A Theoretical Framework for Energy and Momentum Consistency in Subgrid-Scale Parameterization for Climate Models

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ABSTRACT

A theoretical framework for the joint conservation of energy and momentum in the parameterization of subgrid-scale processes in climate models is presented. The framework couples a hydrostatic resolved (planetary scale) flow to a nonhydrostatic subgrid-scale (mesoscale) flow. The temporal and horizontal spatial scale separation between the planetary scale and mesoscale is imposed using multiple-scale asymptotics. Energy and momentum are exchanged through subgrid-scale flux convergences of heat, pressure, and momentum. The generation and dissipation of subgrid-scale energy and momentum is understood using wave-activity conservation laws that are derived by exploiting the (mesoscale) temporal and horizontal spatial homogeneities in the planetary-scale flow. The relations between these conservation laws and the planetary-scale dynamics represent generalized nonacceleration theorems. A derived relationship between the wave-activity fluxes—which represents a generalization of the second Eliassen–Palm theorem—is key to ensuring consistency between energy and momentum conservation. The framework includes a consistent formulation of heating and entropy production due to kinetic energy dissipation.

1. Introduction

The equations governing fluid dynamics are fundamentally conservative, representing conservation of momentum, energy, and mass (Batchelor 1967). The equations of climate modeling, which are based on dynamical approximations to the governing equations of fluid dynamics, preserve the conservation properties to avoid introducing spurious sources or sinks of the conserved quantities (Lorenz 1967). The numerical solution to the equations of climate modeling requires the parameterization of processes that occur on scales smaller than can be represented by the discrete model grid. Physical processes occurring on these subgrid scales are important to the resolved energy and momentum budgets and necessarily respect the same conservation principles.

Self-consistency of conservation properties in subgrid-scale parameterization is an important issue because parameterizations of subgrid-scale processes can lead to spurious long-term trends if energy and momentum conservation are not respected (Boville and Bretherton 2003). Self-consistency is also important for ensuring the robustness of the parameterized response to climate perturbations (Shaw and Shepherd 2007). The errors introduced by non-self-consistent parameterizations of horizontal and vertical diffusion can be orders of magnitude larger than those due to implicit numerical diffusion (Burkhardt and Becker 2006; Becker 2003). One important consistency issue is the inclusion of thermodynamic heating resulting from the transfer of kinetic energy from the resolved scales to subgrid scales and eventually to a turbulent microscale, irreversibly. Becker (2003) showed that in the planetary boundary layer, this heating can be as large as 1.5 W m^{-2} in the global mean. Many models treat this energy transfer locally; that is, any resolved-scale kinetic energy tendency due to subgrid-scale momentum flux convergence or vertical diffusion is assumed to be balanced locally by a thermodynamic energy tendency of the opposite sign; that is,

$$Q = -\frac{\partial K}{\partial t} = -\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \qquad (1)$$

[ECHAM3 atmospheric GCM (see http://www.mpimet.mpg.de/fileadmin/models/echam/echam3_DKRZ-Report No.6.pdf); Boville and Bretherton 2003; the European Centre for Medium-Range Weather Forecasts (ECMWF)

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Integrated Forecast System (see http://www.ecmwf.int/research/ifsdocs/CY28r1/Physics/index.html, section 3.6)], where the tendencies in (1) are understood to be those arising from subgrid-scale parameterizations. This assumption of local conservation is evidently not appropriate for nonlocal transfers of energy and momentum by subgrid-scale processes. For example, where parameterized gravity wave momentum flux convergence acts to increase the resolved kinetic energy, local energy conservation would imply a negative thermodynamic energy tendency, which is in violation of the second law of thermodynamics. (It is not possible to extract heat from a single reservoir to perform useful work.) Similarly, mixing induced by wave breaking will generally require energy input to balance the gain in potential energy, so the extent of mixing is constrained by the overall energetics. For these reasons, an understanding of the energy transfers between the subgrid scale and the resolved scale is necessary to ensure consistency. Here we address the problem of developing a theoretical framework for subgrid-scale parameterization in climate models that consistently conserves both energy and momentum in a general manner and respects the second law of thermodynamics.

Any framework for subgrid-scale parameterization in climate models must involve coupling the equations of climate modeling, the hydrostatic primitive equations (Lorenz 1967)—which are valid for length and time scales

$$L_p \approx O(100 - 10000\ \text{km}), \tag{2a}$$

$$H_p \approx O(10\ \text{km}), \quad \text{and} \tag{2b}$$

$$t_p \approx O(\Omega^{-1}) \tag{2c}$$

(Zeytounian 1990), where Ω is the rotation rate of the earth (note the lower bound on the length scale, which is set by the numerical discretization)—to equations governing subgrid-scale dynamics with approximate length and time scales

$$L_m \approx O(10\ \text{km}), \tag{3a}$$

$$H_m \approx O(10\ \text{km}), \quad \text{and} \tag{3b}$$

$$t_m \approx O(N^{-1}), \tag{3c}$$

where N is the buoyancy frequency. Dynamics occurring on subgrid scales include convection and gravity wave propagation and are nonhydrostatic. Therefore, the framework must account for the interaction between two flows: a hydrostatically balanced resolved flow operating on long and slow spatiotemporal scales, and a nonhydrostatic subgrid-scale flow operating on short and fast spatiotemporal scales.

The tools of systematic multiple-scale perturbation theory from applied mathematics (Kevorkian and Cole 1981) are naturally suited for developing a framework to study the interaction of physical phenomena occurring on multiple spatial and temporal scales (i.e., the resolved and subgrid scales). Klein (2000) and Majda and Klein (2003) have shown how this theory can be used to systematically derive balanced models in the midlatitudes and tropics. Here, the goal is not to derive an asymptotic framework from first principles. Rather, the goal is to find an asymptotic framework that leads to the equations of interest and to use that framework to define a self-consistent treatment of energy and momentum in the context in which there is an imposed separation of horizontal length and time scales but vertical coupling within each column. This is the case of relevance to climate models. Note that the assumed time scale separation imposes statistical stationarity on the subgrid-scale processes.

Section 2 introduces the equations and relevant nondimensionalization. We introduce the multiple-scale framework in section 3, including the derivation of momentum, thermodynamic, and continuity equations for the resolved and subgrid-scale flow including the resolved-scale total energy budget. The horizontal space and time scale separation between the resolved and subgrid scales is used to define wave-activity conservation laws on the subgrid scale in section 4, which are used to close the interaction terms. In section 5 the subgrid-scale dynamics are reduced to a subset satisfying the anelastic constraint, which is an important regime for applications (most subgrid-scale parameterizations are formulated using the anelastic equations), and implications for the wave-activity conservation law closures are discussed. We show how the dissipation of subgrid-scale kinetic energy can be assured to lead to an increase in thermodynamic energy and to entropy production. The paper concludes with a summary and discussion in section 6.

2. Preliminaries

The multiscale models developed in subsequent sections are derived systematically from the Navier–Stokes, thermodynamic, and continuity equations in planar coordinates:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} + \boldsymbol{\Omega} \times \rho\mathbf{v} + \nabla p = -\rho g \hat{\mathbf{z}} + \tilde{\mathbf{S}}_{\mathbf{v}}, \tag{4a}$$

$$c_p \rho \pi \left[\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla)\theta\right] = \tilde{S}_\theta, \quad \text{and} \tag{4b}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4c}$$

TABLE 1. Scaling parameters and their values.

Parameter	Value
$\lvert\boldsymbol{\Omega}\rvert$	$O(10^{-4}\ \mathrm{s}^{-1})$
g	$O(10\ \mathrm{m\ s}^{-2})$
p_{ref}	$O(10^5\ \mathrm{kg\ m}^{-1}\ \mathrm{s}^{-2})$
ρ_{ref}	$O(1\ \mathrm{kg\ m}^{-3})$
H	$O(10^4\ \mathrm{m})$
N	$O(10^{-2}\ \mathrm{s}^{-1})$
c_{ref}	$O(10^2\ \mathrm{m\ s}^{-1})$
T_{ref}	c_{ref}^2/R
U	NH

where in the usual way ρ, p, $\mathbf{v}$, θ, and π are the density, pressure, velocity, potential temperature, and Exner function ($\pi = p^{\kappa}$, where $\kappa = R/c_p$ with R being the dry gas constant and c_p the specific heat at constant pressure), respectively; $\hat{\mathbf{z}}$ is the unit vector in the vertical direction; and $\tilde{\mathbf{S}}_{\mathbf{v}}$ and $\tilde{S}_{\theta}$ are (dimensional) source–sink terms meant to represent the interaction with a turbulent microscale in which resolved-scale kinetic energy is ultimately dissipated and converted into thermodynamic energy (see section 4). Although a complete treatment of the problem will require treatment of the effects of moisture, as a first step we treat only dry dynamics.

