NOTES AND CORRESPONDENCE

Comments on "Piecewise Potential Vorticity Inversion: Elementary Tests"

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1. Introduction

Egger (2008) constructs some idealized experiments to test the usefulness of piecewise potential vorticity inversion (PPVI) in the diagnosis of Rossby wave dynamics and baroclinic development. He concludes that "PPVI does not help us to understand the dynamics of linear Rossby waves. It provides local tendencies of the streamfunction that are unrelated to the true ones. In the same way, the motion of baroclinic waves in shear flow cannot be understood by using PPVI. Moreover, the 'effect' of boundary temperatures as determined by PPVI is unrelated to the flow evolution." He goes further by arguing that we should not consider velocities as "induced" by PV anomalies defined by carving up the global domain. However, these conclusions partly reflect the limitations of his idealized experiments and the manner in which the PV components were partitioned from one another.

He considered two different experiments in which the flow consists of a zonal basic state plus a perturbation that is wavelike in the zonal direction. In the first, potential vorticity (PV) anomalies are isolated by partitioning the wave in the zonal direction (e.g., by isolating a single trough from the remainder of the domain). He demonstrated that the wind obtained by inverting the PV within the trough, assuming zero PV outside, is not simply related to the full flow field and does not help one to argue about the subsequent evolution. The wind found by PPVI depends on the boundaries imposed to partition the periodic domain. In the second experiment, PV anomalies are isolated by partitioning the full PV perturbation in the vertical. In this experiment, each PV component is wavelike (zonally) and so does not suffer from the limitations highlighted by the first experiment. Egger then goes on to argue that the winds obtained by inverting a single PV component, assuming zero PV in its surroundings, are also not easily related to the subsequent evolution of the system.

Here, we focus on the second point and argue that PV components can be isolated in such a way that their evolution can be rationalized using simple propagation and interaction arguments. The crucial point is that the piecewise partition of PV components (by demanding that they occupy distinct spatial domains) is not appropriate, except in special circumstances, for the reasons discussed by Egger. However, there are more appropriate ways to partition PV structures that in general have spatial overlap and evolve as described by simple ordinary differential equations (ODEs).

2. Suitable partition of Rossby wave components

For simplicity, we restrict our attention to sinusoidal perturbations of a zonally symmetric shear flow. Rossby waves can be defined as any wave in a PV contour. Because PV is materially conserved for adiabatic, frictionless flows, the basic state zonal flow tends to advect the PV pattern. However, such waves propagate relative to the flow along PV contours (perpendicular to the basic state PV gradient). The propagation mechanism involves the invertibility property of PV: the PV distribution can be inverted to obtain the flow, given suitable boundary conditions. For a monochromatic plane wave in PV, the meridional wind obtained by inversion is also a monochromatic plane wave shifted by phase $\pi/2$ relative to PV (i.e., in quadrature). The phase shift implies that the flow advects the basic

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state PV such that the PV pattern has a phase propagation without a change in amplitude.

a. Basic states with isolated PV gradients

In situations where the basic state contains isolated spikes in the PV gradient, it is clear that a Rossby wave can exist at each of these locations. This is the situation in the Rayleigh or Eady (1949) models of shear instability. Hoskins et al. (1985) illustrated the mechanism of shear instability in terms of the interaction between two Rossby waves located in two regions with opposing signs of PV gradient. The mechanism of counterpropagating Rossby waves (CRWs) was originally proposed by Lighthill (1963) in the context of shear instability at the top of the boundary layer and made quantitative by Bretherton (1966a) for the two-layer baroclinic model. The evolution from any initial perturbations at one or both of the PV gradients is described exactly by two ODEs for the amplitude and phase of the two CRWs (Davies and Bishop 1994). In these simple models, the Rossby wave components are separated spatially, as dictated by the basic state PV gradient structure. The appropriate application of PPVI is obvious.

General initial value problems can include PV anomalies in which the PV gradient is zero, such as the interior of the Eady model. Clearly, these could not have been formed conservatively and they cannot propagate relative to the flow. However, PV anomalies are readily produced by nonconservative effects such as latent heat release, and it is important to consider the effects of such PV on transient development (Farrell 1982).

