sin(latitude), where Ω is the earth's angular velocity.

The objective of this article is to clarify the circumstance in which a distinct kind of wave oscillation emerges whose frequencies are very close to the inertial frequency. Because this particular kind of normal mode appears only due to the presence of boundary conditions in the vertical, it may be appropriate to call these waves boundary-induced inertial (BII) modes as demonstrated through the normal mode analyses of a homogeneous and incompressible model and a Boussinesq model with thermal stratification. Thus, it can be understood that the BII modes can coexist with the acoustic and inertio-gravity modes when the effect of compressibility is added to the effects of buoyancy and complete Coriolis force in the compressible, stratified, and rotating model.

1. Introduction

Current atmospheric models for weather prediction and climate simulation are mostly based on the hydrostatic primitive equations (e.g., Phillips 1973) that are derived from the Eulerian system of equations with some ‘‘traditional’’ assumptions. One of the major assumptions is referred to as ‘‘shallowness approximation’’ based on the notion that the vertical extent of the atmosphere of our interest is rather small compared with the earth's radius. Another assumption is that the vertical acceleration is negligible in the vertical equation of motion. This is a reasonable approximation as far as large-scale motions are concerned and has a benefit of eliminating the vertical propagation of acoustic waves, so that a small vertical grid increment does not overly restrict the choice of time step in explicit discretized calculations.

With the increase of computing power and data storage capacity, we will be able to refine the model resolutions to incorporate smaller-scale motions, as well as to extend the top of model atmosphere to improve the calculation of flows and the quality of observational data assimilations. An extensive discussion is presented by White and Bromley (1995) as a critique to hydrostatic primitive equation (HPE) formulations. A similar critical review is presented also by Marshall et al. (1997) concerning the use of hydrostatic and shallowness approximations for ocean modeling. One of the dynamical consequences of the shallowness approximation is the omission of Coriolis terms involving 2Ω cos(latitude), where Ω is the angular rotation rate of the earth, which appear in the vertical and zonal equations of motion (Phillips 1966). White and Bromley (1995) argue through a scale analysis that the effects of cos(latitude) or simply cosine Coriolis terms may attain magnitudes of as much as 10% of major terms for both planetary-scale and diabatically driven tropical motions. These terms are now included in the numerical prediction models developed at the Met Office (Davies 2000). However, no physical mechanism is offered through which the cosine Coriolis terms may play important roles.

The dynamical effects of cosine Coriolis terms on fluid motions have been studied mostly in connection with boundary layer flows. For example, Wippermann (1969) and Leibovich and Lele (1985) investigated these effects in the problem of Ekman layer instability. Since vertical and horizontal perturbation speeds are compa-

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rable in the Ekman layer, the cosine Coriolis terms are on the same order of magnitude of the sine Coriolis terms and both are relevant to the stability problem, although the cosine Coriolis terms play little role in the unperturbed basic Ekman layer velocity field. Wang et al. (1996) considered the effect of cosine Coriolis terms on entraining equatorial ocean boundary layers. They concluded that the importance of cosine Coriolis terms may depend on the thermal stratification of the boundary layer. If stratification is neutral or unstable without a thermocline, the effects of cosine Coriolis terms can be significant, as suggested by Hassid and Galperin (1994).

Concerning the dynamical effects of cosine Coriolis terms on atmospheric oscillations, Eckart (1960) investigated the normal modes of a stratified and compressible atmosphere in Cartesian coordinates on a tangent plane with a complete representation of the Coriolis force. He discussed the solutions of Lamb waves as the external mode, but left the solutions of internal modes unexplored. Eckart (1960, 134–135) concluded,

> These calculations are by no means complete. It would be possible to discuss the properties of the simple waves—their phase and group velocities, the impedances, etc.,—but the algebraic complexities would be great. These two sample calculations indicate one thing, however: there are effects that depend on [2Ω cos(latitude)], and these can be very marked for frequencies in the neighbourhood of [frequency = 2Ω sin(latitude)].

It is well known that there are two kinds of oscillations in the stratified and compressible model on a tangent plane, including only the vertical component of the earth's rotation vector, which is treated as constant (e.g., Eckart 1960; Monin and Obukhov 1959; Gill 1982). One corresponds to high-frequency acoustic waves and the other to low-frequency inertio-gravity waves. However, when we carried out the internal mode solutions of the Eckart model with cosine Coriolis terms under an isothermal condition, we found that there is another kind of wave oscillation whose frequencies are very close to the inertial frequency [2Ω sin(latitude)], in addition to the acoustic and inertio-gravity modes. [There are Lamb waves too, in this case, which are somewhat unusual, and Eckart (1960) discussed their properties in detail.] We are not aware of any discussion on this kind of near-inertial frequency waves in the literature, except that Egger (1999) pointed out the necessity of cos Coriolis terms to give rise to the oscillations whose frequencies are very close to the inertial frequency. [See appendix.] We will come back to this point in section 5.

The objective of this article is to clarify the circumstance in which near-inertial frequency waves appear as a distinct kind. We begin our discussion in section 2 on the role of cosine Coriolis terms in a homogeneous and incompressible model on a tangent plane. Through the normal mode analysis of this model, it is shown that there is a possibility of having a distinct kind of wave oscillations due to the presence of boundary conditions in the vertical in addition to inertial waves. In section 3, we analyze the normal modes of a Boussinesq model with thermal stratification. We notice that the distinct kind of modes found in section 2 also appear in the model and their frequencies are very close to the inertial frequency. Therefore, it may be appropriate to call this distinct kind of mode the boundary-induced inertial (BII) modes, because the role of the boundary conditions is essential. In section 4, the stratified and compressible model with a complete representation of Coriolis effects is considered. Since compressibility is now added to the Coriolis and buoyancy effects, it is natural to expect the emergence of acoustic modes in addition to those modes discussed in section 3. The properties of the modal frequencies of the compressible and stratified model are discussed in section 5. Conclusions and further discussions are presented in section 6.

2. Incompressible and homogeneous model

We consider the small-amplitude oscillations of an incompressible and homogeneous (the density ρ = constant) model, including the vertical and horizontal components of the Coriolis vector 2Ω in the Cartesian coordinates (x, y, z, t) on a tangent plane, with x, y, and z directed eastward, northward, and upward, and t being time. The equations of motion for the velocity components (u, v, w) corresponding to (x, y, z) with the pressure p and the mass continuity equation are written as

$$\frac{\partial u}{\partial t} - f_V v + f_H w + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + f_V u + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \tag{2.2}$$

$$\frac{\partial w}{\partial t} - f_H u + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \quad \text{and} \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.4}$$

where

$$f_V = 2\Omega \sin\phi, \qquad f_H = 2\Omega \cos\phi, \tag{2.5}$$

with ϕ being the latitude of the coordinate center. Here, f_V and f_H are assumed to be constant.