To nondimensionalize the equations, we choose a general set of scaling parameters and make specific scaling choices for the remaining parameters (see Table 1). The dimensionless equations are then

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\mathrm{Ro}}\mathbf{e}\times\rho\mathbf{v} + \frac{1}{M^2}\nabla p = -\frac{1}{\mathrm{Fr}^2}\rho\hat{\mathbf{z}} + \mathbf{S}_{\mathbf{v}}, \tag{5a}$$

$$\frac{\rho\pi}{\kappa}\left[\frac{\partial\theta}{\partial t} + (\mathbf{v}\cdot\nabla)\theta\right] = S_{\theta}, \tag{5b}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{5c}$$

with

$$\mathbf{e} = \frac{\sin\phi\hat{\mathbf{z}} + \cos\phi\hat{\mathbf{y}}}{\sin\phi_0}, \tag{6}$$

where ϕ is the latitude on the tangent plane centered at ϕ_0 and $\hat{\mathbf{y}}$ is the unit vector in the meridional direction. In (5), M is the Mach number, Fr is the Froude number, Ro is the Rossby number, $\mathbf{S}_{\mathbf{v}}$ and S_{θ} are the nondimensional versions of $\tilde{\mathbf{S}}_{\mathbf{v}}$ and $\tilde{S}_{\theta}$, and time is scaled advectively. The three nondimensional numbers—Rossby, Mach, and Froude—are defined as

$$\mathrm{Ro} = \frac{U}{\Omega H}, \quad M = \frac{U}{c_{\mathrm{ref}}}, \quad \text{and} \quad \mathrm{Fr} = \frac{U}{\sqrt{gH}}. \tag{7}$$

Note that in our scaling the length scale in the Rossby number is H, a length scale relevant for the mesoscale, rather than a planetary length scale. For the chosen reference length scale, the Mach and Froude numbers are equivalent (i.e., for $H = p_{\mathrm{ref}}/g\rho_{\mathrm{ref}}$ then $gH = p_{\mathrm{ref}}/\rho_{\mathrm{ref}} = c_{\mathrm{ref}}^2$).

The final equations are the dimensionless equation of state and definition of potential temperature:

$$p = \rho T, \quad \theta = Tp^{-\kappa} \quad \text{or} \quad \rho\theta = p^{1/\gamma}, \tag{8}$$

where $\gamma = c_p/c_v$ so that $\kappa - 1 = -1/\gamma$.

Each variable is assumed to be decomposed into two component scales: a planetary-scale p and a mesoscale m; that is,

$$f(\mathbf{x}_p, \mathbf{x}_m, z_p, z_m, t_p, t_m) = f_p(\mathbf{x}_p, z_p, t_p) + f_m(\mathbf{x}_p, \mathbf{x}_m, z_p, z_m, t_p, t_m), \tag{9}$$

where $\mathbf{x}_p = (H/L_p)\mathbf{x}$, $z_p = (H/H_p)z$, $t_p = (H/L_p)t$, $\mathbf{x}_m = (H/L_m)\mathbf{x}$, $z_m = (H/H_m)z$, and $t_m = (H/L_m)t$. We choose our scalings bearing in mind the dynamical models required on each scale and according to (2) and (3) we set $H/L_p = t_p/t = \epsilon^2$, with $\epsilon \approx 0.1$, $H/H_p = 1$, $H/L_m = t_m/t = 1$, and $H/H_m = 1$, whence $z_p = z_m = z$ and we refer only to z.[1] Given that time is scaled advectively and that $t_m/t = 1$, with $t_m \sim O(N^{-1})$ according to (3), implies a reference velocity scale $U \sim NH$ and thus $\mathrm{Fr}_{\mathrm{int}} = U/NH \sim 1$, where $\mathrm{Fr}_{\mathrm{int}}$ is the internal Froude number. According to the choice of the reference velocity scale, the Mach and the (external) Froude numbers in (7) are near unity. Given the multiple space and time-scale dependence of a field f, the time derivative and total horizontal gradient are then

$$\frac{\partial f}{\partial t} = \epsilon^2\frac{\partial f}{\partial t_p} + \frac{\partial f}{\partial t_m}, \tag{10a}$$

$$\nabla^H f = \epsilon^2\nabla_p^H f + \nabla_m^H f. \tag{10b}$$

The asymptotic ansatz we will employ is

$$\mathbf{u} = \mathbf{u}_p(\mathbf{x}_p, z, t_p) + \epsilon\mathbf{u}_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m), \tag{11a}$$

$$w = \epsilon w_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m) + \epsilon^2 w_p(\mathbf{x}_p, z, t_p), \tag{11b}$$

[1] In keeping with our motivation, we consider only two time scales. We do not include the $O(\epsilon^{-1})$ time scale in our framework, which is the mesoscale advective time scale. Its neglect does not fundamentally change what is derived in the subsequent sections.

$$p = p_p(\mathbf{x}_p, z, t_p) + \epsilon M^2 p_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m), \tag{11c}$$

$$\rho = \rho_p(\mathbf{x}_p, z, t_p) + \epsilon M^2 \rho_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m), \tag{11d}$$

$$T = T_p(\mathbf{x}_p, z, t_p) + \epsilon M^2 T_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m), \tag{11e}$$

$$\theta = \theta_p(\mathbf{x}_p, z, t_p) + \epsilon M^2 \theta_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m), \tag{11f}$$

where $\mathbf{v} = (\mathbf{u}, w)$ and each field is expanded in ϵ for example,

$$f_m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m; \epsilon) = \sum_i \epsilon^i f_m^{(i)}(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m). \tag{12}$$

In the above ansatz, all thermodynamic variables are decomposed using a standard mesoscale average over t_m and $\mathbf{x}_m$ (but not z) with the property that $\overline{f_m} = 0$ and $\overline{f_p} = f_p$, which is an ansatz to eliminate secular growth. Note that this corresponds to an area-weighted average. For the velocity, we adopt a mass-weighted average decomposition such that $\rho_p \mathbf{u}_p = \overline{\rho \mathbf{u}}$, $\rho_p w_p = \overline{\rho w}$, and $\overline{\rho \mathbf{v}_m} = 0$ since this is more appropriate when considering conserved quantities such as energy and momentum. The ansatz (11) accounts for the fact that the mesoscale fields are small compared to planetary-scale fields and it includes an anisotropy in the planetary-scale velocities to ensure that vertical advection balances horizontal advection. We also choose to scale the mesoscale thermodynamic variables by M^2; limits of this parameter are considered in section 5. In terms of the thermodynamics, the ansatz (11) implies a leading-order thermodynamic state that varies only on the slow and long planetary scales and acts as a background state for the mesoscale. The source–sink terms $\mathbf{S}_\mathbf{v}$ and S_θ are assumed to take the following forms:

$$\mathbf{S}_\mathbf{v} = \epsilon \mathbf{S}_\mathbf{v}^m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m), \tag{13a}$$

$$S_\theta = \epsilon M^2 S_\theta^m(\mathbf{x}_p, \mathbf{x}_m, z, t_p, t_m) + \epsilon^2 S_\theta^p(\mathbf{x}_p, z, t_p). \tag{13b}$$

We assume that the mesoscale source–sink terms represent flux divergences of microscale fluctuations and hence vanish, as do all mesoscale fields, under the mesoscale average (i.e., $\overline{\mathbf{S}_\mathbf{v}^m} = \overline{S_\theta^m} = 0$). This implies that the microscale fluctuations directly affect the mesoscale dynamics but only affect the planetary scale indirectly. This completes the multiple scale formalism.

Given the introduction of multiple length scales, it is appropriate to write the Rossby number as

$$\mathrm{Ro} = \frac{L_p}{H}\mathrm{Ro}_p \sim \epsilon^{-2}\mathrm{Ro}_p, \tag{14}$$

where $\mathrm{Ro}_p = U/\Omega L_p \sim 1$ is the usual planetary-scale Rossby number. Thus, it is clear that on the planetary scale, horizontal advection, vertical advection, and rotation are of the same order. The limit of small $\epsilon^2 = H/L_p$ as represented in (12) is taken with M, Fr, and Ro_p held fixed.

3. Equations for the mesoscale planetary-scale interaction

Proceeding as usual with the asymptotics [plugging (11) expanded according to (12) into (5)], we obtain a hydrostatically balanced reference state at $O(1)$; that is,

$$\frac{1}{M^2}\frac{\partial p_p}{\partial z} = -\frac{1}{M^2}\rho_p, \tag{15}$$

where we have substituted M for Fr. Here and henceforth all velocities are leading order and superscripts are dropped unless otherwise indicated. The mesoscale dynamical equations are obtained at $O(\epsilon)$. The mesoscale horizontal and vertical momentum equations are

$$\rho_p\frac{\partial \mathbf{u}_m}{\partial t_m} + \rho_p(\mathbf{u}_p \cdot \nabla_m^H)\mathbf{u}_m + \rho_p w_m \frac{\partial \mathbf{u}_p}{\partial z} + \nabla_m^H p_m = \mathbf{S}_\mathbf{u}^m, \tag{16a}$$

$$\rho_p\frac{\partial w_m}{\partial t_m} + \rho_p(\mathbf{u}_p \cdot \nabla_m^H)w_m + \frac{\partial p_m}{\partial z} = -\rho_m + S_w^m. \tag{16b}$$

Note that the mesoscale average, which is hydrostatic balance between $p_p^{(1)}$ and $p_p^{(1)}$, has been removed to obtain (16b). The mesoscale potential temperature equation is

$$\frac{\rho_p \pi_p}{\kappa}\left\{M^2\left[\frac{\partial \theta_m}{\partial t_m} + (\mathbf{u}_p \cdot \nabla_m^H)\theta_m\right] + \frac{\partial \theta_p}{\partial z}w_m\right\} = M^2 S_\theta^m. \tag{17}$$

The final equation is the continuity equation,

$$M^2\left[\frac{\partial \rho_m}{\partial t_m} + (\mathbf{u}_p \cdot \nabla_m^H)\rho_m\right] + \nabla_m \cdot (\rho_p \mathbf{v}_m) = 0, \tag{18}$$

where ∇_m is the three-dimensional gradient on the mesoscale. Note that the ansatz (11) together with the assumed scaling has not eliminated compressibility effects on the mesoscale [cf. the first term on the left-hand side of (18)].