Consider initial conditions with interior PV at only one level (zero elsewhere) and sinusoidal in the zonal direction. The inversion of this delta-function structure to obtain the streamfunction gives the Green function of the PV inversion operator, which Heifetz and Methven (2005) called a kernel Rossby wave (KRW). The streamfunction of the KRW extends through the domain in the vertical. Such a KRW is not a solution of the dynamics by itself because it must immediately advect PV where its gradient is nonzero. In the Eady model, a KRW excites the edge wave on the ground and rigid lid—the CRWs of the Eady model. The propagation and interaction of these three components provide a complete description of the perturbation evolution. Dirren and Davies (2004) and de Vries and Opsteegh (2007) have shown the "three-wave" evolution equations for the amplitude and phases of these components.

Generally, one must consider kernels at any level at which the PV is nonzero initially. However, the partial differential equation problem is still reduced to a set of coupled ODEs by the Green function approach. Importantly, the Rossby wave propagation mechanism is the same for all components, and change in the PV anomaly amplitude (and the meridional displacement of air parcels) can only occur through interaction between these components, mediated by the meridional wind associated with each component at other levels. The interaction is inherently nonlocal because wind is induced by a KRW far from its PV kernel.

b. Basic states with continuous PV gradients

This viewpoint was extended by Heifetz et al. (2004a) to flows in which the PV gradient can be everywhere nonzero. The crucial difference is that the perturbation PV associated with a CRW extends throughout the domain. However, it has zero zonal tilt by definition, with the effect that its structure is untilted in all variables obtained through PV inversion. Generally, the pair of CRWs required to describe shear instability have overlapping PV distributions and cannot be identified by "spatial PPVI." Heifetz et al. (2004a) proposed two methods to identify CRW components: the "home base" and "wave activity orthogonality" methods. In the home-base method, a growing normal mode and its decaying complex conjugate are linearly combined to produce two untilted structures, each with zero PV at a chosen location. The choice of home bases is rather subjective, but they must be located where the basic state PV gradient has opposite signs. For example, the home base of CRW-1 is specified where the PV gradient is negative; CRW-2 is then defined by requiring it to have zero PV there. In this way, the full perturbation PV is described only by wave 1 at its own home base. Similarly, wave 1 is defined by requiring it to have zero PV at the home base of wave 2 where the PV gradient is positive. Thus, it is only at these two home bases that the full PV is attributable to a single CRW. In the wave activity orthogonality method, the CRW pair is uniquely obtained from a growing normal mode and its conjugate by a linear combination such that the two CRWs are orthogonal with respect to pseudomomentum and a second integral condition related to pseudoenergy. Pseudomomentum and pseudoenergy are globally conserved wave properties (for adiabatic, frictionless flow). By this method, the PV may be shared between the CRWs at any location, but they are orthogonal in an integral sense.

Using either method, the evolution from initial conditions containing any linear combination of two CRW structures is given exactly by two ODEs describing their amplitudes and phases. These CRW evolution equations are homomorphic to those for the Eady model. The phase speed of each component consists of

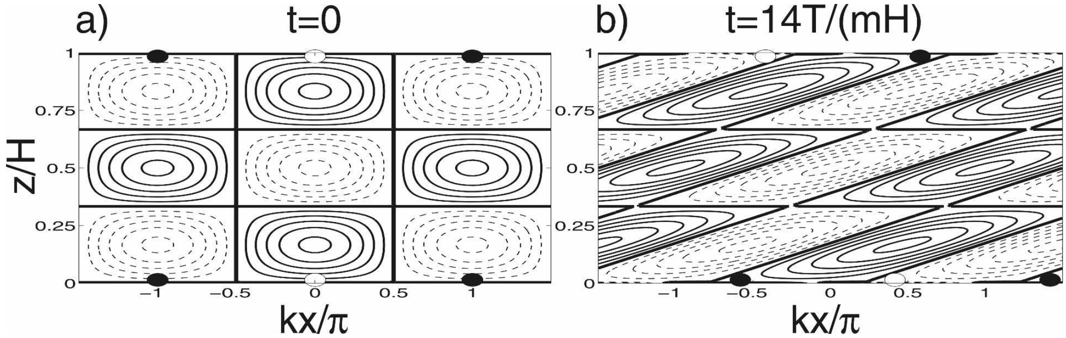

FIG. 1. Cross section showing evolution of PV in the Eady model from initial conditions specified by Egger (2008) in a frame moving with the midlevel flow so that the central −ve PV anomaly is stationary. The filled (open) circles denote positive (negative) boundary–PV anomalies that can propagate on the negative PV gradient at the ground and the positive PV gradient at the lid ($z = H$); $T = N/(f_0\Lambda)$ is the time scale related to the Eady growth rate and $1/m = f_0/(Nk)$ is the Rossby height scale.