We consider the solutions of (2.1)–(2.4) in a three-dimensional domain which is periodic in x and y and bounded by the bottom at $z = 0$ and the top at $z = z_T$, where we assume that

$$w = 0 \quad \text{at } z = 0 \quad \text{and} \quad z_T. \tag{2.6}$$

Since Eqs. (2.1)–(2.4) are of constant coefficients, we seek wave solutions in the form

$$(u, v, w, \rho^{-1}p) = (U, iV, iW, P)\exp[i(mx + ny - \sigma t)], \tag{2.7}$$

where U, V, W, P are functions of z only, and m and n

are the wavenumbers in x and y. Also, σ denotes the frequency and can be shown to be real. Here, $i = \sqrt{-1}$.

Thus, we obtain from (2.1)–(2.4) the following:

$$-\sigma U - f_V V + f_H W + mP = 0, \quad (2.8)$$

$$f_V U + \sigma V + inP = 0, \quad (2.9)$$

$$-f_H U + \sigma W + \frac{dP}{dz} = 0, \quad \text{and} \quad (2.10)$$

$$mU + inV + \frac{dW}{dz} = 0. \quad (2.11)$$

From (2.8) and (2.9), we can express U and V in terms of W and P as

$$(f_V^2 - \sigma^2)U = -\sigma f_H W - (\sigma m + if_V n)P, \quad (2.12)$$

$$(f_V^2 - \sigma^2)V = f_H f_V W + i(\sigma n - if_V m)P, \quad (2.13)$$

where we assume that $\sigma \neq 0$ and $(f_V^2 - \sigma^2) \neq 0$ to obtain nontrivial transient solutions.

We now eliminate U from (2.10) using (2.12) and get

$$\sigma[(f_V^2 + f_H^2) - \sigma^2]W + (f_V^2 - \sigma^2)\frac{dP}{dz} + f_H(\sigma m + if_V n)P = 0. \quad (2.14)$$

Likewise, we eliminate U and V from (2.11) and get

$$(f_V^2 - \sigma^2)\frac{dW}{dz} - f_H(\sigma m - if_V n)W - \sigma(m^2 + n^2)P = 0. \quad (2.15)$$

We then derive the vertical structure equation of W by eliminating P between (2.14) and (2.15) and obtain

$$\frac{d^2W}{dz^2} + \frac{2if_V f_H n}{f_V^2 - \sigma^2}\frac{dW}{dz} + \left[\frac{\sigma^2(m^2 + n^2) - n^2 f_H^2}{f_V^2 - \sigma^2}\right]W = 0. \quad (2.16)$$

Our task now is to solve (2.16) under the boundary conditions

$$W = 0 \quad \text{at } z = 0 \quad \text{and} \quad z = z_T, \quad (2.17)$$

which are derived from (2.6).

To eliminate the first-order term of W in (2.16), we use the following transformation:

$$W(z) = \eta(z)\exp(i\Gamma_2 z), \quad (2.18)$$

where

$$\Gamma_2 = -\frac{f_H f_V n}{f_V^2 - \sigma^2}. \quad (2.19)$$

Substitution of (2.18) into (2.16) leads to the following equation for η:

$$\frac{d^2\eta}{dz^2} + \left[\Gamma_2^2 + \frac{\sigma^2(m^2 + n^2) - n^2 f_H^2}{f_V^2 - \sigma^2}\right]\eta = 0. \quad (2.20)$$

The solutions of (2.20) that satisfy the boundary conditions (2.17) are given by

$$\eta(z) = A_k \sin(kz) \quad \text{and} \quad (2.21)$$

$$k = k_i(\pi/z_T), \quad k_i = 1, 2, \ldots, \quad (2.22)$$

where A_k denotes the coefficient, k the vertical wavenumber, and k_i the vertical wave index.

To determine the coefficient A_k, we express $P(z)$ as

$$P(z) = \xi(z)\exp(i\Gamma_2 z). \quad (2.23)$$

By substituting (2.18) and (2.23) into (2.15), and remembering that $(f_V^2 - \sigma^2) \neq 0$, we get

$$\frac{d\eta}{dz} - \Gamma_1\eta = \frac{\sigma(m^2 + n^2)}{f_V^2 - \sigma^2}\xi, \quad (2.24)$$

where

$$\Gamma_1 = \frac{f_H \sigma m}{f_V^2 - \sigma^2}. \quad (2.25)$$

Now, substitution of (2.21) into (2.24) yields

$$\sigma\xi(z) = \frac{f_V^2 - \sigma^2}{m^2 + n^2}A_k(k\cos kz - \Gamma_1 \sin kz). \quad (2.26)$$

The value of ξ at $z = 0$, namely $\xi(0)$, can be obtained from the distribution of $\rho^{-1}p$ at $z = 0$. Therefore, the coefficient A_k can be determined from (2.26) in terms of $\xi(0)$ as

$$A_k = \sigma\xi(0)\frac{m^2 + n^2}{f_V^2 - \sigma^2}\frac{1}{k}. \quad (2.27)$$

Thus, (2.21) can be expressed, using (2.27) as,

$$\eta(z) = \sigma\xi(0)\frac{m^2 + n^2}{f_V^2 - \sigma^2}\left(\frac{\sin kz}{k}\right). \quad (2.28)$$

Notice that the value of η at the limit of $k \to 0$, which occurs when $z_T \to \infty$ in (2.22), is finite and nonzero unless $\sigma = 0$. Therefore, $k = 0$ can be considered the lowest vertical internal mode.

The frequency of normal modes can be determined by substituting (2.21) into (2.20). The result becomes, again remembering $(f_V^2 - \sigma^2) \neq 0$,

$$(m^2 + n^2 + k^2)\sigma^4 - [(m^2 + n^2 + k^2)f_V^2 + n^2 f_H^2 + k^2 f_V^2]\sigma^2 + k^2 f_V^4 = 0. \quad (2.29)$$

The solutions of (2.29) are given by

$$\sigma^2 = \frac{1}{2}\left(f_V^2 + \frac{n^2 f_H^2 + k^2 f_V^2}{m^2 + n^2 + k^2}\right) \pm \frac{1}{2}\left[\left(f_V^2 + \frac{n^2 f_H^2 + k^2 f_V^2}{m^2 + n^2 + k^2}\right)^2 - \frac{4k^2 f_V^4}{m^2 + n^2 + k^2}\right]^{1/2}. \quad (2.30)$$

There are two real solutions σ^2 of different magnitudes. Each solution of σ^2 has a pair of positive and negative values of σ with the same magnitude.

