Linear Anelastic Equations for Atmospheric Vortices

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ABSTRACT

A linear anelastic-vortex model is derived using assumptions appropriate to waves on vortices with scales similar to tropical cyclones. The equation set is derived through application of a multiple-scaling technique, such that the radial variations of the thermodynamic fields are incorporated into the reference state. The primary assumption required for the model is that the horizontal variations in the thermodynamic variables describing the reference state are appreciably longer than the waves on the vortex. This new version of the anelastic system makes no approximation to the requirements for hydrostatic and gradient wind balance, or the buoyancy frequency, in the core of the vortex. A small but measurable improvement in the performance of the new equation set is demonstrated through simulations of gravity waves and vortex–Rossby waves in a baroclinic vortex.

1. Introduction

The anelastic approximation has been successfully used for many years in a variety of meteorological applications. The primary motivation for the anelastic approximation is to remove the meteorologically insignificant sound waves from the fully compressible equations without requiring hydrostatic balance.

Ogura and Philips (1962) were the first to introduce the anelastic approximation to the study of deep convection. Their scaling analysis was based on the following assumptions: 1) the disturbance time scale is set by the buoyancy frequency, and 2) the perturbations of potential temperature are small and about an isentropic reference state. Wilhelmson and Ogura (1972) extended this model by incorporation of vertical variations in the potential temperature of the reference state. This allowed for the application of the anelastic system to environments that are not isentropic, but at the cost of losing energy conservation. Lipps and Hemler (1982) revisited the anelastic system and found that restricting the Wilhelmson and Ogura (1972) system to include only slow vertical variations in the potential temperature of the reference state led to an energy-conserving set. Other energy-conserving sets have since been found that are based on assumptions concerning the scale of the variations in the thermodynamic variables and time scale of the perturbations (e.g., Durran 1989; Bannon 1996). In addition, Scinocca and Shepherd (1992) have rigorously derived, from a Hamiltonian principle, finite-amplitude, local wave-activity conservation laws for a two-dimensional version of the anelastic equations of Lipps and Hemler (1982).

The use of the anelastic equations to study the dynamics of strong atmospheric vortices, like tropical cyclones, has been hindered by the radial dependencies in the vortex. Other meteorological problems have similar issues. Wave dynamics in large-scale flows at midlatitudes is an example of a phenomenon for which the anelastic equations have been used for many years and where it is advantageous to study mean flow conditions with vertical shears and horizontal temperature gradients (e.g., Yamazaki and Peltier 2001; Lott 2003). The general procedure, however, has been to incorporate horizontal gradients as a small deviation from the vertically varying reference state (as shown in section 2). Because of the importance of the strength of the warm

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core in tropical cyclones, they cannot reasonably be considered small deviations from a reference state that varies only in the vertical. A different procedure must be adopted for vortices of this type.

The significance of radial variation in the reference state for a strong, baroclinic vortex is shown in Fig. 1. Figure 1a shows an axisymmetric wind field modeled after a modestly strong tropical cyclone (near hurricane strength) with a maximum azimuthal wind speed of 30 m s\(^{-1}\) at a radius of 60 km. The pressure and temperature fields that hold this vortex in hydrostatic and gradient wind balance [see Eqs. (3.5a) and (3.5d)] were computed through an iterative scheme as described in Nolan et al. (2001). Figure 1b shows the potential temperature perturbation, relative to the far-field sounding, that holds this vortex in balance. Figure 1c shows the square of the buoyancy frequency,

\[
N^2 = \frac{g}{\theta} \frac{d\theta}{dz},
\]

associated with the warm core in the upper part of the center of the vortex. In the core of the vortex, \(N^2\) varies from its value in the far field \([N^2(r \rightarrow \infty) = 10^{-4} \text{ s}^{-2}]\), with regions of stronger and weaker stability below and above the warm core, respectively. As a vortex of this type increases in strength, the warm core and its associated deviations in \(N^2\) become stronger. For example, a vortex of identical structure, but with a maximum wind speed of 40 m s\(^{-1}\), has \(N^2\) decrease to nearly to 0 above the warm core, indicating very low static stability. The warm core of this vortex becomes convectively unstable if the wind speed is increased beyond 42 m s\(^{-1}\).

For consistency with the equations of motion, the equations for hydrostatic and gradient wind balance in an anelastic system are slightly different from the unapproximated balance equations. These equations for the anelastic and unapproximated forms of the vortex-anelastic equations are presented in later sections. To illustrate the differences, we show in Figs. 2a,b vertical profiles of the warm core and the static stability for vortices with maximum winds of 30 and 40 m s\(^{-1}\), for the unapproximated balance equations and for the anelastic balance equations. The anelastic system underestimates the strength of the warm core, and overestimates the static stability above the warm core. This effect can be important. Note in Fig. 2b that for the 40 m s\(^{-1}\) vortex the difference in buoyancy frequency at \(z \sim 8\) km is approximately 30\%. In addition, if the wind speed is increased slightly beyond 42 m s\(^{-1}\) the vortex will become unstable for the unapproximated balance equations but not for the anelastic approximations.

The desire to use the simplifications afforded by the anelastic approximation to study tropical cyclone dynamics has given rise to the use of ad hoc anelastic equation sets (e.g., Nolan and Montgomery 2002, hereafter NM02; Nolan and Grasso 2003, hereafter NG03).
It is the goal of this article to place these models, which use radially varying reference states, on a firmer footing. In particular, we will describe through scale analysis the conditions that assure that models that use radially varying reference states may be safely regarded as accurate approximations to the fully compressible equations.