terms for advection by the basic flow, self-propagation, and propagation associated with interaction through advection of the PV gradient by the winds associated with the other CRW. The growth in air parcel displacements occurs only through the interaction term. As with the qualitative picture of Hoskins et al. (1985), the key features of the evolution are determined by the phase difference between the Rossby waves. Furthermore, new work by the authors has shown that if initial conditions consist of a PV delta function (one KRW), the perturbation evolution can be described to a good approximation by only three components, the initial KRW and the two CRWs that it excites (e.g., Fig. 2). Thus, general initial value problems on zonal jets can be interpreted in the same manner as for the highly simplified Eady model. However, the CRW components have amplitude throughout the cross-stream plane and are not spatially separated.

3. Anticipation of evolution using Rossby wave components

The power of partitioning perturbations into Rossby wave components (CRWs and KRWs) lies in the ability to discuss from physical principals the way in which the full perturbation must evolve, without requiring explicit calculation. For example, consider the baroclinic initial value problem posed in section 2.3 of Egger (2008). The initial conditions include interior PV where the basic state PV gradient is zero, in addition to potential temperature anomalies along the upper and lower boundaries. Because there is a potential temperature gradient in the Eady model, the waves on the two boundaries can propagate as Rossby edge waves. The temperature wave on the lid is initially in antiphase with the temperature wave at the ground. Using the Bretherton (1966b) delta function technique, warm anomalies on the ground are equivalent to positive boundary–PV anomalies, whereas warm anomalies on the lid are equivalent to negative boundary–PV anomalies. This means that the boundary–PV wave on the lid is in phase with the PV wave at the ground initially (circles in Fig. 1a).

Because the basic state PV gradient is zero, the interior PV structure is simply advected by the shear flow ($U = \Lambda z$) and tilts eastward with height. Thus, the phase tilt (in radians) of the interior PV wave between the two boundaries increases with time as

$$k\Delta x = k\Lambda Ht = m\frac{f_0\Lambda}{N}Ht = \frac{mHt}{T}, \qquad (1)$$

where k is the horizontal wavenumber, $1/m = f_0/(Nk)$ is the Rossby height scale, f_0 is the Coriolis parameter, N is the Brunt–Väisälä frequency, and $T = N/(f_0\Lambda)$ is the characteristic time scale of the Eady problem. Figure 1b shows the PV at time $t \approx 14.4T/(mH)$ when $\Delta x \approx 14.4\lambda/(2\pi) \approx 2.3\lambda$, where λ is the wavelength.

In contrast, the two edge waves are influenced by the flow induced by each other and the interior PV because these induced meridional flows can advect the PV gradient at the boundaries. Initially, the circulation anomalies at a boundary attributable to both edge waves are in antiphase with the circulation attributable to the interior PV. This is a "hindering configuration," meaning that the tendency for the upper wave to propagate westward (relative to the flow) and the lower wave to propagate eastward is reduced by the interaction with interior PV. Indeed, for these particular initial conditions the cancellation is exact because there is zero perturbation streamfunction on the boundaries (Egger 2008) and the CRWs are simply advected by the shear flow. However, after time $t \approx \pi T/(mH)$, the positive PV anomaly on the lid overtakes the negative PV anomaly on the ground and effectively there is a westward tilt between the edge waves, meaning that they will make each other grow through meridional advection. As the edge wave amplitudes increase, the interior PV will become less influential because the interior PV cannot change amplitude. Also, the tilting of the interior structure reduces its associated circulation anomalies at the boundaries and its hindering influence decreases. Eventually [in this problem, $t > 4\pi T/(mH)$], the edge waves approach the phase-locked configuration of the growing normal mode. In this configuration, the upper CRW is displaced to the west of the lower CRW and the phase speed of both waves (including interaction) is $\Lambda H/2$.