Before proceeding to the next order, it will be useful to plug (11) into the equation of state, the definition of potential temperature (8), and the Exner function:

$$\begin{aligned} p = \rho T &\rightarrow (p_p + \epsilon M^2 p_m) = (\rho_p + \epsilon M^2 \rho_m)(T_p + \epsilon M^2 T_m) \\ &\rightarrow p_p = \rho_p T_p, \quad p_m = \rho_p T_m + \rho_m T_p, \end{aligned} \tag{19}$$

$$\theta = Tp^{-\kappa} \rightarrow (\theta_p + \epsilon M^2\theta_m) = (T_p + \epsilon M^2 T_m)(p_p + \epsilon M^2 p_m)^{-\kappa}$$
$$\rightarrow (\theta_p + \epsilon M^2\theta_m) \approx (T_p + \epsilon M^2 T_m)(p_p)^{-\kappa}(1 - \epsilon M^2 \kappa p_m/p_p)$$
$$\rightarrow \theta_p = T_p p_p^{-\kappa}, \quad \theta_m = p_p^{-\kappa} T_m - \kappa \frac{\theta_p}{p_p} p_m = \frac{\theta_p}{\gamma T_p} T_m - \kappa \frac{\theta_p}{\rho_p}\rho_m, \tag{20}$$

and

$$\pi = p^{\kappa} \rightarrow (\pi_p + \epsilon M^2 \pi_m) = (p_p + \epsilon M^2 p_m)^{\kappa}$$
$$\rightarrow (\pi_p + \epsilon M^2 \pi_m) \approx p_p^{\kappa}(1 + \epsilon M^2 \kappa p_m/p_p)$$
$$\rightarrow \pi_p = p_p^{\kappa}, \quad \pi_m = \kappa p_p^{\kappa-1} p_m = \kappa \frac{p_m}{\rho_p \theta_p}. \tag{21}$$

According to these expansions, the leading-order mesoscale momentum Eq. (16) can be written without approximation as

$$\rho_p \frac{\partial \mathbf{u}_m}{\partial t_m} + \rho_p(\mathbf{u}_p \cdot \nabla_m^H)\mathbf{u}_m + \rho_p w_m \frac{\partial \mathbf{u}_p}{\partial z} + \rho_p \nabla_m^H\left(\frac{\theta_p}{\kappa}\pi_m\right) = \mathbf{S}_{\mathbf{u}}^m, \tag{22a}$$

$$\rho_p \frac{\partial w_m}{\partial t_m} + \rho_p(\mathbf{u}_p \cdot \nabla_m^H) w_m + \rho_p \frac{\partial}{\partial z}\left(\frac{\theta_p}{\kappa}\pi_m\right) = \frac{\rho_p}{\theta_p}\theta_m + \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\pi_m + S_w^m, \tag{22b}$$

where (22b) can be obtained from (16b) using

$$-\rho_m = \frac{\rho_p}{\theta_p}\theta_m - \frac{1}{\gamma\kappa}\frac{\rho_p}{\pi_p}\pi_m, \tag{23}$$

which is derivable from (19), (20), and (21), as well as

$$\frac{\partial p_m}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\rho_p \theta_p}{\kappa}\pi_m\right) = \rho_p \frac{\partial}{\partial z}\left(\frac{\theta_p}{\kappa}\pi_m\right) + \frac{\theta_p}{\kappa}\frac{\partial \rho_p}{\partial z}\pi_m \tag{24}$$

$$= \rho_p \frac{\partial}{\partial z}\left(\frac{\theta_p}{\kappa}\pi_m\right) - \frac{1}{\kappa}\left(\rho_p \frac{\partial \theta_p}{\partial z} + \frac{1}{\gamma}\frac{\rho_p}{\pi_p}\right)\pi_m, \tag{25}$$

where the last line is obtained upon taking the vertical derivative of $\rho_p\theta_p = {p_p}^{1/\gamma}$, which can be obtained from (19) and (20).

The planetary-scale dynamical equations are $O(\epsilon^2)$. To obtain them, we apply the mesoscale average to the $O(\epsilon^2)$ equations and account for the mesoscale average applied to the $O(\epsilon)$ equations.[2] The horizontal momentum equation on the planetary scale is

$$\rho_p \frac{\partial \mathbf{u}_p}{\partial t_p} + \rho_p(\mathbf{v}_p \cdot \nabla_p)\mathbf{u}_p + \mathbf{e}_z \times \rho_p \mathbf{u}_p + \frac{1}{M^2}\nabla_p^H p_p = -\frac{\partial}{\partial z}(\rho_p \overline{\mathbf{u}_m w_m}), \tag{26}$$

where $\mathbf{e}_z = (\mathbf{e} \cdot \hat{\mathbf{z}})\hat{\mathbf{z}}$, and is derived using (18). The planetary-scale potential temperature equation is

$$\frac{1}{M^2}\frac{\rho_p \pi_p}{\kappa}\left[\frac{\partial \theta_p}{\partial t_p} + (\mathbf{v}_p \cdot \nabla_p)\theta_p\right] = -\frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) - \frac{\rho_p}{\theta_p}\overline{\theta_m w_m} - \frac{\theta_p}{\kappa}\overline{\pi_m \nabla_m \cdot (\rho_p \mathbf{v}_m)} - \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m} + \frac{1}{M^2}S_\theta^p \tag{27}$$

and is derived using the relations

$$M^2 \frac{1}{\rho_p}\overline{\rho_m S_\theta^m} = \frac{\pi_p}{\kappa}\left[\overline{\theta_m \nabla_m \cdot (\rho_p \mathbf{v}_m)} + \frac{\partial \theta_p}{\partial z}\overline{\rho_m w_m}\right], \tag{28}$$

$$M^2 \frac{1}{\pi_p}\overline{\pi_m S_\theta^m} = \frac{\rho_p}{\kappa}\left[\frac{\theta_p}{\rho_p}\overline{\pi_m \nabla_m \cdot (\rho_p \mathbf{v}_m)} + \frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m}\right], \tag{29}$$

which can be derived by multiplying (17) by ρ_m and π_m, respectively, and using (18). Note that (27) has been divided by M^2 so that when calculating the total energy budget the pressure-work term on the planetary scale can be associated with the corresponding term in the planetary-scale kinetic energy budget. We choose to write the planetary-scale potential temperature equation in the above form so that mesoscale kinetic energy conversion terms can be identified. We note that the first

[2] The mesoscale average of (22) contributes to $O(\epsilon^2)$ according to the mass-weighted average decomposition: $\overline{\rho \mathbf{v}_m} = 0$ implies $\overline{\rho_p \mathbf{v}_m} = -\epsilon M^2 \overline{\rho_m \mathbf{v}_m} \sim O(\epsilon)$. Thus, the $\overline{\rho_p w_m}$ and $\overline{\rho_m w_m}$ contributions combine at $O(\epsilon^2)$ to satisfy $\overline{\rho w_m} = 0$.

two terms on the right-hand side of (27) can be associated with diabatic effects on the mesoscale because

$$\frac{\rho_p \pi_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\theta_m w_m} = M^2 \overline{\theta_m S_\theta^m}, \tag{30}$$

which is derivable by multiplying (17) by θ_m [analogous to (28) and (29)]. Finally, the planetary-scale continuity equation is

$$\frac{\partial \rho_p}{\partial t_p} + \mathbf{\nabla}_p \cdot (\rho_p \mathbf{v}_p) = 0. \tag{31}$$

Given that most climate models use enthalpy as their prognostic thermodynamic variable, it is beneficial to convert the planetary-scale potential temperature equation into an equation for planetary-scale enthalpy. Expanding the left-hand side of (27) and using the leading-order expression in (20), we obtain

$$\begin{aligned}&\frac{1}{M^2}\frac{\rho_p}{\kappa}\left[\frac{\partial T_p}{\partial t_p} + (\mathbf{v}_p \cdot \mathbf{\nabla}_p)T_p\right] - \frac{1}{M^2}\left[\frac{\partial p_p}{\partial t_p} + (\mathbf{v}_p \cdot \nabla_p^H)p_p\right] + \frac{1}{M^2}\rho_p w_p \\ &= -\frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) - \frac{\rho_p}{\theta_p}\overline{\theta_m w_m} - \frac{\theta_p}{\kappa}\overline{\pi_m \mathbf{\nabla}_m \cdot (\rho_p \mathbf{v}_m)} - \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m} + \frac{1}{M^2}S_\theta^p.\end{aligned} \tag{32}$$

It is clear from (26) and (32) that mesoscale fluxes of momentum, potential temperature, and pressure drive the planetary-scale flow, which obeys hydrostatic dynamics (15). The quasi-linear mesoscale [(17), (18), and (22)] is coupled nonlinearly through these eddy fluxes to the planetary scale. This interaction has implications for the planetary-scale energy and momentum budgets.