We begin in section 2 where we describe the traditional approach to creating a linear anelastic-vortex model and discuss its limitations. In section 3 we derive, using scaling assumptions appropriate to atmospheric vortices, linear anelastic equations valid for horizontally and vertically varying reference states. In section 4 we apply this new equation set to the study of unbalanced, asymmetric perturbations to tropical cyclones, and in section 5 we provide a brief summary.

2. A traditional anelastic-vortex model

In this section we will briefly review the anelastic approximation as applied to atmospheric vortices. There exist several anelastic systems in the literature. We choose that of Bannon (1996, hereafter B96) because of its superior performance in simulating Lamb's problem for hydrostatic adjustment, and its conservation of energy and potential vorticity. The anelastic approximation requires the following conditions: 1) the buoyancy force is a dominant physical process, 2) the vertical displacement of an air parcel is comparable to or less than the density-scale height, and 3) the horizontal variations of thermodynamic variables at any height are small compared to the static value. See B96 for a more in-depth discussion of the approximations leading to this form of the anelastic equation set.

With the appropriate application of these conditions the following anelastic model is obtained. We define the anelastic thermodynamic fields as deviations from a static anelastic reference state:

\[
\begin{align*}
\rho & = \rho_s(z) + \rho'(r, \lambda, z, t), \\
\theta & = \theta_s(z) + \theta'(r, \lambda, z, t), \\
T & = T_s(z) + T'(r, \lambda, z, t), \\
\rho & = \rho_s(z) + \rho'(r, \lambda, z, t).
\end{align*}
\]

The radial, azimuthal, and vertical directions will be denoted by \( r \), \( \lambda \), and \( z \), respectively. Time will be denoted by \( t \).

Requirement 3 above demands that the primed fields in (2.1) be small such that

\[
\left| \frac{\rho'}{\rho_s} \right|, \left| \frac{\theta'}{\theta_s} \right|, \left| \frac{T'}{T_s} \right|, \left| \frac{\rho'}{\rho_s} \right| \ll 1.
\]

Using the decompositions in (2.1) the anelastic momentum equations in cylindrical coordinates may be written as

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \lambda} + w \frac{\partial u}{\partial z} - \frac{u^2}{r} - f v &= - \frac{1}{\rho_s} \frac{\partial \rho'}{\partial r}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \lambda} + w \frac{\partial v}{\partial z} + \frac{u v}{r} + f u &= - \frac{1}{r \rho_s} \frac{\partial \rho'}{\partial \lambda}, \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial \lambda} + w \frac{\partial w}{\partial z} &= - \frac{\partial}{\partial z} \left( \frac{\rho'}{\rho_s} \right) + g \frac{\partial \theta'}{\partial \lambda},
\end{align*}
\]
along with the anelastic mass continuity equation:

$$\frac{1}{r} \frac{\partial \rho_r u}{\partial r} + \frac{1}{r} \frac{\partial \rho_\lambda v}{\partial \lambda} + \frac{\partial \rho_w w}{\partial z} = 0, \quad (2.4)$$

and the equation for the potential temperature field:

$$\frac{\partial \theta'}{\partial t} + u \frac{\partial \theta'}{\partial r} + v \frac{\partial \theta'}{\partial \lambda} + w \frac{\partial \theta}{\partial z} = 0. \quad (2.5)$$

The variables $u$, $v$, and $w$ are the radial, azimuthal, and vertical wind, respectively; $p$, $\rho$, $T$, and $\theta$ are density, pressure, temperature, and potential temperature, respectively; and $g$ is the gravitational acceleration.

As discussed by Bannon (1995, 1996) a critical component to assuring that the nonlinear system in (2.3)–(2.5) conserves energy is the proper choice of thermodynamic relations. The anelastic system of B96 uses a linearized form of the ideal gas law:

$$\frac{p'}{\rho_s} = \frac{p'}{\rho_s} + \frac{T'}{T_s}, \quad (2.6)$$

and a diagnostic relation between the potential temperature, density, and pressure:

$$\frac{\theta'}{\theta_s} = \frac{\rho'}{\rho_s g H_p} - \frac{\rho'}{\rho_s}, \quad (2.7)$$

where $H_p$ is the density-scale height defined by

$$H_p = \left( \frac{-1}{\frac{1}{\rho_s} \frac{\partial p}{\partial z}} \right)^{-1}. \quad (2.8)$$

To use these equations to study vortex dynamics, we define the anelastic perturbation fields in the following way:

$$\rho' = \bar{\rho}(r, z) + \rho''(r, \lambda, z, t), \quad (2.9a)$$

$$\theta' = \bar{\theta}(r, z) + \theta''(r, \lambda, z, t), \quad (2.9b)$$

$$T' = \bar{T}(r, z) + T''(r, \lambda, z, t), \quad (2.9c)$$

$$p' = \bar{p}(r, z) + p''(r, \lambda, z, t), \quad (2.9d)$$

$$v = \bar{v}(r, z) + v''(r, \lambda, z, t), \quad (2.9e)$$

$$u = u''(r, \lambda, z, t), \quad w = w''(r, \lambda, z, t), \quad (2.9f,g)$$

where the vortex is contained in the fields with the overbar and the “waves” are contained in the “double-primed” quantities. Upon assuming that the wave quantities are small in the sense that