In the case of $f_V \neq 0$ and $f_H = 0$, we find from (2.30) that

$$\sigma_+^2 = f_V^2, \tag{2.31}$$

corresponding to the plus sign in front of the radical of (2.30). For the minus sign, we have

$$\sigma_-^2 = \frac{k^2 f_V^2}{m^2 + n^2 + k^2}. \tag{2.32}$$

Obviously, we must reject the σ_+^2 solutions (2.31), which contradict with the assumption that $(f_V^2 - \sigma^2)$ should not vanish. However, the system (2.8)–(2.11) can allow the solutions of $\sigma^2 = f_V^2$ if $m = n = 0$ with $P = W = 0$ and $U = \pm V$. This kind of solution is referred to as the *inertial oscillations* (e.g., Durran 1993). Therefore, in the case of $f_V \neq 0$ and $f_H = 0$, the only valid wave oscillations when $m^2 + n^2 \neq 0$ are represented by $\pm\sigma_-$ of (2.32) and are referred to usually as the *inertial waves* (e.g., Tolstoy 1973).

In the general case of $f_V \neq 0$ and $f_H \neq 0$, we see from (2.30) that two kinds of solutions, represented by the higher value of σ_+^2 and the lower value of σ_-^2, exist unless the meridional mode $n = 0$. If $n = 0$, only the traditional type of (2.32) with $n = 0$ is obtained. This can be seen easily from (2.29) as the f_H appears only in association with n.

It may be instructive to derive approximations to the two kinds of the solutions of (2.29) when $f_V^2 \gg f_H^2$ and n^2, m^2, and k^2 are on the same order of magnitudes. Since σ_+^2 is close to f_V^2 and σ_-^2 to $k^2 f_V^2/(m^2 + n^2 + k^2)$, we can find approximate expressions of σ_+^2 and σ_-^2 by a perturbation method as follows:

$$\sigma_+^2 \doteq f_V^2\left[1 + \frac{n^2 f_H^2}{(m^2 + n^2) f_V^2}\right] \quad \text{and} \tag{2.33}$$

$$\sigma_-^2 \doteq \frac{k^2 f_V^2}{m^2 + n^2 + k^2}\left[1 + \frac{n^2 f_H^2}{(m^2 + n^2) f_V^2}\right]. \tag{2.34}$$

By comparing (2.33) with (2.31), we find that $\pm\sigma_+$ are now valid wave solutions because of the presence of f_H terms.

The actual values of σ_+^2 and σ_-^2 of the general case can be calculated from (2.30). The maximum of σ^2 becomes $f_V^2 + f_H^2 = (2\Omega)^2$ for $m = k = 0$ with $n \neq 0$, and the minimum of σ^2 approaches zero for large values of m and n. Thus, two kinds of σ^2 represented by $\pm\sigma_+$ and $\pm\sigma_-$ together produce a spectrum of discrete frequencies, spanning from 2Ω to -2Ω, namely

$$(2\Omega)^2 \geq \sigma_+^2 > \sigma_-^2 > 0 \tag{2.35}$$

for various combinations of zonal, meridional, and vertical wavenumbers.

We should emphasize that the emergence of the two kinds of σ^2 in the general case arises from the imposition of the boundary condition (2.17). If we disregard these boundary conditions, then the plane wave solutions of Eq. (2.16) for W can be expressed proportionally to $\exp(ikz)$ and substitution of this form into (2.16) yields only that

$$\sigma^2 = \frac{(k f_V + n f_H)^2}{m^2 + n^2 + k^2}. \tag{2.36}$$

This dispersion equation is clearly different from that given by (2.29). It is worthwhile to note that the frequency σ, as given by (2.36), depends on the sign of k/n if $f_H \neq 0$. Therefore, upward and downward propagating plane wave solutions of the form $\exp[i(mx + ny \pm kz)]$ cannot be combined to satisfy the boundary condition $W = 0$ at $z = 0$ because they have different frequencies. Thus, the solutions of (2.16) must have the form $\exp[i(mx + ny) + i(\Gamma_2 \pm k)]$ to satisfy the boundary condition $W = 0$ at $z = 0$, because then the frequencies of incident and reflected waves at the boundary $z = 0$ are identical as given by the solutions of the dispersion equation (2.29). In the case of $f_H = 0$, (2.36) is reduced to (2.32) and the imposition of the boundary conditions does not create the need of different solutions from the plane wave solutions.

Before leaving this section, we should point out that, if $\sigma = 0$, the system of (2.8)–(2.11) is satisfied by the following steady-state ‘‘geostrophic’’ solutions:

$$f_V U = -inP, \qquad f_V V = mP,$$
$$W = 0, \qquad f_H U = \frac{dP}{dz}. \tag{2.37}$$

On the other hand, if $\sigma^2 = f_V^2$ and $m^2 + n^2 \neq 0$, then we can derive a first-order homogeneous equation for W from (2.8)–(2.11). But, due to the boundary condition that $W = 0$ at $z = 0$, we find that $W = 0$. Then, we can show that $P = 0$, $V = 0$, and $U = 0$, successively. If $\sigma^2 = f_V^2$ and $m = n = 0$, we find that $U = \pm V$ from (2.9), $W = 0$ from (2.8). Then, P can be determined from (2.10) as the solution of $dP/dz = f_H U$ with a suitable boundary condition on z (Kamenkovich and Kulakov 1977).

3. Boussinesq model

In order to include the effect of buoyancy in the previous example, we consider the following Boussinesq model often adopted in oceanography in connection with the study of inertial motions (e.g., Munk and Phillips 1968; Pollard 1970), but we include both the vertical and horizontal components of the Coriolis vector:

$$\frac{\partial u}{\partial t} - f_V v + f_H w + \frac{1}{\rho_o}\frac{\partial p}{\partial x} = 0, \tag{3.1}$$

$$\frac{\partial v}{\partial t} + f_V u + \frac{1}{\rho_o}\frac{\partial p}{\partial y} = 0, \tag{3.2}$$

$$\frac{\partial w}{\partial t} - f_{\rm H}u + \frac{1}{\rho_o}\frac{\partial p}{\partial z} = s, \quad (3.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \text{and} \quad (3.4)$$

$$\frac{\partial s}{\partial t} + N^2 w = 0, \quad (3.5)$$

where

$s = g(\rho_o - \rho)/\rho_o$: buoyancy, (3.6)

$\rho(x, y, z, t)$: density,

ρ_o: volume mean density (constant),

$\overline{\rho}(z)$: horizontal mean density,

$$N(z) = \left(-\frac{g}{\rho_o}\frac{d\overline{\rho}}{dz}\right)^{1/2}: \text{ Brunt–Väisälä frequency}, \quad (3.7)$$

and g denotes gravity constant. The boundary conditions are the same as (2.6).