The planetary-scale momentum budget, derived by adding (26) to $\mathbf{u}_p$ times (31), is

$$\begin{aligned}&\frac{\partial}{\partial t_p}(\rho_p \mathbf{u}_p) + \mathbf{\nabla}_p \cdot (\rho_p \mathbf{v}_p \circ \mathbf{u}_p) + \mathbf{e}_z \times \rho_p \mathbf{u}_p + \frac{1}{M^2}\nabla_p^H p_p \\ &= -\frac{\partial}{\partial z}(\rho_p \overline{\mathbf{u}_m w_m}),\end{aligned} \tag{33}$$

where $\circ$ is the tensor product. Because the important transfers between the planetary scale and mesoscale are assumed to occur in the vertical direction, angular momentum conservation applies within each model vertical column and hence within each latitude band. Moreover, we have made a shallow atmosphere approximation. Therefore, angular momentum conservation in this case is equivalent to zonal momentum conservation. (Recall that in the shallow-atmosphere approximation the angular momentum does not vary with z.) It is clear from (33) that integrating over z leads to conservation of zonal momentum.

The contribution of mesoscale momentum flux convergences to the planetary-scale energy budget can be understood by calculating the total energy budget on the planetary scale. Upon taking the inner product of (22) with $\mathbf{v}_m$ and taking a mesoscale average, we obtain the mesoscale kinetic energy equation on the planetary scale:

$$\begin{aligned}&\rho_p \overline{\mathbf{u}_m w_m} \cdot \frac{\partial \mathbf{u}_p}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\rho_p \theta_p}{\kappa}\overline{\pi_m w_m}\right) \\ &= \frac{\rho_p}{\theta_p}\overline{\theta_m w_m} + \frac{\theta_p}{\kappa}\overline{\pi_m \mathbf{\nabla}_m \cdot (\rho_p \mathbf{v}_m)} \\ &\quad + \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m} + \overline{\mathbf{v}_m \cdot \mathbf{S}_\mathbf{v}^m}.\end{aligned} \tag{34}$$

Mesoscale kinetic energy is changed by planetary-scale shear, pressure work, the conversion of planetary-scale enthalpy to mesoscale kinetic energy [cf. first three terms on the right-hand side of (34) and the second through fourth terms on the right-hand side of (32)], and the cascade of kinetic energy to smaller (micro) scales [the last term in (34)]. We obtain the planetary-scale kinetic energy equation upon taking the inner product of (26) with $\mathbf{u}_p$ and adding it to K_p multiplied by (31):

$$\begin{aligned}&\frac{\partial}{\partial t_p}(\rho_p K_p) + \mathbf{\nabla}_p \cdot (\rho_p \mathbf{v}_p K_p) + \frac{1}{M^2}\mathbf{u}_p \cdot \nabla_p^H p_p \\ &= -\mathbf{u}_p \cdot \frac{\partial}{\partial z}(\rho_p \overline{\mathbf{u}_m w_m}),\end{aligned} \tag{35}$$

where $K_p = |\mathbf{u}_p|^2/2$. As we expect for hydrostatic dynamics, K_p does not include w_p. Planetary-scale kinetic energy is changed locally by pressure work and mesoscale momentum fluxes. Upon adding $T_p/\kappa M^2$ times (31) to (32), we obtain the planetary-scale internal energy equation:

$$\frac{1}{M^2}\frac{1}{\kappa}\left[\frac{1}{\gamma}\frac{\partial}{\partial t_p}(\rho_p T_p) + \nabla_p \cdot (\rho_p \mathbf{v}_p T_p)\right] - \frac{1}{M^2}(\mathbf{u}_p \cdot \nabla_p^H)p_p + \frac{1}{M^2}\rho_p w_p$$
$$= -\frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) - \frac{\rho_p}{\theta_p}\overline{\theta_m w_m} - \frac{\theta_p}{\kappa}\overline{\pi_m \nabla_m \cdot (\rho_p \mathbf{v}_m)} - \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m} + \frac{1}{M^2}S_\theta^p. \tag{36}$$

Planetary-scale internal energy is changed locally by pressure work, the conversion to planetary-scale potential energy, mesoscale entropy fluxes, the conversion to mesoscale kinetic energy, and planetary-scale sources and sinks. Defining the geopotential as $\Phi_p = z/M^2$, the planetary-scale potential energy equation can be calculated using (31) as

$$\frac{\partial}{\partial t_p}(\rho_p \Phi_p) + \nabla_p \cdot (\rho_p \mathbf{v}_p \Phi_p) - \frac{1}{M^2}\rho_p w_p = 0. \tag{37}$$

Planetary-scale potential energy is changed locally by conversion to planetary-scale internal energy. Finally, the total energy equation on the planetary scale is obtained by adding (34), (35), (36), and (37):

$$\frac{\partial}{\partial t_p}\left[\rho_p\left(K_p + \frac{1}{M^2}\frac{1}{\kappa\gamma}T_p + \Phi_p\right)\right] + \nabla_p \cdot \left[\rho_p \mathbf{v}_p\left(K_p + \frac{1}{M^2}\frac{1}{\kappa}T_p + \Phi_p\right)\right]$$
$$= -\frac{\partial}{\partial z}\left(\rho_p \mathbf{u}_p \cdot \overline{\mathbf{u}_m w_m} + \frac{\rho_p \theta_p}{\kappa}\overline{\pi_m w_m}\right) - \frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) + \overline{\mathbf{v}_m \cdot \mathbf{S}_\mathbf{v}^m} + \frac{1}{M^2}S_\theta^p. \tag{38}$$

It is clear from (38) that total energy is globally conserved apart from the last two terms on the right-hand side. Total energy on the planetary scale is changed locally by mesoscale momentum and pressure fluxes, mesoscale potential temperature fluxes, the cascade of mesoscale kinetic energy to smaller scales, and planetary-scale sources and sinks. The last three terms in (38) are directly attributable to nonconservative effects. In the case of the mesoscale potential temperature flux, it is the direct result of diabatic effects on the mesoscale according to (30).

4. Understanding the interaction across scales: Wave-activity conservation laws

The planetary scale and mesoscale interact through eddy flux convergences of mesoscale momentum, potential temperature, and pressure, which drive the planetary scale via quasi-linear mesoscale dynamics. A full understanding of the exchange of energy and momentum across scales requires dynamical equations on the mesoscale describing how mesoscale energy and momentum evolve through the interaction with the planetary scale. Because of the existence of the planetary-scale background, the mesoscale energy and momentum are themselves not the best quantities to examine for this purpose, as they are not conserved under adiabatic dynamics. Here we use wave-activity conservation laws to understand such multiscale interactions. By a wave-activity conservation law, we mean a relation of the form

$$\frac{\partial A}{\partial t_m} + \nabla_m \cdot \mathbf{F} = D, \tag{39}$$

where A is the wave-activity density and $\mathbf{F}$ its flux, both being quadratic for linear dynamics, and D is the wave-activity source–sink term. In the conservative case $D = 0$. Note that after performing a mesoscale average, (39) becomes

$$\frac{\partial}{\partial z}\overline{F_{(z)}} = \overline{D} \tag{40}$$

and hence any vertical wave-activity flux must be driven by source–sink terms somewhere in the vertical column.

Wave-activity conservation laws play a central role in the study of fluid dynamical disturbances to a specified background state. In the case of the large-scale circulation of the atmosphere, the Eliassen–Palm wave activity has been crucial to theoretical analysis (Andrews et al. 1987; Shepherd 2003; Vallis 2006). Wave-activity conservation laws can be generally derived using the Hamiltonian structure of geophysical fluid dynamics (Shepherd 1990). This framework should, in principle, allow one to consider general disturbances to a background flow, without making any Wentzel–Kramers–Brillouin (WKB)-type assumptions, and be extendable to finite amplitude. In the case of the dynamics derived in the previous section, these conservation laws can be derived in the usual way because the planetary scale acts as a horizontally and temporally homogeneous background flow for the mesoscale dynamics according to the ansatz (11). (The scales are, however, coupled in the vertical, as is assumed in the parameterization of subgrid scales in climate models.) In particular, the x_m, y_m, and t_m symmetries in the background flow lead to pseudomomentum and pseudoenergy conservation laws (Shepherd 1990).

Shaw and Shepherd (2008, hereafter SS08) derived wave-activity conservation laws for three-dimensional disturbances to a horizontally homogeneous background flow with disturbances governed by the anelastic or Boussinesq equations. In the current nomenclature, the mesoscale is considered the disturbance and the planetary scale is the background flow. The mesoscale dynamics derived in section 3 are not explicitly anelastic; however, the results of SS08 can be extended to the situation considered here. A detailed derivation can be found in appendix A. The anelastic form of the mesoscale fluxes and multiscale interactions, and their connection to the wave-activity conservation laws, is presented in the next section.