$$\frac{\rho''}{\bar{\rho}}, \frac{\theta''}{\bar{\theta}}, \frac{T''}{\bar{T}}, \frac{p''}{\bar{p}}, \frac{v''}{\bar{v}} \ll 1, \quad (2.10)$$

we may linearize around the reference state and overbar fields. Note that the vortex must satisfy anelastic approximations to the gradient wind balance and hydrostatic balance:

$$\frac{\bar{v}^2}{r} + \frac{f \bar{v}}{\rho_s \frac{\partial p}{\partial r}} = 0, \quad (2.11a)$$

$$\frac{\partial}{\partial z} \left( \frac{\bar{p}}{\rho_s} \right) = g \frac{\bar{\theta}}{\theta_s}, \quad (2.11b)$$

as well as the following diagnostic relations for the thermodynamics:

$$\frac{\bar{p}}{\rho_s} = \frac{\bar{p}}{\rho_s} + \bar{T}, \quad (2.12a)$$

$$\frac{\bar{v}}{\theta_s} = \frac{\bar{p}}{\rho_s g H_p} - \frac{\bar{p}}{\rho_s}, \quad (2.12b)$$

The waves are governed by a linearized form of the anelastic equations of B96 in cylindrical coordinates:

$$\frac{\partial u''}{\partial t} + \frac{\bar{\nu}}{r} \frac{\partial u''}{\partial \lambda} = - \frac{1}{\rho_s} \frac{\partial}{\partial r} \left( \frac{p''}{\rho_s} \right), \quad (2.13a)$$

$$\frac{\partial v''}{\partial t} + \frac{\bar{\nu}}{r} \frac{\partial v''}{\partial \lambda} + w'' \frac{\partial \bar{v}}{\partial z} - \bar{\nu} u'' = - \frac{1}{\rho_s} \frac{\partial}{\partial \lambda} \left( \frac{p''}{\rho_s} \right), \quad (2.13b)$$

$$\frac{\partial w''}{\partial t} + \frac{\bar{\nu}}{r} \frac{\partial w''}{\partial \lambda} + \frac{\theta''}{\theta_s} \bar{w} - \bar{\nu} u'' = - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \frac{p''}{\rho_s} \right), \quad (2.13c)$$

$$\frac{1}{r} \frac{\partial \rho_r u''}{\partial r} + \frac{1}{r} \frac{\partial \rho_\lambda v''}{\partial \lambda} + \frac{\partial \rho_w w''}{\partial z} = 0, \quad (2.13d)$$

where the modified Coriolis parameter is

$$\zeta = \frac{2 \bar{\nu}}{r} + f, \quad (2.14a)$$

and the absolute vorticity of the background state is

$$\bar{\eta} = \frac{1}{r} \frac{\partial \bar{\theta}}{\partial r} + f. \quad (2.14b)$$

There are two major drawbacks to applying the system presented in (2.13) to the study of waves on atmospheric vortices. The first is that the vortex is defined to satisfy anelastic approximations to gradient wind balance, hydrostatic balance, and the thermodynamic diagnostic relations. The second is the assumption that
the buoyancy frequency of the background state is accurately approximated as

\[ N_B^2 = \frac{g}{\theta_0} \frac{\partial}{\partial z} (\theta_s + \bar{\theta}). \]  

(2.15)

There are two potential errors in (2.15): 1) because of the approximations in (2.11), \( \theta_s + \bar{\theta} \) is an approximation to the vortex potential temperature and 2) because the thermodynamic variations of the vortex are not included in the denominator of (2.15), the near-core buoyancy frequency will be overestimated for warm-core vortices (\( N_B^2 > N^2 \)) and underestimated for cold-core vortices (\( N_B^2 < N^2 \)). These errors in the buoyancy frequency will lead directly to errors in the vertical motion field. We show in section 4 that this error in the buoyancy frequency leads to small errors in the simulation of the hydrostatic adjustment of an initially imbalanced potential temperature perturbation. By formulating a model that makes no approximation to the buoyancy frequency, or the requirements for hydrostatic and gradient wind balance, in the next section we obtain an improved linear anelastic system.

3. The vortex-anelastic model

The following analysis will make use of a judiciously chosen multiple-scale analysis to derive a linear anelastic equation set directly from the fully compressible fluid equations on the \( f \) plane. The interested reader may consult Klein (2000) and Majda and Klein (2003) for a more comprehensive discussion of the application of multiple-scale techniques to atmospheric modeling.

a. Equations of motion

The fully compressible fluid equations on an \( f \) plane and in cylindrical coordinates are

\[ \begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + & \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial z} (u^2) - f v = - \frac{1}{\rho} \frac{\partial p}{\partial r}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + & \frac{1}{r} \frac{\partial}{\partial r} (rv) + \frac{\partial}{\partial z} (uv) + \frac{\partial w}{\partial t} + f u = - \frac{1}{\rho} \frac{\partial p}{\partial \lambda} , \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + & \frac{1}{r} \frac{\partial}{\partial r} (rw) + \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g, \\
\frac{\partial p}{\partial t} + & \frac{1}{r} \frac{\partial}{\partial r} (rp) + \frac{1}{r} \frac{\partial}{\partial \lambda} (pw) + \frac{\partial}{\partial z} (pw) = 0, \\
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + & \frac{1}{r} \frac{\partial}{\partial r} (\theta r) + \frac{\partial}{\partial \lambda} (\theta r) + \frac{\partial}{\partial z} (\theta) = 0, \\
\theta = T \left( \frac{\rho_0}{\rho} \right)^{\kappa}, & p = \rho T. 
\end{align*} \]  

(3.1a) (3.1b) (3.1c) (3.1d) (3.1e) (3.1f)

The symbols representing the dependent and independent variables in (3.1) are defined in section 2. The parameter \( R \) is the gas constant for dry air, \( p_{oo} \) is a reference pressure typically taken to be \( 10^5 \) Pa, \( \kappa = R/c_p^2 \), and \( c_p \) is the specific heat at constant pressure.