We seek the solutions of (3.1)–(3.5) in the form

$$(u, v, w, \rho_o^{-1}p, s) = (U, iV, iW, P, S)\exp[i(mx + ny - \sigma t)]. \quad (3.8)$$

By substituting (3.8) into (3.1)–(3.5), we obtain

$$-\sigma U - f_{\rm V}V + f_{\rm H}W + mP = 0, \quad (3.9)$$

$$f_{\rm V}U + \sigma V + inP = 0, \quad (3.10)$$

$$-f_{\rm H}U + \sigma W - S + \frac{dP}{dz} = 0, \quad (3.11)$$

$$mU + inV + \frac{dW}{dz} = 0, \quad \text{and} \quad (3.12)$$

$$\sigma S - N^2 W = 0. \quad (3.13)$$

The boundary conditions are the same as (2.17).

From this point, we assume that N in (3.13) is constant. After elimination of the variables U, V, S, and P, we can derive the vertical structure equation for W in a similar manner as we derived (2.16). The result is

$$\frac{d^2W}{dz^2} + \frac{2if_{\rm V}f_{\rm H}n}{f_{\rm V}^2 - \sigma^2}\frac{dW}{dz} + \left[\frac{(\sigma^2 - N^2)(m^2 + n^2) - n^2 f_{\rm H}^2}{f_{\rm V}^2 - \sigma^2}\right]W = 0. \quad (3.14)$$

It is clear that (3.14) reduces to (2.16) in the case of $N = 0$, corresponding to the reduction of the Boussinesq model (3.9)–(3.13) to the homogeneous model (2.8)–(2.11). Thus, by substituting (2.18) into (3.14), we obtain

$$\frac{d^2\eta}{dz^2} + \left[\Gamma_2^2 + \frac{(\sigma^2 - N^2)(m^2 + n^2) - n^2 f_{\rm H}^2}{f_{\rm V}^2 - \sigma^2}\right]\eta = 0. \quad (3.15)$$

The solutions of (3.15) that satisfy the boundary conditions (2.17) are given by the same form as (2.21). Therefore, by substituting the eigensolutions in the form of (2.21) into (3.15) and remembering that $(f_{\rm V}^2 - \sigma^2) \neq 0$, we get

$$(m^2 + n^2 + k^2)\sigma^4 - [(m^2 + n^2)N^2 + (m^2 + n^2 + k^2)f_{\rm V}^2 + n^2 f_{\rm H}^2 + k^2 f_{\rm V}^2]\sigma^2 + [(m^2 + n^2)N^2 + k^2 f_{\rm V}^2]f_{\rm V}^2 = 0. \quad (3.16)$$

Since (3.16) is a quadratic equation for σ^2 similar to (2.29), we can easily obtain the explicit form of solutions like (2.30). However, because usually $N^2 \gg \Omega^2$, we can obtain fairly accurate solutions by approximation.

For the high-frequency approximate solutions of (3.16), we get

$$\sigma_g \doteq \pm\left[f_{\rm V}^2 + \frac{(m^2 + n^2)N^2 + n^2 f_{\rm H}^2 + k^2 f_{\rm V}^2}{m^2 + n^2 + k^2}\right]^{1/2}. \quad (3.17)$$

It is clear that these frequencies correspond to the inertio-gravity modes, modified by the $f_{\rm H}$ terms, as explained below.

In the case of $f_{\rm V} \neq 0$ and $f_{\rm H} = 0$, we can factor out (3.16) by $(\sigma^2 - f_{\rm V}^2)$, which should not vanish. Therefore, we have the only solutions in the form

$$\sigma_g = \pm\left[\frac{(m^2 + n^2)N^2 + k^2 f_{\rm V}^2}{m^2 + n^2 + k^2}\right]^{1/2}, \quad (3.18)$$

corresponding to the inertio-gravity modes (e.g., Monin and Obukhov 1959; Gill 1982).

For the low-frequency solutions of (3.16), because normally $N^2 \gg \Omega^2$, we expect by inspection of (3.16) that σ^2 is close to $f_{\rm V}^2$. Therefore, by substituting $\sigma = f_{\rm V} + \Delta\sigma$ into (3.16) and retaining only the first-order terms of $\Delta\sigma$, we obtain the following approximate solutions:

$$\sigma_I \doteq \pm|f_{\rm V}|(1 - f_{\rm H}^2 n^2 E_1^{-1}), \quad (3.19)$$

where

$$E_1 = 2[(m^2 + n^2)(N^2 - f_{\rm V}^2) + n^2 f_{\rm H}^2]. \quad (3.20)$$

Because (3.19) is an approximate solution, σ_I does not depend on the vertical wavenumber k at this level of approximation. Nevertheless, as can be verified numerically, the formula (3.19) gives accurate results in general. Moreover, the fact that σ_I is present only when $n \neq 0$ is the exact result. If $n = 0$, $(\sigma^2 - f_{\rm V}^2)$ can be factored out from (3.16). Since we have assumed that $(\sigma^2 - f_{\rm V}^2)$ should not vanish, no wave solution corresponding to σ_I exists if $n = 0$. The magnitude of σ_I is very close to $|f_{\rm V}|$, because the correction term $f_{\rm H}^2 n^2 E_1^{-1}$ in (3.19) is normally on the order of 10^{-4}.

We should again reiterate that the imposition of the boundary conditions is responsible to produce the two kinds of solutions in this model. If we disregard the boundary conditions, then the plane wave solutions of Eq. (3.14) can be expressed proportionally to exp(ikz), and substitution of this form into (3.14) gives only

$$\sigma^2 = \frac{(m^2 + n^2)N^2 + (kf_V + nf_H)^2}{m^2 + n^2 + k^2}. \quad (3.21)$$

These frequencies correspond to those of inertio-gravity waves including the effect of f_H terms. However, when the vertical boundary conditions are imposed, the eigensolutions of (3.14) must take the form proportional to $\exp[i(\Gamma_2 + k)z]$ to satisfy the boundary conditions and the dispersion equation becomes (3.16), which is clearly different from (3.21). Note that (3.21) does depend on the sign of k/n if $f_H \neq 0$, while (3.16) does not. Moreover, the high-frequency solutions σ_g^2 of (3.16) have the form similar to (3.21), while the low-frequency solutions σ_I^2 emerge as a variation to the forbidden wave solutions of $\sigma^2 = f_V^2$. Since the wave oscillations corresponding to σ_I^2 are uniquely associated with the boundary conditions, it may be appropriate to refer to this kind of normal mode as the BII mode.