The mesoscale source–sink terms are responsible for the generation and dissipation of mesoscale pseudoenergy and pseudomomentum: taking the mesoscale average of wave activities derived in appendix A, (A10) and (A21), yields

$$\frac{\partial}{\partial z}\overline{F^{\mathcal{E}}_{(z)}} = \frac{\partial}{\partial z}\left(\rho_p \mathbf{u}_p \cdot \overline{\mathbf{u}_m w_m} + \frac{\rho_p \theta_p}{\kappa}\overline{\pi_m w_m}\right) = \overline{D^{\mathcal{E}}}, \tag{41a}$$

$$\frac{\partial}{\partial z}\left[\overline{F^{\mathcal{P}_x}_{(z)}}\hat{\mathbf{x}} + \overline{F^{\mathcal{P}_y}_{(z)}}\hat{\mathbf{y}}\right] = \frac{\partial}{\partial z}(\rho_p \overline{\mathbf{u}_m w_m}) = \overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}, \tag{41b}$$

where $F^{\mathcal{E}}_{(z)}$, $F^{\mathcal{P}_x}_{(z)}$, and $F^{\mathcal{P}_y}_{(z)}$ are the vertical components of the pseudoenergy and x and y pseudomomentum fluxes with corresponding source–sink terms $D^{\mathcal{E}}$, $D^{\mathcal{P}_x}$, and $D^{\mathcal{P}_y}$, defined in (A11) and (A22), which satisfy the constraint

$$\begin{aligned}\overline{D^{\mathcal{E}}} - \overline{\mathbf{v}_m \cdot \mathbf{S}^m_{\mathbf{v}}} - M^2\left[\frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S^m_\theta} + \frac{1}{\pi_p}\overline{\pi_m S^m_\theta}\right] \\ = \mathbf{u}_p \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right).\end{aligned} \tag{42}$$

Note that the vertical fluxes are identical to those derived by Hines and Reddy (1967) without using the Hamiltonian framework but assuming WKB-type conditions in the vertical. This shows that previous work using these fluxes applies more generally than under the circumstances considered by Hines and Reddy (1967). This generality is reflected in the fact that the vertical fluxes are directly associated with the wave-activity densities (A10) and (A21), which take full account of the background shear, as does the right-hand side of (41).

The vertical wave-activity fluxes in (41) are important in driving the planetary scale. In particular, the pseudomomentum (41b) forces the planetary-scale momentum via (33), whereas the pseudoenergy (41a) contributes to forcing the total energy on the planetary scale via (38). The wave-activity conservation laws provide a means of relating the mesoscale fluxes in (33) and (38) to source–sink terms on the mesoscale. The other mesoscale terms on the right-hand side of (38) have already been related to source–sink terms on the mesoscale. In the absence of wave-activity sources–sinks, the subgrid-scales do not contribute to the planetary-scale budgets and thus (41a) and (41b) can be thought of as ''non-acceleration'' theorems (Charney and Drazin 1961; Eliassen and Palm 1961) for the effects of subgrid-scale disturbances.

We now proceed by making use of a relationship that connects the planetary-scale energy and momentum budgets. There exists a general relationship between the pseudoenergy and pseudomomentum wave activities, $A^{\mathcal{E}} = cA^{\mathcal{P}}$ (where c is the phase velocity in the direction of symmetry associated with $A^{\mathcal{P}}$), which is derivable from Noether's theorem (see SS08). This relationship holds for a monochromatic wave. However, because $A^{\mathcal{E}}$, $A^{\mathcal{P}_s}$, and $A^{\mathcal{P}_n}$ are quadratic and the background flow is homogeneous in x_m, y_m, and t_m, the quantities can be decomposed into spectra and the relationship holds for each wavenumber–frequency pair (with its own c). In fact, the wave activities need not correspond to waves at all; we may just define c as $c = A^{\mathcal{E}}/A^{\mathcal{P}}$. This relationship imposes a relationship between the vertical components of the pseudoenergy and pseudomomentum fluxes:

$$\overline{F^{\mathcal{E}}_{(z)}} = c_s\overline{F^{\mathcal{P}_s}_{(z)}} = c_s\hat{\mathbf{s}} \cdot \left[\overline{F^{\mathcal{P}_x}_{(z)}}\hat{\mathbf{x}} + \overline{F^{\mathcal{P}_y}_{(z)}}\hat{\mathbf{y}}\right], \tag{43}$$

where c_s is the streamwise phase velocity. As noted by SS08, (43) is the generalization of the first Eliassen–Palm theorem (Lindzen 1990) to three dimensions with a veering background flow.[3] Relations (42), (41), and (43) are key to ensuring the consistency of energy and momentum conservation on the mesoscale. In particular, the relations (41) and (43) imply

$$\overline{D^{\mathcal{E}}} = c_s\hat{\mathbf{s}} \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right). \tag{44}$$

Combining (44) with (42), we obtain

[3] Eliassen and Palm (1961) derived their first and second theorems for the case of steady, two-dimensional gravity waves, assuming conservative dynamics. However, they did not derive conservation laws that relate the vertical fluxes to conserved quantities. Subsequent authors (Bretherton 1966; Hines and Reddy 1967; Lindzen 1973) generalized the Eliassen–Palm theorems to nonsteady disturbances, but they appealed to WKB-type conditions in the vertical (large Richardson number) to define a wave packet with phase velocity c and the associated wave action.

$$-\overline{\mathbf{v}_m \cdot \mathbf{S}_{\mathbf{v}}^m} - M^2\left[\frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S_\theta^m} + \frac{1}{\pi_p}\overline{\pi_m S_\theta^m}\right] = -\overline{\mathbf{v}_m \cdot \mathbf{S}_{\mathbf{v}}^m} - \frac{\rho_p}{\theta_p}\overline{\theta_m w_m} - \frac{\theta_p}{\kappa}\overline{\pi_m \nabla_m \cdot (\rho_p \mathbf{v}_m)} - \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m}$$
$$= -\overline{D^{\mathcal{E}}} + \mathbf{u}_p \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) = (\mathbf{u}_p - c_s\hat{\mathbf{s}}) \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right), \quad (45)$$

where the second equality follows from (29) and (30).

Relation (45) represents the generalization of the second Eliassen–Palm theorem to three dimensions with a veering background flow and to dissipative dynamics. Lindzen (1973) extended the second Eliassen–Palm theorem to include diabatic forcing but assumed WKB conditions in the vertical. The plane-parallel version of (45) with $\mathbf{S}_{\mathbf{v}}^m = 0$, in particular the first line and last equality, corresponds exactly to (8) in Lindzen (1973) and to (8.24) in Lindzen (1990). Here we have shown that the second Eliassen–Palm theorem can be derived systematically using the Hamiltonian structure of geophysical fluid dynamics in the general context of the interaction between a mesoscale (subgrid-scale) flow and a planetary-scale (resolved) flow. Furthermore, we have generalized the relation to three dimensions with a veering background flow, and the wave-activity source–sink terms take full account of the background vertical shear (there is no WKB-type requirement on the background flow).

We now discuss the implications of (45) for conservation of total energy and the second law of thermodynamics. Two requirements of our theory are that total energy is conserved when there are no external sources or sinks and that the second law of thermodynamics (entropy production by internal dynamics) is satisfied. To prove that total energy is conserved, we must show that the integral of (38) over the entire domain $dV = dx_p\, dy_p\, dz$ vanishes under appropriate boundary conditions. Before applying the integral, we must partition the planetary-scale thermal source/sink term S_θ^p into the rate of work done by the mesoscale source/sink term $\mathbf{S}_{\mathbf{v}}^m$ and the contribution from planetary-scale external sources/sinks. In particular, we set

$$\frac{1}{M^2}S_\theta^p = \overline{(\mathbf{d}_m \cdot \nabla_m) \cdot \mathbf{v}_m} + Q_\theta^p$$
$$= -\overline{\mathbf{v}_m \cdot \mathbf{S}_{\mathbf{v}}^m} + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] + Q_\theta^p, \quad (46)$$

where $\mathbf{d}_m$ is the symmetric shear stress tensor on the mesoscale with $(\mathbf{d}_m)_z = [(d_m)_{xz}, (d_m)_{yz}, (d_m)_{zz}]$ and $\mathbf{S}_{\mathbf{v}}^m = \nabla_m \cdot \mathbf{d}_m$; Q_θ^p is a mass-weighted diabatic source/sink term. The second term on the right-hand side involving $\mathbf{d}_m$ is recognized as the transfer of the turbulent shear stress by the mesoscale velocity. This partitioning isolates the external (radiative) heating from the internal heating due to kinetic energy dissipation. (To incorporate latent heat release would require a full treatment of moist processes to ensure total energy conservation). We choose not to introduce an explicit stress tensor formulation of $\mathbf{S}_{\mathbf{v}}^m$ so as to keep the framework general enough to admit different closures. (The closure will depend on the particular physical processes being parameterized.)