To apply the multiple-scaling technique, we must first nondimensionalize these equations. To nondimensionalize (3.1) we choose to scale the dependent and independent variables in the following way:

\[ (r, r\lambda, \xi, \tau) = L \left( \hat{r}, \hat{r}\lambda, \hat{\xi}, \frac{1}{\sqrt{\nu}} \right), \]  

(3.2a)

\[ (u, v, w) = V (\hat{u}, \hat{v}, \hat{w}), \]  

(3.2b)

\[ (p, p, T, \theta) = \left( \rho_{oo} \hat{p}, \rho_{oo} \hat{p}, \frac{p_{oo}}{R \rho_{oo}} \frac{T}{\theta_0}, \frac{p_{so}}{R \rho_{oo}} \theta \right), \]  

(3.2c)

where carats over a variable denotes a nondimensional quantity. In (3.2) it is assumed that the chosen scaling parameters will lead to nondimensional quantities that are \( O(1) \) in the particular application.

Using (3.2) in (3.1), and for ease of presentation subsequently dropping the carats, results in a nondimensional set of equations governing the vortex dynamics:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \lambda} + \frac{w}{\partial z} - \frac{v^2}{r} - \frac{1}{Ro} v = - \frac{1}{M^2} \frac{\partial p}{\partial r}, \]  

(3.3a)

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \lambda} + \frac{w}{\partial z} + \frac{uv}{r} + \frac{1}{Ro} u = - \frac{1}{M^2} \frac{\partial p}{\partial \lambda}, \]  

(3.3b)

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \lambda} + \frac{w}{\partial z} = - \frac{1}{M^2} \frac{\partial p}{\partial z} - \frac{1}{F_r}, \]  

(3.3c)

\[ \frac{\partial p}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rp) + \frac{1}{r} \frac{\partial}{\partial \lambda} (pw) + \frac{\partial}{\partial z} (pw) = 0, \]  

(3.3d)

\[ \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + \frac{v}{r} \frac{\partial \theta}{\partial \lambda} + \frac{w}{\partial z} = 0, \]  

(3.3e)

\[ \theta = \frac{p^{1-\kappa}}{p}, \quad p = \rho T, \]  

(3.3f,g)

where \( M = V/c_s \) is the Mach number, \( c_s = \sqrt{p_{oo}/\rho_{oo}} \) is the sound wave phase speed, \( Ro = V/fL \) is the Rossby number, \( F_r = V/c_{s_k} \) is the barotropic Froude number, and \( c_{s_k} = \sqrt{gL} \) is the long-wave, barotropic gravity wave phase speed. For the typical scales of waves on atmospheric vortices, say \( L = 10 \) km, \( c_k \sim c_s \). For this
reason the barotropic Froude number should not be considered an indicator of the actual speed of gravity waves in this baroclinic model. For example, in the simulations of section 4, the gravity waves propagate at a maximum speed of approximately 50 m s\(^{-1}\).

**b. Multiple-scale analysis**

To derive the vortex-anelastic equation set we must first choose the relationships between the nondimensional parameters. We will assume that the wind speed scale, \(V\), is such that the Mach and Froude numbers are small. For example, the tropical cyclone in Fig. 1 is consistently scaled by \(V = 30 \text{ m s}^{-1}\), which implies that \(M, F_\tau \sim 1/10\). Defining a small parameter, \(\varepsilon\), such that \(M = F_\tau = \varepsilon \ll 1\), allows for the expansion of the dependent variables in a small single parameter. We will also require that \(Ro = O(1)\) in order to retain the Coriolis force at the lowest order. However, for strong tropical cyclones, \(Ro = O(10)\) and therefore the Coriolis force will not be significant when compared with inertial and pressure gradient forces. We prefer to include the Coriolis force in the vortex-anelastic equation set and allow the particular application to scale the Coriolis force appropriately.