4. Compressible and stratified model

Finally, we consider small-amplitude oscillations of a stratified and compressible model in the Cartesian coordinates (x, y, z, t) on a tangent plane, including the vertical and horizontal components of the Coriolis vector. This problem was initiated by Eckart (1960) who discussed the solutions of Lamb waves, but the discussion on the internal modes was not completed. Here, we obtain the solutions of internal modes of this problem using the same approach as in the previous two sections, but the inclusion of the compressibility of fluid adds some complexity in treatment.

The basic system consists of linearized equations for the momentum, the mass continuity, and the law of thermodynamics. The basic states are assumed to be at rest with temperature $T_o(z)$, pressure $p_o(z)$, and density $\rho_o(z)$ that are in hydrostatic equilibrium $dp_o/dz = -\rho_o g$, where T_o is defined through the equation of state, $p_o = \rho_o R T_o$, and R denotes the gas constant.

The basic equations for the perturbation variables of the velocity components (u, v, w), the pressure p, and the density ρ are expressed by

$$\rho_o \frac{\partial u}{\partial t} - f_V \rho_o v + f_H \rho_o w = -\frac{\partial p}{\partial x}, \quad (4.1)$$

$$\rho_o \frac{\partial v}{\partial t} + f_V \rho_o u = -\frac{\partial p}{\partial y}, \quad (4.2)$$

$$\rho_o \frac{\partial w}{\partial t} - f_H \rho_o u + \frac{g}{C_s^2} p - s = -\frac{\partial p}{\partial z}, \quad (4.3)$$

$$\frac{1}{\rho_o C_s^2}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{g}{C_s^2} w = 0, \quad \text{and} \quad (4.4)$$

$$\frac{1}{\rho_o N^2}\frac{\partial s}{\partial t} + w = 0, \quad (4.5)$$

where a new variable s is defined by

$$s = \frac{g}{C_s^2} p - g\rho. \quad (4.6)$$

The use of the variable s instead of ρ by (4.6) helps the derivation of perturbation energy equation, and s is related to perturbation of the logarithm of potential temperature (Gill 1982).

The basic-state parameters are

$$C_s^2 = \gamma R T_o \quad \text{and} \quad N^2 = -g\left(\frac{1}{\rho_o}\frac{d\rho_o}{dz} + \frac{g}{C_s^2}\right), \quad (4.7)$$

where C_s denotes the speed of sound, with γ defining the ratio of the two specific heat values at constant pressure and at constant volume $\gamma = C_p/C_v$, and N is the Brunt–Väisälä frequency.

We seek the solutions of (4.1)–(4.5) in the form

$$\begin{vmatrix} \rho_o^{1/2} u \\ \rho_o^{1/2} v \\ \rho_o^{1/2} w \\ \rho_o^{-1/2} p \\ \rho_o^{-1/2} s \end{vmatrix} = \begin{vmatrix} U(z) \\ iV(z) \\ iW(z) \\ P(z) \\ S(z) \end{vmatrix} \exp[i(mx + ny - \sigma t)] \quad (4.8)$$

and obtain

$$\sigma U + f_V V = mP + f_H W, \quad (4.9)$$

$$f_V U + \sigma V = -inP, \quad (4.10)$$

$$\sigma W - f_H U - S = -\frac{dP}{dz} - \Gamma P, \quad (4.11)$$

$$\frac{\sigma}{C_s^2} P - (mU + inV) = \frac{dW}{dz} - \Gamma W, \quad \text{and} \quad (4.12)$$

$$\sigma S = N^2 W, \quad (4.13)$$

where the parameter Γ is defined by

$$\Gamma = \frac{1}{2\rho_o}\frac{d\rho_o}{dz} + \frac{g}{C_s^2} = \frac{1}{2}\left(\frac{g}{C_s^2} - \frac{N^2}{g}\right). \quad (4.14)$$

By eliminating U, V, and S, we obtain

$$\sigma\left(\frac{d}{dz} + \Gamma_1 - i\Gamma_2\right) P = (N^2 - \lambda) W, \quad (4.15)$$

$$\left(\frac{d}{dz} - \Gamma_1 - i\Gamma_2\right) W = \sigma\left(\frac{1}{C_s^2} - \frac{1}{\mu}\right) P, \quad (4.16)$$

where

$$\Gamma_1 = \Gamma + \frac{f_H \sigma m}{f_V^2 - \sigma^2}, \quad (4.17)$$

$$\Gamma_2 = \frac{-f_H f_V n}{f_V^2 - \sigma^2}, \quad (4.18)$$

$$\lambda = \sigma^2\left(\frac{f_V^2 + f_H^2 - \sigma^2}{f_V^2 - \sigma^2}\right), \qquad (4.19)$$

$$\mu = \frac{-(f_V^2 - \sigma^2)}{m^2 + n^2}. \qquad (4.20)$$

Equations (4.15) and (4.16) are solved under the boundary conditions that

$$W = 0 \quad \text{at } z = 0 \quad \text{and} \quad z = z_T. \qquad (4.21)$$

From this point we will consider the case of isothermal basic state. Thus, the following basic parameters become constant:

$$N^2 = \frac{\kappa g}{H}, \quad C_s^2 = \frac{gH}{1 - \kappa}, \quad \Gamma = \frac{1 - 2\kappa}{2H}, \qquad (4.22)$$

where

$$\kappa = \frac{R}{C_p}\left(= \frac{2}{7}\right) \quad \text{and} \quad H = \frac{RT_o}{g}. \qquad (4.23)$$

Now, (4.15) and (4.16) become equations of constant coefficients. After elimination of P between them, we obtain

$$\frac{d^2W}{dz^2} - 2i\Gamma_2\frac{dW}{dz} - \left[\Gamma_1^2 + \Gamma_2^2 + \left(\frac{1}{C_s^2} - \frac{1}{\mu}\right)(N^2 - \lambda)\right]W = 0. \qquad (4.24)$$

By introducing the transformation

$$W = \eta(z)e^{i\Gamma_2 z} \qquad (4.25)$$

into (4.24), we obtain for $\eta(z)$ in the form

$$\frac{d^2\eta}{dz^2} + \left[\left(\frac{1}{\mu} - \frac{1}{C_s^2}\right)(N^2 - \lambda) - \Gamma_1^2\right]\eta = 0. \qquad (4.26)$$