Note that the fastest growing normal mode of the Eady problem has scale $mH \approx 1.6$ and growth rate $\sigma \approx 0.31/T$. Figure 1b is shown at time $t = 9T \approx 2.8/\sigma$, which for this wave scale is approximately $14.4T/(mH)$. Davies and Bishop (1994) have shown that phase-locking is approached from any initial conditions within a few e-folding time scales. Furthermore, in the Eady model (apart from near the short-wave cutoff) the CRWs must lock in a configuration[1] with a large westward phase tilt between $\pi/2$ and π. Therefore, we know that the CRWs must be close to such a phase-locked configuration in Fig. 1b.

As discussed in section 2b, the decomposition into Rossby wave components also applies when there is a nonzero interior PV gradient. The key difference is that the CRWs must contain interior PV in addition to the boundary–PV anomalies of the edge waves. Figure 2 shows the evolution from the same initial conditions as for Fig. 1 but on a basic state with a uniform interior PV gradient β. The initial perturbation includes anomalies at the boundaries and therefore projects onto the upper and lower CRW structures. However, the initial projection is small and most of the interior PV falls into the remainder (the third row) obtained by subtracting the two CRWs from the full perturbation. The amplitude and phase evolution of these three components are similar to those in the f-plane case. Crucially, the "remainder PV" behaves as though it were passive and tilts eastward with height at rate (1) so that it closely resembles the interior PV from the f-plane case. Although on a β plane a kernel Rossby wave (which has PV at only one level) "excites" PV waves at all other levels, the response is coherent and can be described almost completely in terms of a superposition of the upper and lower CRW structures. Consequently, although the full PV perturbation appears to have complex structure in the β-plane case (Fig. 2), it is readily understood when decomposed into a CRW pair and remaining PV.

It is hard to imagine a different conceptual framework that would enable one to anticipate so much of the evolution from physical principles.

Note that in the above discussion the crucial distinction when considering the PV components is between the two CRWs (which are simply edge waves in the Eady model) and the remaining PV structure. In contrast, Egger isolates the uppermost interior PV anomaly and labels it "stratospheric." However, the evolution of this structure is not described by the CRW evolution equations because it exists where the PV gradient is zero and is simply sheared apart by the flow. Arbitrary spatial divisions of the domain do not isolate suitable PV components.

4. Application of "PV thinking" to the atmosphere

As highlighted by Egger (2008) there are potential pitfalls to the application of "PV thinking" to case studies of atmospheric development. However, we believe that if it is possible to isolate the appropriate Rossby wave components, the anticipation of flow development is possible using physically based propagation and interaction arguments.

In the literature, piecewise potential vorticity inversion has typically been used diagnostically to identify the circulation and static stability anomalies associated with a PV anomaly. To ascertain flow evolution, the PV components and their associated flow structures must be defined in a specific manner. One such approach is to use CRWs as the components because the full perturbation evolution is then described by a simple set of ODEs for the amplitudes and phases of these components.

Egger (2008) proposes that one should determine the streamfunction tendency associated with each PV component and suggests that PPVI is only useful if the result (in the domain outside that defining the PV anomaly) is similar to the streamfunction tendency of the full perturbation. We disagree with this tenet. The CRW evolution equations make it explicit that the phase speed of each component has three terms associated with advection by the basic state flow, self-propagation, and propagation through interaction. The propagation terms are proportional to $Q_y v_j/q_i$, where Q_y is the PV gradient, q_i is the PV amplitude of CRW-i,

[1] Heifetz et al. (2004b) have shown that CRWs in the Charney (1947) model must always lock in this *hindering* configuration. Furthermore, the fastest growing mode of the Charney model also has growth rate $0.31/T$ (Lindzen and Farrell 1980). Therefore, there is robustness in these predictions to changes in the basic state.