We expand the dependent variables in the following perturbation series:

\[
\begin{align*}
  u &= u_0(r, \varepsilon^2 r, z) + \varepsilon u_1(r, \varepsilon^2 r, \lambda, z, \varepsilon^2 z, t) + \ldots, \\
  v &= v_0(r, \varepsilon^2 r, z) + \varepsilon v_1(r, \varepsilon^2 r, \lambda, z, \varepsilon^2 z, t) + \ldots, \\
  w &= w_0(r, \varepsilon^2 r, z) + \varepsilon w_1(r, \varepsilon^2 r, \lambda, z, \varepsilon^2 z, t) + \ldots, \\
  p &= p_0(\varepsilon^2 r, z) + \varepsilon^3 p_1(r, \varepsilon^2 r, \lambda, z, \varepsilon^2 z, t) + \ldots, \\
  \theta &= \theta_0(\varepsilon^2 r, \varepsilon^2 z) + \varepsilon^3 \theta_1(r, \varepsilon^2 r, \lambda, z, \varepsilon^2 z, t) + \ldots.
\end{align*}
\]

By making the ansatz (3.4) we concisely define our a priori assumptions about the relative magnitudes of all the dependent variables as well as the space and time-scale assumptions of the independent variables. In the subsequent analysis we will find that the lowest-order terms represent the “vortex” and the next order terms represent the “waves.” The kinematic variables at lowest order have been chosen to include both slow and fast radial variations, but the thermodynamic variables include only slow radial variations. This particular choice for the lowest-order kinematic and thermodynamic fields ensures the vortex will be in gradient and hydrostatic balance. In addition, consistent with early work on the anelastic system (e.g., Lipps and Hemler 1982), the potential temperature varies slowly in the vertical.

Using the expansions (3.4) in (3.3) and balancing like orders in the expansion parameter, \(\varepsilon\), results in a sequence of equations governing the vortex and disturbances to that vortex. The lowest-order balance results in the diagnostic balances for the vortex:

\[
\begin{align*}
  \frac{v_0^2}{r} + \frac{1}{Ro} v_0 &= \frac{1}{\rho_0} \frac{\partial p_0}{\partial r}, \\
  u_0 &= 0, \quad w_0 = 0, \\
  \frac{\partial p_0}{\partial z} &= -\rho_0, \\
  \theta_0 &= \rho_0^{1-\kappa}, \quad p_0 = \rho_0 T_0.
\end{align*}
\]

Because the centrifugal force term in Eq. (3.5a) contains “fast” radial variations and the pressure gradient force contains “slow” radial variations the lowest-order azimuthal wind \((u_0)\) requires two radial scales. These two spatial scales have distinct physical interpretations. The fast radial scale represents the cyclostrophic balance and dominates in the eyewall region inward toward the vortex center. The slow radial scale represents the geostrophic balance and dominates outside of the eyewall as \(r \to \infty\). In addition, note that in contrast to the linear anelastic system of section 2, the vortex in (3.5) satisfies gradient wind, hydrostatic balance, Poisson’s equation, and the ideal gas equation without approximation [i.e., \(\theta_0 \neq \theta_1 + \theta; \text{ cf. (3.5)} \text{ to (2.8) and (2.9)}\)].

The next order balance obtains equations that govern the motion of small disturbances to the vortex in (3.5). First, however, we must consider the \(O(\varepsilon)\) vertical momentum equation:

\[
\frac{\partial w_1}{\partial t} + \frac{v_0}{r} \frac{\partial w_1}{\partial r} = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial z} - \frac{\rho_1}{\rho_0},
\]

Without approximation (3.6) may be written as

\[
\frac{\partial w_1}{\partial t} + \frac{v_0}{r} \frac{\partial w_1}{\partial r} = -\frac{\partial}{\partial z} \left( \frac{p_1}{\rho_0} \right) - \frac{p_1}{\rho_0} + \frac{p_1}{\rho_0 H_\rho},
\]

where \(H_\rho\) differs from that in section 2 and is now the nondimensional density-scale height:

\[
H_\rho = \left( \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right)^{-1}.
\]

Additionally, note that in contrast to the density-scale height of section 2, Eq. (3.8) contains the variations in scale due to the vortex.

Common to most anelastic systems the last two terms
on the right-hand side of (3.7) are expressed in terms of the perturbation potential temperature field. To accomplish this for the vortex-anelastic system, first note that when the expansions in (3.4) are used in the definition of the potential temperature,

$$\theta = \frac{p^{1-\kappa}}{\rho},$$  

(3.9)

we obtain

$$\frac{\theta_1}{\theta_0} = (1 - \kappa) \frac{p_1}{p_0} - \frac{\rho_1}{\rho_0},$$  

(3.10)

Using (3.10) in (3.7) allows the buoyancy terms in the vertical momentum equation to be written as

$$\frac{\partial \omega_1}{\partial t} + \frac{v_0 \omega_1}{r \partial \lambda} = - \frac{\partial}{\partial z} \left( \frac{p_1}{p_0} \right) + \frac{\theta_1}{\theta_0} + \delta \frac{p_1}{\rho_0},$$  

(3.11a)

where

$$\delta = \frac{1}{H_p} - \frac{(1 - \kappa)}{H_p},$$  

(3.11b)

and the nondimensional pressure-scale height is

$$H_p = \left( \frac{-1}{p_0} \frac{d p_0}{d z} \right)^{-1} = \frac{p_0}{\rho_0}.$$  

(3.11c)

To neglect the last term on the right-hand side of (3.11a) we must show that $\delta \ll 1$. Vertically differentiating (3.5e) reveals that a self-consistent relationship between $p_{0r}$, $\rho_0$, and $\theta_0$ requires

$$\delta = \frac{1}{H_p} - \frac{(1 - \kappa)}{H_p} = \frac{1}{H_p} = O(\varepsilon^2),$$  

(3.12)

where the nondimensional potential temperature-scale height is

$$H_\theta = \left( \frac{1}{\theta_0} \frac{d \theta_0}{d z} \right)^{-1}.$$  

(3.13)

Because of our vertical-scaling assumptions in (3.4d, e), the nondimensional pressure- and density-scale heights must be $O(1)$. Hence, the condition in (3.12) implies $|H_p - (1 - \kappa)H_\theta| \leq O(\varepsilon^2)$. This condition, which is simply a result of hydrostatic balance, requires the application of the vortex-anelastic system to atmospheric vortices whose pressure-scale height is less than the density-scale height by a factor of $1 - \kappa \sim 0.7$. This is consistent with B96 and, consequently, is consistent with using (2.7) in the vortex-anelastic equation set.