As seen from the two previous sections, the solutions of (4.26) that satisfy the boundary conditions (4.21) are proportional to $\sin(kz)$. Thus, substitution of (2.21) into (4.26) gives the following equation for frequency σ, using the definition of Γ_1 by (4.17), λ by (4.19), and μ by (4.20),

$$\left(\frac{m^2 + n^2}{f_V^2 - \sigma^2} + \frac{1}{C_s^2}\right)\left[N^2 - \sigma^2\left(\frac{f_V^2 + f_H^2 - \sigma^2}{f_V^2 - \sigma^2}\right)\right] + \Gamma^2 + k^2 + \frac{2f_H m\Gamma\sigma}{f_V^2 - \sigma^2} + \frac{f_H^2 m^2\sigma^2}{(f_V^2 - \sigma^2)^2} = 0. \qquad (4.27)$$

5. Frequency equations for the compressible and stratified model

a. General case with $f_V \neq 0$ and $f_H \neq 0$

In order to derive the algebraic equation of σ from (4.27), we must assume that $f_V^2 \neq \sigma^2$. By multiplying (4.27) by $C_s^2(f_V^2 - \sigma^2)^2$, we obtain

$$\begin{aligned}\sigma^6 &- [N^2 + C_s^2(\Gamma^2 + k^2) + 2f_V^2 + C_s^2(m^2 + n^2) + f_H^2]\sigma^4 + 2f_H m\Gamma C_s^2\sigma^3 \\ &+ \{f_V^4 + 2f_V^2[N^2 + C_s^2(\Gamma^2 + k^2)] + f_V^2C_s^2(m^2 + n^2) + C_s^2N^2(m^2 + n^2) + f_H^2(f_V^2 + C_s^2n^2)\}\sigma^2 \\ &- 2f_H m\Gamma C_s^2 f_V^2\sigma - \{f_V^4[N^2 + C_s^2(\Gamma^2 + k^2)] + C_s^2N^2f_V^2(m^2 + n^2)\} = 0. \qquad (5.1)\end{aligned}$$

The above is a sixth-order equation of σ, and we must resort to solving it numerically. Before doing so, we consider some simpler cases to identify the nature of oscillations associated with the six roots of (5.1).

b. Traditional case of $f_V \neq 0$ and $f_H = 0$

If $f_H = 0$, Eq. (5.1) can be factored out by $(f_V^2 - \sigma^2)$, which is assumed to be nonzero. Therefore, the following quartic equation of σ is obtained:

$$\sigma^4 - [f_V^2 + N^2 + C_s^2(\Gamma^2 + k^2) + C_s^2(m^2 + n^2)]\sigma^2 + \{f_V^2[N^2 + C_s^2(\Gamma^2 + k^2)] + C_s^2N^2(m^2 + n^2)\} = 0. \qquad (5.2)$$

Equation (5.2) has the following two pairs of solutions:

$$\left.\begin{matrix}\sigma_a^2\\ \sigma_g^2\end{matrix}\right\} = \frac{1}{2}C_s^2[m^2 + n^2 + k^2 + \Gamma^2 + C_s^{-2}(f_V^2 + N^2)]\left\{1 \pm \left[1 - \frac{4[N^2(m^2 + n^2) + (k^2 + \Gamma^2 + C_s^{-2}N^2)f_V^2]}{C_s^2[m^2 + n^2 + k^2 + \Gamma^2 + C_s^{-2}(f_V^2 + N^2)]^2}\right]^{1/2}\right\}, \qquad (5.3)$$

where σ_a corresponds to the solutions with the plus sign in front of the radical and σ_g with the minus sign.

The characteristics of (5.3) have been discussed extensively by Eckart (1960), Monin and Obukhov (1959), and Gill (1982). The high-frequency oscillations corresponding to $\pm\sigma_a$ are known as acoustic modes. The low-frequency oscillations corresponding to $\pm\sigma_g$ are referred to as inertio-gravity modes. Of course, the acoustic modes too are modified by the presence of f_V in (5.2).

For a later discussion, it is useful to give the solutions of (5.2) in the case of $m = n = 0$, which is reduced to

$$\sigma^4 - [f_V^2 + C_s^2(\Gamma^2 + k^2) + N^2]\sigma^2 + f_V^2[C_s^2(\Gamma^2 + k^2) + N^2] = 0. \quad (5.4)$$

The solutions of (5.4) are

$$\sigma_a^2 = N^2 + C_s^2(\Gamma^2 + k^2), \qquad \sigma_g^2 = f_V^2. \quad (5.5)$$

However, the low-frequency solutions σ_g^2 are not acceptable, because these violate the basic assumption of $\sigma^2 \neq f_V^2$. Thus, only vertically propagating acoustic waves are present, if $m = n = 0$, but $k \neq 0$.

c. *Special case of $f_V = 0$ and $f_H \neq 0$*

At the equator, f_V vanishes and f_H becomes 2Ω. In this case, the frequency equation (5.1) is reduced to the following quartic equation by again assuming that σ does not vanish:

$$\sigma^4 - [f_H^2 + N^2 + C_s^2(\Gamma^2 + k^2) + C_s^2(m^2 + n^2)]\sigma^2 + 2f_H m\Gamma C_s^2\sigma + [C_s^2N^2(m^2 + n^2) + f_H^2C_s^2n^2] = 0. \quad (5.6)$$

It is obvious that there are two pairs of large and small values of σ having positive and negative signs with slightly different magnitudes. These high- and low-frequency pairs correspond to the acoustic and gravity waves, respectively, modified by the earth's rotation.

For the zonal motions $m = 0$, the roots of (5.6) can be expressed by

$$\left.\begin{matrix}\sigma_a^2\\ \sigma_g^2\end{matrix}\right\} = \frac{1}{2}C_s^2[n^2 + k^2 + \Gamma^2 + C_s^{-2}(f_H^2 + N^2)]\left\{1 \pm \left[1 - \frac{4n^2(N^2 + f_H^2)}{C_s^2[n^2 + k^2 + \Gamma^2 + C_s^{-2}(f_H^2 + N^2)]^2}\right]^{1/2}\right\}. \quad (5.7)$$

Furthermore, in the case of $n = 0$ in addition to $m = 0$, (5.7) yields

$$\sigma_a^2 = N^2 + C_s^2(\Gamma^2 + k^2) + f_H^2, \qquad \sigma_g^2 = 0. \quad (5.8)$$

Thus, only vertically propagating acoustic waves are present if $m = n = 0$, but $k \neq 0$.

d. *Special case of $f_V \neq 0$ and $f_H \neq 0$ with $m = n = 0$ but $k \neq 0$*