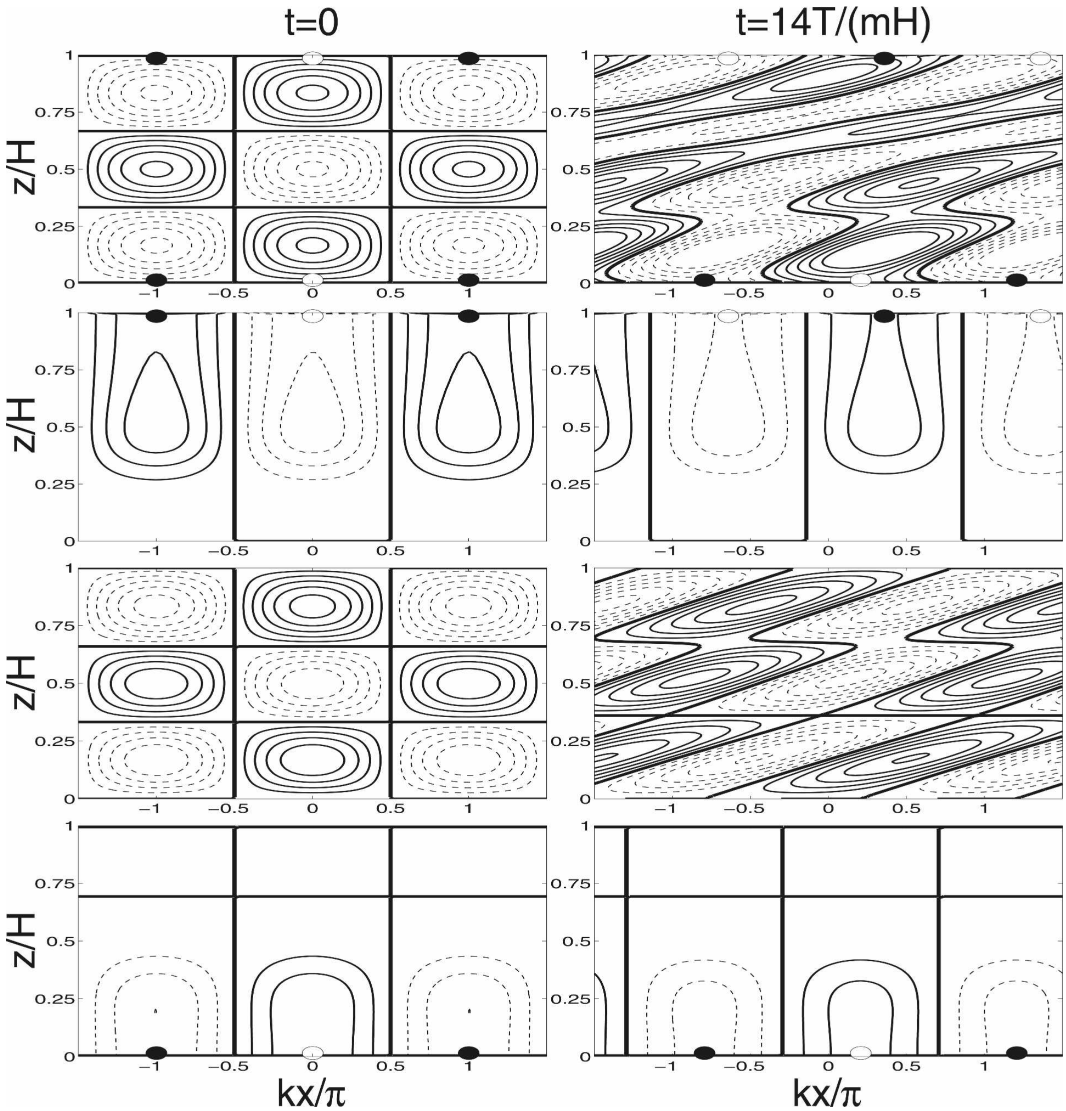

FIG. 2. Perturbation evolution on the Eady basic state modified by adding the uniform PV gradient β. The top row shows full perturbation PV as in Fig. 1. The second and fourth rows show the perturbation decomposed into the upper CRW and lower CRW, and the third row shows the remaining PV. The contour interval is the same in all panels, except for the CRWs at $t = 0$, where the contour interval is divided by 10.

and v_j is the meridional wind associated with CRW-j, all evaluated at the home base of CRW-i. Self-propagation refers to $j = i$ and interaction to $j \neq i$. Although the interaction terms may be small relative to advection and self-propagation, they are the only means for growth in PV amplitude to occur. Thus, the partition is fundamental in the understanding of the shear instability mechanism.

Ultimately, the acid test of the "PV thinking" approach lies in the qualitative prediction of evolution from a given situation, not in the instantaneous diagnosis of the "interaction" between PV anomalies. Physical insight also enables anticipation of changes in wave behavior associated with changes in background state properties. This may be helpful in predicting the changes in storm strength and structure under climate change scenarios, or at least it may provide a means to investigate the physical basis underlying changes simulated by climate models.

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