Upon assuming (3.12) holds, the $O(\varepsilon)$ kinematic and $O(\varepsilon^2)$ thermodynamic balances obtains the vortex-anelastic equation set:

$$\frac{\partial u_1}{\partial t} + \frac{v_0 \partial u_1}{r \partial \lambda} - \overline{\xi} v_1 = - \frac{\partial}{\partial z} \left( \frac{p_1}{p_0} \right),$$  

(3.14a)

$$\frac{\partial v_1}{\partial t} + \frac{v_0 \partial v_1}{r \partial \lambda} + w_1 \frac{\partial v_1}{\partial z} + \overline{\tau} u_1 = - \frac{1}{r} \frac{\partial}{\partial \lambda} \left( \frac{p_1}{p_0} \right),$$  

(3.14b)

$$\frac{\partial w_1}{\partial t} + \frac{v_0 \partial w_1}{r \partial \lambda} = - \frac{\partial}{\partial z} \left( \frac{p_1}{p_0} \right) + \frac{\theta_1}{\theta_0},$$  

(3.14c)

$$\frac{1}{r} \frac{\partial p_0 \partial \xi_1}{\partial r} + \frac{1}{r} \frac{\partial p_0 \partial \tau_1}{\partial \lambda} + \frac{\theta_1}{\theta_0} = 0,$$  

(3.14d)

$$\frac{\partial \theta_1}{\partial t} + \frac{v_0 \partial \theta_1}{r \partial \lambda} + u_1 \frac{\partial \theta_0}{\partial r} + w_1 \frac{\partial \theta_0}{\partial z} = 0.$$  

(3.14e)

This new set of equations constitutes an extension of the linear anelastic equations of section 2 by incorporating the thermodynamic variations of the vortex into three key areas: 1) the pressure gradient force in all three dimensions, 2) the vertical and radial mass flux terms in the anelastic mass continuity equation, and 3) the vertical buoyancy term. While the form of (3.14e) is identical to that of (2.13e), note that because $\theta_0 \neq \theta + \overline{\theta}$ the radial and vertical temperature advection is potentially quite different. Consequently, because $\theta_0$ satisfies the unapproximated balance equations and $\theta + \overline{\theta}$ satisfies an approximation to balance, (3.14e) is superior to (2.13e).

Note that because the extensions mentioned above involve only the thermodynamic terms it is the slow radial variations that have been incorporated into the modified terms. Therefore, the equation set (3.14) may be interpreted, with respect to the fast radial variations, as a locally traditional anelastic model of the form of that in section 2, but for which the buoyancy frequency is exact.

c. Energetics of the vortex-anelastic system

Because the motions described by the vortex-anelastic equations are linear about a nonresting reference state they do not conserve energy; the waves may exchange their energy with the reference state (the background vortex). We define the kinetic ($E_k$) and available potential ($E_A$) energies of the waves as

$$E_k = \varepsilon^2 \frac{1}{2} \int \rho_0 (u_1^2 + v_1^2 + w_1^2) r \, dr \, d\lambda \, dz,$$  

(3.15a)

$$E_A = \varepsilon^2 \frac{1}{2} \int N_0^2 \rho_0 \frac{\theta_1^2}{\theta_0^2} r \, dr \, d\lambda \, dz,$$  

(3.15b)

where
and the buoyancy frequency is $N_0^2 = (\partial \theta_0 / \partial z) / \theta_0$. It is apparent from (3.16) that the vertical buoyancy terms act solely to exchange perturbation kinetic and available potential energy within the wave. The wave may gain or lose perturbation kinetic and available potential energy to/from the vortex with the remaining terms in (3.16). The radial production term in (3.16b) arose from the term in the equation for potential temperature that measures the effect of the radial displacement of a perturbation within the radially varying temperature field. This term increases (decreases) the available potential energy of the wave for parcel motion that moves warm (cold) parcels radially away from the center of a warm-core vortex. In the anelastic system of section 2, the radial production term of (3.16b) and the perturbation exchange terms in (3.16a)–(3.16b) are overestimated by warm-core vortices and underestimated for cold-core vortices.

4. Comparisons

To evaluate the accuracy of the vortex-anelastic equations and its predecessors, we present simulations that depict the linearized, nonhydrostatic dynamics of both gravity waves and vortex–Rossby waves in the core of a baroclinic vortex. These simulations are similar to those presented by NM02 and NG03 to study the effects of asymmetric heating on a balanced, warm-core vortex. The equations presented in NM02 simulate the evolution of nonhydrostatic, asymmetric perturbations to a balanced vortex, and are easily modified to conform to the B96 form of the anelastic equations, and the vortex-anelastic (VA) equations.$^1$

Since the anelastic equations are an approximation to the fully compressible set, we compare these simulations to those with identical initial conditions using a fully compressible model. The relevant details of each model, the initial conditions, and the results follow.