Egger (1999) examined the characteristics of oscillations in a stratified and compressible model identical to ours, including the f_V and f_H terms, except that he only treated the case of both zonal and meridional wavenumbers being zero; that is, $m = n = 0$ but $k \neq 0$. In this special case, Eq. (5.1) is factored out by $(f_V^2 - \sigma^2)$ which should not vanish, and the remaining fourth-order equation becomes

$$\sigma^4 - [f_V^2 + f_H^2 + C_s^2(\Gamma^2 + k^2) + N^2]\sigma^2 + f_V^2[C_s^2(\Gamma^2 + k^2) + N^2] = 0. \quad (5.9)$$

This agrees with the quartic equation derived from Eq. (18) of Egger (1999). To compare the form of (5.9) with his Eq. (18), we introduce

$$\sigma_{a0}^2 = C_s^2\Gamma^2 + N^2 = \frac{N^2}{4\kappa(1 - \kappa)} = \frac{49}{40}N^2 \quad (5.10)$$

for $\kappa = 2/7$. The quantity σ_{a0} is referred to as the acoustic cutoff frequency (e.g., Tolstoy 1973; Gossard and Hooke 1975). Hence, (5.9) becomes

$$\sigma^4 - (4\Omega^2 + \sigma_{a0}^2 + C_s^2k^2)\sigma^2 + f_V^2(\sigma_{a0}^2 + C_s^2k^2) = 0. \quad (5.11)$$

As noted by Egger (1999), the above equation gives two pairs of frequencies with plus and minus sign. One pair of high frequencies can be approximated by $\pm(4\Omega^2 + \sigma_{a0}^2 + C_s^2k^2)^{1/2}$. They represent the vertically propagating acoustic waves modified by the earth's rotation. The other pair of low frequencies are close to $\pm|f_V|$ and represent a kind of inertial wave. As Egger (1999) stated, the role of cos Coriolis terms, as well as compressibility and gravity are needed to recover inertial waves in the case of $m = n = 0$ but $k \neq 0$. This point can be seen by comparing (5.9) with (5.4). The presence of f_H^2 in (5.9) is responsible for giving a slight shift of σ^2 away from f_V^2. Thus, in contrast to the situation of (5.5), the vertically propagating inertial oscillations can emerge even in the case of $m = n = 0$ but $k \neq 0$.

e. *Boundary-induced inertial modes*

Earlier in section 3, we noted that there are two kinds of normal modes in the Boussinesq model. One kind represents the inertio-gravity modes and an approximate form of the solutions is given by (3.17). The other represents the BII modes whose frequencies are very close to f_V and an approximate form of frequencies is given by (3.19) with (3.20). Therefore, it is likely that there exist two roots of (5.1) whose values are close to $\pm|f_V|$. We can derive an approximate form of these roots under normal conditions of $N^2 \gg \Omega^2$. The result is

$$\sigma_I \doteq f_V(1 - f_H^2n^2E_2^{-1}) \quad (5.12)$$

where

$$E_2 = 2[(m^2 + n^2)(N^2 - f_V^2) + f_H^2(n^2 - C_s^{-2}f_V^2) + 2m\Gamma f_H f_V]. \quad (5.13)$$

We can see from (5.12) that one σ_I corresponds to f_V and another σ_I corresponds to $-f_V$. Their magnitudes are slightly smaller than $|f_V|$, because the terms $f_H^2 n^2 E_2^{-1}$, which is assumed to be much smaller than unity, is normally positive. Moreover, the magnitude of positive σ_I is slightly larger than the magnitude of the corresponding negative σ_I, except for the case of $m = 0$ in which case both magnitudes are equal.

It is clear that the forms (3.19) and (5.12) are similar, and E_1 and E_2 are related. In fact, in the Boussinesq model ($C_s \to \infty$ and $\Gamma = 0$), E_2 reduces to E_1. Therefore, the waves represented by the frequency σ_I of (5.12) are the BII modes in the sense defined in section 3, modified now by the presence of compressibility and Γ.

We have verified through the numerical solutions of (5.1) that the formula (5.12) with (5.13) gives accurate approximations to the numerical values that correspond to the remaining two roots of (5.1) besides the two pairs of acoustic frequencies σ_a and inertio-gravity frequencies σ_g. If $n = 0$, $(\sigma^2 - f_V^2)$ can be factored out from (5.1), and the roots of the remaining quartic equation are identified as the two pairs of acoustic and inertio-gravity modes with $n = 0$. Since we have assumed that $(\sigma^2 - f_V^2)$ should not vanish, no solution corresponding to σ_I exists if $n = 0$.

f. Lamb waves

For isothermal atmospheres there are additional special solutions of Eqs. (4.15) and (4.16) with the boundary conditions (4.21). They are

$$W = 0, \qquad P = e^{-(\Gamma_1 - i\Gamma_2)z}. \tag{5.14}$$

Equation (4.16) is satisfied by $\mu = C_s^2$. Hence, using the definition (4.20) we find

$$\sigma^2 = C_s^2(m^2 + n^2) + f_V^2. \tag{5.15}$$

In order to ensure that the eigensolutions, known as Lamb waves, decrease exponentially with height, we must choose $\Gamma_1 > 0$, namely

$$\sigma f_H m \leq \Gamma C_s^2(m^2 + n^2). \tag{5.16}$$

This condition has been pointed out by Eckart (1960). Note that the waves corresponding to negative σ satisfy (5.16) automatically, but the waves with positive σ must satisfy the following condition:

$$[C_s^2(m^2 + n^2) + f_V^2]^{1/2} f_H m \leq \Gamma C_s^2(m^2 + n^2), \tag{5.17}$$

which is obtained by combining (5.15) and (5.16).

Eckart (1960) examines this unique character of Lamb waves. Note that the amplitude of Lamb waves decreases exponentially, but the direction of propagation is not horizontal. The angle α with the horizontal is given as

$$\tan\alpha = \frac{\Gamma_2}{(m^2 + n^2)^{1/2}} = \frac{n f_H f_V}{C_s^2(m^2 + n^2)^{3/2}} \tag{5.18}$$

by using (4.18) and (5.15). For the terrestrial atmosphere, the condition (5.17) is well satisfied and the angle α is negligibly small except for very large-scale motions, never exceeding the matter of few degrees.

6. Conclusions and further discussion

Motivated by the desire to quantitatively assess the role of the horizontal component of the earth's rotation, which is neglected in primitive equation models, normal mode analysis has been performed for a compressible and stratified isothermal atmosphere that includes the cos (latitude) Coriolis terms, referred to as f_H terms. It was found that there are three kinds of internal modes. Each kind consists of positive and negative frequencies whose magnitudes are slightly different unless the product $m\Gamma$ vanishes. The three kinds are acoustic, inertio-gravity, and boundary-induced inertial (BII) modes.