Applying (46) to (38), integrating over dV (assuming vanishing boundary conditions), and assuming $Q_\theta^p = 0$ we obtain

$$\int \frac{\partial}{\partial t_p}\left[\rho_p\left(K_p + \frac{1}{M^2}\frac{1}{\kappa\gamma}T_p + \Phi_p\right)\right]dV = 0. \quad (47)$$

Thus, total energy is conserved when $Q_\theta^p = 0$.

The second requirement that must be satisfied is the second law of thermodynamics. The planetary-scale entropy equation is

$$\frac{1}{M^2}\left[\frac{\partial}{\partial t_p}(\rho_p \ln\theta_p) + \nabla_p \cdot (\rho_p \mathbf{v}_p \ln\theta_p)\right] = -\frac{\partial}{\partial z}\left(\frac{\rho_p}{\theta_p}\overline{\theta_m w_m}\right) - \frac{\rho_p}{\theta_p^2}\frac{\partial \theta_p}{\partial z}\overline{\theta_m w_m} - \frac{\kappa}{T_p}\left[\frac{\theta_p}{\kappa}\overline{\pi_m \nabla_m \cdot (\rho_p \mathbf{v}_m)}\right.$$
$$\left. + \frac{\rho_p}{\kappa}\frac{\partial \theta_p}{\partial z}\overline{\pi_m w_m}\right] - \frac{\kappa}{T_p}\left\{\overline{\mathbf{v}_m \cdot \mathbf{S}_{\mathbf{v}}^m} - \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] - Q_\theta^p\right\}, \quad (48)$$

which is obtained by multiplying (27) by $\kappa/\pi_p\theta_p$, adding $\ln\theta_p$ multiplied by (31), and using (46). To ensure that the planetary-scale entropy increases as a result of exchanges of energy and momentum with the mesoscale, we require that the vertical integral of (48) be positive definite when $Q_\theta^p = 0$—that is, when the atmosphere is a closed system (a condition that is required for the second law to hold). (In practice, $\int \kappa Q_\theta^p/T_p\, dV < 0$ balances the entropy production.) Under such conditions, ensuring the positivity of the right-hand side of

(48) (with $Q^p_\theta = 0$) ensures that there is an irreversible increase in planetary-scale entropy, consistent with the second law. The right-hand side of (48) can be related to the wave-activity conservation law source/sink terms using (45) such that entropy production requires

$$\begin{aligned}
&\int \frac{\kappa}{T_p}\left\{(\mathbf{u}_p - c_s\hat{\mathbf{s}})\cdot\left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) - \frac{\partial T_p}{\partial z}\frac{\rho_p}{\kappa\theta_p}\overline{\theta_m w_m} + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m\cdot(\mathbf{d}_m)_z}]\right\}dz \\
&= \int \frac{\kappa}{T_p}\left\{-\overline{D^{\mathcal{E}}} + \mathbf{u}_p\cdot\left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) - \frac{\partial T_p}{\partial z}\frac{\rho_p}{\kappa\theta_p}\overline{\theta_m w_m} + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m\cdot(\mathbf{d}_m)_z}]\right\}dz \\
&= -\int \frac{\kappa}{T_p}\left\{\overline{\mathbf{v}_m\cdot\mathbf{S}^m_{\mathbf{v}}} - \frac{\partial}{\partial z}[\overline{\mathbf{v}_m\cdot(\mathbf{d}_m)_z}] + M^2\frac{1}{\theta_p}\overline{\theta_m S^m_\theta} + M^2\frac{1}{\pi_p}\overline{\pi_m S^m_\theta}\right\}dz \\
&= -\int \frac{\kappa}{T_p}\left\{\overline{\mathbf{v}_m\cdot\mathbf{S}^m_{\mathbf{v}}} - \frac{\partial}{\partial z}[\overline{\mathbf{v}_m\cdot(\mathbf{d}_m)_z}] + \frac{\pi_p}{\kappa}\frac{\rho_p}{\theta_p}\frac{\partial\theta_p}{\partial z}\overline{\theta_m w_m} + \frac{\theta_p}{\kappa}\overline{\pi_m\nabla_m\cdot(\rho_p\mathbf{v}_m)} + \frac{\rho_p}{\kappa}\frac{\partial\theta_p}{\partial z}\overline{\pi_m w_m}\right\}dz > 0. \qquad (49)
\end{aligned}$$

Further interpretation is relegated to the next section where we discuss the implications under the anelastic constraint.

5. Anelastic dynamics

In section 3 planetary-scale momentum and total energy budgets were derived, including contributions from mesoscale fluxes of momentum, heat, density, and pressure. The dynamics on the mesoscale, although nonhydrostatic, did not satisfy the anelastic constraint $\nabla_m\cdot(\rho_p\mathbf{v}_m) = 0$. Understanding how the planetary-scale momentum and total energy budgets are affected by an anelastic constraint on the mesoscale is relevant for climate models because the theoretical formulations of many state-of-the-art subgrid-scale parameterizations are based on the anelastic equations. A reduction of the planetary-scale momentum and energy budgets derived in section 3 can be made by neglecting those terms affected by the anelastic constraint. In particular, a reduced set of planetary-scale equations with mesoscale fluxes satisfying the anelastic constraint can be obtained by neglecting terms in (27) proportional to $M^2\overline{\pi_m S^m_\theta}$ [identified using (29)]. Upon neglecting the relevant terms and proceeding as in section 3, the planetary-scale momentum and total energy equations become

$$\frac{\partial}{\partial t_p}(\rho_p\mathbf{u}_p) + \nabla_p\cdot(\rho_p\mathbf{v}_p\circ\mathbf{u}_p) + \mathbf{e}_z\times\rho_p\mathbf{u}_p + \frac{1}{M^2}\nabla^H_p p_p = -\frac{\partial}{\partial z}(\rho_p\overline{\mathbf{u}_m w_m}), \qquad (50a)$$

$$\begin{aligned}
&\frac{\partial}{\partial t_p}\left[\rho_p\left(K_p + \frac{1}{M^2}\frac{1}{\kappa\gamma}T_p + \Phi_p\right)\right] + \nabla_p\cdot\left[\rho_p\mathbf{v}_p\left(K_p + \frac{1}{M^2}\frac{1}{\kappa}T_p + \Phi_p\right)\right] \\
&= -\frac{\partial}{\partial z}\left(\rho_p\mathbf{u}_p\cdot\overline{\mathbf{u}_m w_m} + \frac{\rho_p\theta_p}{\kappa}\overline{\pi_m w_m}\right) - \frac{\partial}{\partial z}\left(\frac{\rho_p\pi_p}{\kappa}\overline{\theta_m w_m}\right) + \overline{\mathbf{v}_m\cdot\mathbf{S}^m_{\mathbf{v}}} + \frac{1}{M^2}S^p_\theta. \qquad (50b)
\end{aligned}$$

Clearly, the neglect of these terms does not alter the conservation of momentum and energy on the planetary scale. It also does not change the mesoscale contributions to the right-hand sides of the two budgets.

The neglect of these terms to obtain the reduced planetary-scale dynamics can be justified by considering the small Mach number limit of the dynamics presented in section 3. [Note that section 3 considered an $O(1)$ Mach number.] It is clear from (18) that in the limit of small Mach number, the anelastic constraint $\nabla_m\cdot(\rho_p\mathbf{v}_m) = 0$ is recovered. In this limit, the integrity of the mesoscale thermodynamic equation (17) can be preserved if the stratification on the planetary scale is assumed to be $O(M^2)$. A weak stratification is a well-known assumption of the anelastic equations (Lipps and Hemler 1982; Klein 2000). It is clear that these assumptions result in the neglect of terms proportional to $M^2\overline{\pi_m S^m_\theta}$ in the planetary-scale enthalpy and mesoscale kinetic energy equations.

The mesoscale potential temperature flux is not neglected because (30) remains valid under the anelastic approximation when $\partial\theta_p/\partial z = O(M^2)$. As in the previous sections, its source is diabatic effects on the mesoscale.

Zeytounian (1990) derived the small-Mach number limit of the hydrostatic primitive equations, which he called the reduced hydrostatic primitive equations, by first applying the shallow atmosphere approximation (equivalent to $\epsilon \ll 1$ in section 3) and then applying the small Mach number limit (see his section 7.4). The small Mach number limit was applied to the hydrostatic primitive equations in pressure coordinates, and the leading-order pressure and enthalpy contributions were $O(M^2)$. Our reduced planetary-scale dynamics are in agreement with (7.57) of Zeytounian (1990).

The reduced wave-activity conservation laws are obtained by applying the small Mach number and weak stratification limit to the compressible pseudoenergy and streamwise and normal pseudomomentum densities in appendix A. The anelastic wave-activity conservation laws derived in appendix B correspond exactly to those derived in SS08. Upon performing a mesoscale average over the anelastic wave-activity conservation laws (B3) and (B6), we obtain

$$\frac{\partial}{\partial z}\overline{F^{\mathcal{E}}_{(z)}} = \frac{\partial}{\partial z}\left(\rho_p \mathbf{u}_p \cdot \overline{\mathbf{u}_m w_m} + \frac{\rho_p \theta_p}{\kappa}\overline{\pi_m w_m}\right) = \overline{D^{\mathcal{E}}}, \tag{51a}$$

$$\frac{\partial}{\partial z}[\overline{F^{\mathcal{P}_x}_{(z)}}\hat{\mathbf{x}} + \overline{F^{\mathcal{P}_y}_{(z)}}\hat{\mathbf{y}}] = \frac{\partial}{\partial z}(\rho_p \overline{\mathbf{u}_m w_m}) = \overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}, \tag{51b}$$

with the wave-activity source–sink terms defined in (B4) and (B7) satisfying the constraint

$$\overline{D^{\mathcal{E}}} - \overline{\mathbf{v}_m \cdot \mathbf{S}^m_{\mathbf{v}}} - M^2 \frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S^m_\theta} = \mathbf{u}_p \cdot (\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}). \tag{52}$$

Analogously to (41), (51) can be thought of as nonacceleration theorems for the effects of subgrid-scale disturbances satisfying the anelastic constraint.