a. The linear anelastic model

The linear model solves for the time evolution of perturbations of the form

$$ u_t = \hat{u}(r,z,t)e^{i\omega k} $$

and so on for $v_1$, $w_1$, $\theta_1$, and $p_1$. The numerical method of solution is described in detail in NM02 and NG03. Of relevance here is that the linear model is solved on a fully-staggered Arakawa C grid in the $r$–$z$ plane, with 30 points in the vertical direction, and 60 points in the radial direction. The vertical levels are equally spaced between $z = 0$ and 20 km, while in the radial direction the grid is stretched, with nearly constant 3-km grid spacing between 0 and 90 km, expanding out to 15-km grid spacing near the outer boundary at $r = 400$ km. The upper, outer, and lower boundaries are treated as free-slip, insulating walls, while the $r = 0$ boundary is treated as the center of a polar coordinate system. Gravity wave reflection from the upper and outer boundaries is suppressed (and essentially eliminated) with Rayleigh damping sponge layers. The upper-level damping layer extends from $z = 16$ to $z = 20$ km, with a constant damping time scale of 2 min.

The internal diffusivity for momentum is set to 20 m$^2$ s$^{-1}$, while the diffusivity for potential temperature is 3 times larger. This matches the factor of 3 between the thermal and momentum diffusivities in the compressible model, which is used in most mesoscale modeling systems.

b. The nonlinear, fully compressible model

The compressible model is the dynamic core of version 2.1.1 of the Weather Research and Forecast (WRF) Model. The WRF uses high-order advection schemes on an Arakawa C grid, with $\eta = p_h / p_{hs}$ as a vertical coordinate, where $p_h$ and $p_{hs}$ are the hydrostatic pressure and the hydrostatic surface pressures, respectively (Laprise 1992). The time integration uses third-

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$^1$ The asymmetric and symmetric codes are now available to the public, and are known as the three-dimensional perturbation analysis and simulation (3DVPAS). Interested parties should contact D. Nolan at dnolan@rsmas.miami.edu.
order Runge–Kutta time stepping (Wicker and Skamarock 2002). All “model physics” are neglected for these simulations. Full documentation of the WRF model dynamics are available from Skamarock et al. (2005).

The WRF domain uses \( 400 \times 400 \) horizontal grid points with 3-km spacing, and 30 levels in the vertical direction, from \( z = 0 \) to 20 km. The vertical levels are stretched in \( \eta \) coordinates so as to be almost exactly equally spaced at the same altitudes as the levels in the linear model. The outer boundary conditions are periodic in both directions. The domain is large enough so that the fastest gravity wave, which is observed to travel at \( c_g \sim 50 \text{ m s}^{-1} \), does not quite return to the core of the vortex before the simulation is completed. Vertical gravity wave reflection from the top of the domain is suppressed by a Rayleigh damping layer with the same scale and damping rate as in the linear model. The advective time step is set to 20 s, with eight acoustic time steps for each advective step. The internal diffusivity is set to \( 20 \text{ m}^2\text{s}^{-1} \) for momentum and \( 60 \text{ m}^2\text{s}^{-1} \) for temperature.

These values are perhaps larger than one might consider when constructing a numerical test of two different advection schemes. However, these larger values of the “resolved” viscosities help to overcome the differences in the numerical viscosities inherent to the two different modeling systems. In particular, the WRF uses fifth-order, upwind-biased advection in the horizontal direction and third-order upwind-biased advection in the vertical direction, while the linear model uses only second-order centered differences. The larger diffusion helps to ensure that the differences between the solutions are due more to the differing equation sets and less to the different numerical methods.

c. Initial conditions

The models are initialized with a basic-state vortex with the same structure as that shown in Fig. 1, but with the maximum wind speed increased to \( 40 \text{ m s}^{-1} \). The radial profile of the azimuthal wind field \([V(r)]\) for this vortex was constructed from the radial integration of a Gaussian vorticity profile:

\[
\zeta(r) = \zeta_0 \exp \left( - \left( \frac{r}{b} \right)^2 \right), \tag{4.2}
\]

with \( \zeta_0 = 2.34 \times 10^{-3} \text{ s}^{-1} \) and \( b = 53.5 \text{ km} \). The wind field was extended into the vertical with the formula:

\[
u_0(r, z) = V(r) \exp \left[ - \frac{z^6}{\alpha L_Z^6} \right], \tag{4.3}
\]

where \( L_Z = 6 \text{ km} \) indicates the depth of the barotropic part of the vortex, and \( \alpha = 2.5 \) is the vertical decay rate of the wind in the baroclinic region.

An initial perturbation is prescribed in the linear model, and added to the compressible model, which consists of a potential temperature perturbation of the following form:

\[
\theta_i(r, \lambda, z) = A \cos(3\lambda) \exp \left[ - \left( \frac{(r - r_b)^2}{\sigma_r^2} + \frac{(z - z_b)^4}{\sigma_z^4} \right) \right], \tag{4.4}
\]

where \( r_b = 50 \text{ km}, z_b = 6 \text{ km}, \sigma_r = 15 \text{ km}, \sigma_z = 3 \text{ km}, \) and \( A = 0.5 \text{ K} \). From the temperature scaling in (3.2c) and \( \epsilon = 0.1 \), a linear temperature perturbation is \( \epsilon^2 \rho_0 / (R \rho_0) \sim 0.3 \text{ K} \). Hence the temperature perturbation (4.4) should be well approximated by the linear dynamics described by the vortex-anelastic equations. The vertical and horizontal structure of the initial perturba-
tion are shown in Figs. 3a,b. This initial condition is modeled after the wavenumber-3 component of the latent heat release associated with an isolated burst of convection in a developing tropical cyclone (Nolan et al. 2007). The perturbation is completely unbalanced (acoustically as well in the compressible model).

d. Results

As described in some detail in NM02, an unbalanced, asymmetric temperature perturbation goes through a two-part adjustment process. First, there is rapid radiation of gravity waves as the disturbances try to regain hydrostatic balance. In this process, quasi-balanced vorticity perturbations are generated, which then go through a slower adjustment process as they are axisymmetrized by the radial shear of the vortex winds.