A question has been raised why there are six wave frequencies for a system that consists of five time-dependent equations for five dependent variables. Phillips (1990) analyzed the same system of equations by expressing the solutions in the form of plane waves proportional to $\exp[i(mx + ny + kz - \sigma t)]$ and obtained the dispersion equation for σ that is a fifth-order polynomial. One of the roots of the frequency equation is $\sigma = 0$, which corresponds to steady-state solutions. Four remaining roots represent two pairs of acoustic and gravity waves, both of which are modified by the presence of f_V and f_H terms.

This article is written to clarify the reasons why there are three kinds of normal modes in a compressible, stratified, and rotating fluid bounded by rigid horizontal surfaces, but only two kinds of plane wave if the fluid is unbounded.

A unique role of the boundary conditions in influencing the normal modes of rotating fluid motions in contrast to plane wave solutions may be explained succinctly using an incompressible and homogeneous model presented in section 2. First, the vertical structure equation (2.16) of W was derived. Because (2.16) contains the first-order derivative term, we introduced transformation (2.18) to eliminate the first-order term and obtained Eq. (2.20), which is solved under the boundary condition (2.17). The origin of a fourth-order term in the frequency equation (2.29) can be traced to the square term Γ_2^2 in Eq. (2.20). The vertical structure functions W are given in the form of $\exp[i(\Gamma_2 \pm k)z]$.

If the same problem is solved without the boundary conditions to obtain plane wave solutions, the frequency equation becomes (2.36), and the vertical structure function W is given by $\exp(ikz)$. Note that the frequency σ, as given by (2.36), depends on the sign of k/n if $f_H \neq 0$. Therefore, upward and downward propagating plane waves cannot be combined to satisfy the boundary condition $W = 0$ at $z = 0$ because they have different frequencies. In contrast, the incident and reflected waves of the form $\exp[i(\Gamma_2 \pm k)z]$ can be combined to satisfy the vertical boundary conditions, because the frequency

σ, as given by (2.30), does not depend on the sign of k/n.

The same methodology is used to analyze the normal modes of a Boussinesq model in section 3. With the addition of thermal stratification effect in the incompressible and homogeneous model, the dispersion equation became again quartic due to the presence of the boundary conditions in the case of $f_H \neq 0$. One unique aspect of the Boussinesq model is that, under a normal condition of $N^2 \gg \Omega^2$, two sets of the quadratic roots of (3.16) tend to separate well. They are expressed approximately by (3.17) corresponding to the inertio-gravity modes and by (3.19) as the BII modes. Once the circumstance in which the cause of BII modes is understood in the Boussinesq model, it is logical to expect in the compressible, stratified, and rotating model the acoustic modes to emerge in addition to the inertio-gravity and BII modes.

It may be pertinent to comment here on the influence of boundary conditions on the normal modes of rotating fluid. A well-known example is the case of Kelvin mode in a shallow, rotating fluid in a partially bounded channel, oriented parallel to the *x* axis. Pedlosky (1987, section 3.9) presented a step-by-step procedure to derive the eigensolutions and associated eigenfrequencies and discussed the emergence of the Kelvin mode in addition to inertio-gravity modes, known as the Poincaré waves. Pedlosky also examined the physical significance of the third apparent solution of the dispersion equation as an oscillation whose frequency is the Coriolis parameter f_{V} and showed that such a solution is spurious. This example points out the desirability of examining the influence of the lateral conditions, as well as the vertical conditions on the normal modes of fluid models with a complete representation of the Coriolis force. However, such studies may lead to very complicated analytical problems.

Extensive numerical calculations were conducted to solve the sixth-order polynomial frequency equation (5.1) for various values of wavenumbers, *m*, *n*, and *k* under atmospheric conditions (Kasahara 2001, unpublished manuscript). As expected, the effects of all Coriolis terms have little influence on the frequency of acoustic modes. One interesting result is that the f_{H} terms have significantly more effect than the f_{V} terms on the frequency of inertio-gravity modes, except for very large scale motions. {The degree of influence of the earth's rotation on the frequency of the inertio-gravity waves is measured by the ratio of $[\sigma(f_{\mathrm{V}}$ or $f_{\mathrm{H}}) - \sigma(0)]/\sigma(0)$, where $\sigma(f_{\mathrm{V}})$ and $\sigma(f_{\mathrm{H}})$ denote, respectively, the value of σ_g at the pole ($f_{\mathrm{V}} = 2\Omega$ and $f_{\mathrm{H}} = 0$) and the value of σ_g at the equator ($f_{\mathrm{V}} = 0$ and $f_{\mathrm{H}} = 2\Omega$) with $\sigma(0)$ indicating the gravity wave frequency without the effect of earth's rotation.}

Because of normal conditions of $N^2 \gg \Omega^2$, the impacts of earth's rotation on gravity waves are relatively minor. Nevertheless, if accurate calculations of the divergent wind components are desired, then it is unreasonable to neglect the role of f_{H} terms. Likewise, it is not justifiable to neglect the role of f_{H} terms in a stratified model if our interest is in the wave motions whose frequencies are close to the Coriolis frequency. It is not the purpose of this article to analyze the properties of the BII modes, nor to discuss the possibility of this kind of wave motion to exist in the atmosphere and oceans.

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APPENDIX

Note on Additional Recent References

During a stage of the review process of this article, the author's attention was called to two articles published recently by Thurburn et al. (2002a,b) related to the normal modes of compressible and stratified atmospheric models with inclusion of both the sin and cosine Coriolis terms. Thuburn et al. (2002a) deals with the model in spherical geometry and the other is concerned with the model in the Cartesian coordinates on a tangent plane in the domain that is periodic in the zonal and meridional directions, but bounded at the top and bottom.

Thus, Thuburn et al. (2002b) deals with exactly the same model as described in section 4 of this article. Therefore, it is no surprise that the results of their normal mode analysis agree in many aspects with those described in sections 4 and 5 of this article and vice versa. For example, the dispersion equation (5.1) in section 5 of this work, which is a sixth-order equation of frequency σ, can be derived from Eq. (4.1) of Thuburn et al. (2002b) by matching the variables, parameters and symbols used in the two works. Also, the boundary-induced inertial modes discussed in section 5e are identical in principle to those of "new modes" discussed in section 5 of their article. Since their method of solution is somewhat different from that used in this work, agreement between the two results will provide independent verification on the existence of this unique kind of wave modes in the present model configuration.

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