Interestingly, the anelastic constraint only affects the horizontal wave-activity fluxes; the vertical wave-activity fluxes are unchanged. A plausible reason for this is that the hydrostatic primitive equations admit only horizontally propagating sound waves, so any coupling with the mesoscale through compressible dynamics can only occur through the horizontal fluxes. Thus, the anelastic limit, which filters out sound waves, has no effect on the vertical fluxes, which are responsible for forcing the planetary-scale momentum and energy.

Making use of (30) and the relationship (44) between the pseudoenergy and the pseudomomentum, we obtain

$$\begin{aligned}-\overline{\mathbf{v}_m \cdot \mathbf{S}^m_{\mathbf{v}}} - M^2 \frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S^m_\theta} &= -\overline{\mathbf{v}_m \cdot \mathbf{S}^m_{\mathbf{v}}} - \frac{\rho_p}{\theta_p}\overline{\theta_m w_m}\\ &= -\overline{D^{\mathcal{E}}} + \mathbf{u}_p \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right)\\ &= (\mathbf{u}_p - c_s\hat{\mathbf{s}}) \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right),\end{aligned} \tag{53}$$

which is the anelastic version of (45).

As in section 4, the wave-activity conservation laws provide a means of relating the mesoscale flux terms in (50a) and (50b) to mesoscale source–sink terms. [The other terms on the right-hand side of (50b) have already been related to source–sink terms on the mesoscale.] As discussed in the previous section, (53) is the generalization of the second Eliassen–Palm theorem for the case of anelastic dynamics.

Upon applying the same partitioning of the planetary-scale thermodynamic source/sink term as in (46), energy conservation when $Q^p_\theta = 0$ is preserved under the anelastic constraint. Proceeding to the second law of thermodynamics, under the anelastic approximation (48) becomes

$$\begin{aligned}&\frac{1}{M^2}\left[\frac{\partial}{\partial t_p}(\rho_p \ln\theta_p) + \nabla_p \cdot (\rho_p \mathbf{v}_p \ln\theta_p)\right]\\ &\quad = -\frac{\partial}{\partial z}\left(\frac{\rho_p}{\theta_p}\overline{\theta_m w_m}\right) - \frac{\kappa}{T_p}\Bigg\{\overline{\mathbf{v}_m \cdot \mathbf{S}^m_{\mathbf{v}}}\\ &\qquad - \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] - Q^p_\theta\Bigg\},\end{aligned} \tag{54}$$

where we have neglected terms proportional to $M^2\overline{\pi_m S^m_\theta}$ and $M^2\overline{\theta_m S^m_\theta}$ identified using (29) and (30). As in the previous section, the right-hand side of (54) can be related to the anelastic wave-activity conservation law source/sink terms via (53) such that entropy production (for $Q^p_\theta = 0$) requires

$$\begin{aligned}&\int \frac{\kappa}{T_p}\left[(\mathbf{u}_p - c_s\hat{\mathbf{s}}) \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) + \frac{\rho_p}{\theta_p}\overline{\theta_m w_m} + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}]\right] dz\\ &\quad = \int \frac{\kappa}{T_p}\left\{-\overline{D^{\mathcal{E}}} + \mathbf{u}_p\left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) + M^2\frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S^m_\theta} + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}]\right\} dz\\ &\quad = -\int \frac{\kappa}{T_p}\left\{\overline{\mathbf{v}_m \cdot \mathbf{S}^m_{\mathbf{v}}} - \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}]\right\} dz > 0.\end{aligned} \tag{55}$$

If a diffusive (Richardson flux gradient) closure is assumed for $(\mathbf{d}_m)_z$, the right-hand side of (55) can be rewritten according to

$$\begin{aligned}
&-\overline{\mathbf{v}_m \cdot \mathbf{S}_{\mathbf{v}}^m} + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] \\
&= -\overline{\mathbf{v}_m \cdot \frac{\partial}{\partial z}\left(K_z \frac{\partial \mathbf{v}_m}{\partial z}\right)} + \frac{\partial}{\partial z}\left[\overline{\mathbf{v}_m \cdot \left(K_z \frac{\partial \mathbf{v}_m}{\partial z}\right)}\right] \\
&= K_z \overline{\left(\frac{\partial \mathbf{v}_m}{\partial z}\right)^2},
\end{aligned} \tag{56}$$

with K_z being a vertical diffusion coefficient. For $K_z > 0$, the entropy production constraint is clearly satisfied in this case. The terms in the entropy production constraint can be associated with different subgrid-scale energy transfers. The first term on the left-hand side in the first line of (55), proportional to the wave-activity source–sink terms, is associated with nonlocal mesoscale energy transfer, the second with local transfer of mesoscale internal–potential energy to planetary-scale internal energy, and the final term with local kinetic energy transfer from the mesoscale to turbulence.

According to the first Eliassen–Palm theorem (43), upward propagation implies that momentum flux deposition must drag the background flow toward the phase velocity of the waves, such that the nonlocal transfer term in (55) is positive in the wave dissipation region. We obtain the same result from the requirement of entropy production, provided the local transfers are not positive. Our framework, however, shows that a complete statement requires consideration of the local transfers as well as of the wave source region (which is not generally taken into account in gravity wave parameterizations).

It is apparent from (55) that the sign of the various contributions to the entropy budget depends crucially on the structure of the background temperature T_p. If we consider the case of stable stratification on the planetary scale and assume that mesoscale mixing is downgradient, then $\overline{\theta_m w_m} < 0$; thus, the second term on the left-hand side of (55) is negative-definite and requires a compensation in the other terms to ensure that entropy is produced. In this way, the entropy budget constrains the extent of the vertical mixing. Similar considerations apply to the local turbulent transfer term. Furthermore, nonlocal transfers between regions of contrasting T_p are constrained by (55).

The nonlocal transfer term in the entropy production constraint can be related to a planetary-scale thermodynamic (enthalpy) tendency, which is how the constraints of the second law have been usually considered (e.g., Lindzen 1990). This can be shown by making use of the generalized second Eliassen–Palm theorem for the mesoscale dynamics, (53) or (45). In particular, the left-hand sides of (53) or (45) can be substituted into the enthalpy equation (32) to obtain

$$\frac{1}{M^2}\frac{\rho_p}{\kappa}\frac{\partial T_p}{\partial t_p} + \cdots = -\frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) - \overline{D^{\mathcal{E}}} + \mathbf{u}_p \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] + Q_\theta^p \tag{57a}$$

$$= -\frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) + (\mathbf{u}_p - c_s\hat{\mathbf{s}}) \cdot \left(\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}}\right) + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] + Q_\theta^p \tag{57b}$$

$$= -\frac{\partial}{\partial z}\left(\frac{\rho_p \pi_p}{\kappa}\overline{\theta_m w_m}\right) - (\mathbf{u}_p - c_s\hat{\mathbf{s}}) \cdot \rho_p\left(\frac{\partial \mathbf{u}_p}{\partial t_p} + \cdots\right) + \frac{\partial}{\partial z}[\overline{\mathbf{v}_m \cdot (\mathbf{d}_m)_z}] + Q_\theta^p, \tag{57c}$$

where we have used the planetary-scale momentum equation (26) to obtain the second line, and where the dots refer to adiabatic terms. Relation (57c) contrasts with the local conservation of energy (1) discussed in the introduction. Consider the case in which the local transfers [the flux divergence terms in (57)] are neglected, and $Q_\theta^p = 0$. If the remaining nonlocal transfer term is positive-definite, then it is clear that the vertical exchanges of energy and momentum lead to a warming of the planetary-scale temperature (wave absorption warms the background), as argued by Lindzen (1990). On the other hand, if the term proportional to c_s in (53) and (45) is neglected, then an acceleration of the planetary-scale flow by momentum flux convergence leads to a decrease in entropy and local cooling, according to (57), in violation of the second law. This was the example considered in the introduction. Interestingly, the form of the thermodynamic energy tendency is not directly affected by the anelastic constraint on the mesoscale, although the sources of the tendency are [compare the left-hand sides of (53) and (45)].

However, the second law of thermodynamics is a global constraint and cannot be ensured from local considerations as in (57). For complete consistency, it is necessary to consider the entropy constraint (55), as discussed above. In this case the local transfers do not integrate out in the vertical column, as they do in the enthalpy budget.