In the linear models, diffusion acts only on the perturbations from the reference state. However, in the nonlinear, compressible model, internal diffusion causes the vortex to decay (albeit very slowly) away from its initial state. Thus, for the most accurate comparison, all the compressible model data have been subtracted from the fields generated by an identical simulation with no perturbations.

To compare the three models’ representations of radiating gravity waves, in Fig. 4 we show horizontal cross sections of \( w \) at \( t = 2 \) h in a 180 × 180 km box centered on the vortex, at \( z = 8.2 \) km. This altitude was chosen since it cuts through the center of the low-stability region above the warm core, and the WRF model and linear model vertical levels match almost perfectly at this location. For these comparisons, the data from the
linear models, which exist in the form (4.1), have been interpolated onto a staggered grid with identical locations in the \((x-y)\) plane as the WRF data points; the radially varying altitudes of the WRF model levels in the core of the vortex and the fixed altitudes of the levels in the linear models differ by less than 40 m. Figure 4a shows the horizontal cross section of \(w\) for the WRF model, while Figs. 4b,c shows the difference between the WRF model and the vortex-anelastic systems and B96, respectively. Both anelastic approximations underestimate the maximum amplitude of the vertical oscillations in the core of the vortex generated by the adjustment process, but the vortex-anelastic equations are closer in amplitude than the B96 equations. Averaged over the data shown in the plots, the RMS errors for differences between compressible model output and the B96, and vortex-anelastic systems are 9.50, and \(8.31 \times 10^{-3}\) m s\(^{-1}\), respectively. Normalized by the RMS value of the WRF fields, these errors are quite large: 60% and 52%, respectively. However, as is evident from looking at the figures, these large RMS values are not indicative of poor comparisons; rather, they are caused by small phase errors in the azimuthal locations of the waves, as can be seen from the difference plots. A more generous measure of the accuracy of the solutions is given by a correlation analysis, whereby the correlation coefficient of the gridded data in the plots is computed. For these figures, the correlation values between the WRF model and the B96 and vortex-anelastic systems are \(r^2 = 0.940\) and 0.971, respectively. These small differences between the B96 and vortex-anelastic systems are reminiscent of other attempts at improving the anelastic approximation (Nance and Durran 1994).
At later times, $w$ decays very quickly and the motions are dominated by vortex–Rossby wave dynamics. To compare these more closely, in Fig. 5 we present horizontal sections of vertical vorticity at $t = 4$ h over the same domains. The compressible model shows bands of vorticity spiraling outward from the core, with two sets of three coherent vorticity anomalies rotating at different radii in the core of the vortex. Again, the linear anelastic models are quite similar, though they again underestimate the amplitude of the inner-core perturbations, and also how close the perturbations peak near the center of the vortex. The RMS errors for the B96 and vortex-anelastic systems are 2.59 and 2.11 $\times 10^{-6}$ s$^{-1}$, respectively. Normalized by the RMS of the WRF solution, these are 71% and 64%, respectively. Again, the correlation values indicate a much better match, with $r^2 = 0.717$ and 0.782, respectively.

5. Summary

A linear anelastic model appropriate to atmospheric vortices was derived that incorporates the vortex into the reference state. Traditional anelastic systems typically overestimate (underestimate) the buoyancy frequency anomalies for warm-core (cold-core) vortices. While this effect may be quite weak in a barotropic vortex, warm or cold cores in strong baroclinic vortices can substantially change the local buoyancy frequency. This drawback was eliminated by rederiving the anelastic system to create the vortex-anelastic model, which used a multiple-scale technique that allowed for the incorporation of the vortex into the anelastic reference state.

The vortex-anelastic equation set is a useful choice over traditional anelastic systems when the linearized dynamics are important. In these situations, the vortex-anelastic system provides a more accurate approximation without any additional computational cost. The primary strength of the vortex-anelastic model is the enhanced representation of near-core gravity wave radiation during a hydrostatic adjustment process when compared against traditional anelastic systems. As discussed in Nolan et al. (2007), the accurate representation of the adjustment process is necessary to understand the influence of convective asymmetries on developing tropical cyclones. The primary weakness of the vortex-anelastic model is that the equations are linear and therefore incapable of describing finite-amplitude effects. This was a direct result of the inclusion of the vortex in the reference state, which demanded that the reference state be in gradient wind balance rather than static like previous anelastic systems.

An example of the appeal of the vortex-anelastic model over the traditional system was provided. The linear evolution of an initially unbalanced, asymmetric perturbation on a baroclinic vortex was simulated. The vortex-anelastic model showed small but measurable improvements over the classic anelastic system for both vertical velocities and vorticity perturbations in the core of the vortex.

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