6. Summary and discussion

By combining the theories of multiple-scale asymptotics and Hamiltonian geophysical fluid dynamics, we have derived a self-consistent (in terms of energy and momentum) theoretical framework for physical parameterization in climate models. We have derived energy (38) and momentum (33) equations for a nonlinear, hydrostatic resolved (planetary) scale, which include interactions with a quasi-linear, compressible or anelastic subgrid-scale (mesoscale) described by (17), (18), and (22). The temporal and horizontal spatial symmetries in the planetary-scale background flow for the mesoscale allow the construction of wave-activity conservation laws (A10), (A21), (B3), and (B6) on the mesoscale. These conservation laws are used to understand the fluxing of energy and momentum between scales and the ultimate dissipation of mesoscale kinetic energy, conversion to planetary-scale internal energy, and increase in planetary-scale entropy. In particular, they are used to relate mesoscale flux terms in the planetary-scale momentum and energy budgets to mesoscale source–sink terms, thereby providing generalized nonacceleration theorems.

The relationships between the subgrid-scale fluxes (45) and (53) derived here, which result from self-consistency in terms of energy and momentum conservation, are generalizations of the second Eliassen–Palm theorem and place strong constraints on the contributions of subgrid-scale fluxes to the resolved energy and momentum budgets. Any parameterization of subgrid-scale momentum fluxes must satisfy them to ensure the conservation of both energy and momentum and conform to the second law of thermodynamics. This includes wave drag, cumulus, and boundary layer parameterizations. In particular, entropy production is only guaranteed if (49) and (55) are satisfied. These relations also place strong constraints on parameterized fluxes and may be especially useful in cases where observational constraints are lacking. [Note that (45), (53), (49), and (55) are valid whether or not a phase velocity can be usefully defined.] For example, mixing by gravity wave breaking is constrained not only by energy considerations but also by the requirement of entropy production. Current treatments of kinetic energy dissipation that assume local conservation of energy by balancing a kinetic energy tendency locally by a thermodynamic tendency, as in (1), are incorrect according to (57c) and may lead to spurious sources/sinks of energy and a violation of the second law. [The local conservation formulation (1) can also be seen to be in error according to (46) because it neglects the second term on the right-hand side of the second equality, which involves a nonlocal transfer by the turbulent microscale flow.] Ensuring that particular parameterizations satisfy the entropy production constraint in the vertical column, (49) and (55), is the subject of future investigation.

While the motivation of the framework was the parameterization of subgrid-scale processes in climate models, numerical weather prediction models, which are generally of higher spatial resolution, must also parameterize the transfer of energy and momentum between the resolved and subgrid scales. Thus, the relationships between the fluxes derived here are also relevant for such models.

In the context of gravity wave propagation, Lindzen (1973) showed that the thermodynamic energy tendency due to gravity wave dissipation has the form (57b); however, a constant background wind was assumed in the derivation. In the context of gravity wave drag parameterization, the Lindzen (1981) parameterization does not represent the thermodynamic energy tendency according to (57b), although Becker and Schmitz (2002) extended Lindzen's parameterization to include the correct tendency [see their Eq. (15)]. In the Hines (1997) parameterization, an analogous thermodynamic energy tendency is used; however, it is scaled by a fudge factor Φ_5 with a suggested range of $1 \leq \Phi_5 \leq 3$. According to the current analysis, Φ_5 must equal unity to ensure consistency between energy and momentum conservation. In all the above studies, the thermodynamic tendency is only applied in the dissipation (wave breaking) region; the present framework implies that it should be also applied in the generation region (and be associated with the wave sources).

Under the time scale separation assumption of our framework, which is that typically made in the parameterization of subgrid scales in climate models, the mesoscale is forced to be statistically stationary on the planetary scale. This leads to the wave-activity conservation laws being diagnostic relations [e.g., (41) and (51)]. Some climate models include turbulent kinetic energy equations that are prognostic. To include prognostic effects on the mesoscale, the theory would need to incorporate an intermittency factor to preserve the scalings on the planetary scale.

A weakness of our framework is that the mesoscale dynamics are assumed to be quasi-linear, in the sense that they are described by the small-amplitude form of the wave-activity conservation laws. This would seem to limit the applicability of the framework (e.g., convection and boundary layer turbulence are clearly nonlinear processes). The introduction of an intermittency factor should also make it possible to allow a nonlinear mesoscale without disturbing the balances on the planetary scale, and the Hamiltonian framework ensures that the mesoscale wave-activity conservation laws can be extended to finite amplitude. [Extension to moist dynamics should also be possible; see, e.g., Bannon (2003).] Unfortunately, in this case the direct connection between the vertical wave-activity fluxes and the planetary-scale equations is lost because there is an additional term in the vertical fluxes involving advection of the wave-activity density (Scinocca and Shepherd 1992). (This problem also arises for the classical theory of wave–mean-flow interaction involving the Eliassen–Palm wave activity; see Andrews et al. 1987.) Since the planetary-scale equations remain the same in the case of a nonlinear mesoscale, our framework still provides a general understanding of the transfers of energy and momentum between the resolved and subgrid scales in this case. In particular, every pseudomomentum flux must be accompanied by a pseudoenergy flux. Thus, the parameterizations of convective momentum transport of Schneider and Lindzen (1976) and Gregory et al. (1997), which neglect the energetics, are in error according to (57a). Similarly, parameterizations of the turbulent dissipation of resolved-scale kinetic energy in the boundary layer that neglect the associated dissipative heating are also in error. Computing the actual importance of these errors in realistic applications is the subject of future investigation.

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APPENDIX A

Compressible Wave-Activity Conservation Laws

The wave-activity conservation laws used in section 4 can be derived following the Hamiltonian formulation of the three-dimensional, compressible, nonhydrostatic equations found in section 4.5 of Shepherd (1990). The Hamiltonian, horizontal momentum and Casimir invariants of the compressible nonhydrostatic equations, according to (4.61), (4.67), and (4.69) of Shepherd (1990) and using the present nondimensionalization, are

$$\mathscr{H} = \int \left(\frac{\rho}{2}|\mathbf{v}|^2 + \frac{1}{\mathrm{Fr}^2}\rho z + \frac{1}{M^2}\frac{1}{\kappa\gamma}\rho T \right) dV, \tag{A1a}$$

$$\mathscr{M} = \int \rho \mathbf{u}\, dV, \tag{A1b}$$

$$\mathscr{C} = \int \rho C(\theta, q)\, dV, \tag{A1c}$$

where $\kappa\gamma = R/c_v$, with c_v being the specific heat at constant volume, $q = \boldsymbol{\omega} \cdot \nabla\theta/\rho$ is the potential vorticity with $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and $dV = dx\, dy\, dz$. (Recall that rotation is negligible on the mesoscale by assumption.) The third term in the Hamiltonian is recognized as the internal energy of the system. Using these functionals, we proceed to derive the relevant wave-activity conservation laws according to the procedure outlined in section 5 of Shepherd (1990). According to the ansatz (11), we consider the planetary scale as a horizontally and temporally homogeneous but vertically varying background flow for the mesoscale dynamics. Accordingly, the x_m, y_m, and t_m symmetries in the planetary-scale flow lead to pseudomomentum and pseudoenergy conservation laws on the mesoscale.

We begin the derivation by taking the first variation of the above functionals. In the case of the Hamiltonian and the Casimir invariants, we obtain

$$\begin{aligned} \delta\mathscr{H} &= \int \left[\rho\delta\mathbf{v} + \frac{1}{2}|\mathbf{v}|^2\delta\rho + \frac{1}{\mathrm{Fr}^2}z\delta\rho + \frac{1}{M^2}\frac{1}{\kappa\gamma}(T\delta\rho + \rho\delta T) \right] dV \\ &= \int \left\{ \rho\delta\mathbf{v} + \left(\frac{1}{2}|\mathbf{v}|^2 + \frac{1}{\mathrm{Fr}^2}z \right)\delta\rho + \frac{1}{M^2}\frac{1}{\kappa\gamma}\left[T\delta\rho + \rho\left(\gamma\frac{T}{\theta}\delta\theta + \kappa\gamma\frac{T}{\rho}\delta\rho \right) \right] \right\} dV \\ &= \int \left[\rho\delta\mathbf{v} + \left(\frac{1}{2}|\mathbf{v}|^2 + \frac{1}{\mathrm{Fr}^2}z + \frac{1}{M^2}\frac{1}{\kappa}\pi\theta \right)\delta\rho + \frac{1}{M^2}\frac{1}{\kappa}\rho\pi\delta\theta \right] dV, \end{aligned} \tag{A2}$$

$$\begin{aligned} \delta\mathscr{C} &= \int (C\delta\rho + \rho C_\theta\delta\theta + \rho C_q\delta q)\, dV = \int [C\delta\rho + \rho C_\theta\delta\theta + C_q(\delta\boldsymbol{\omega} \cdot \nabla\theta + \boldsymbol{\omega} \cdot \nabla\delta\theta - q\delta\rho)]\, dV \\ &= \int [\nabla \times (C_q\nabla\theta) \cdot \delta\mathbf{v} + (C - qC_q)\delta\rho + (\rho C_\theta - \nabla C_q \cdot \boldsymbol{\omega})\delta\theta]\, dV, \end{aligned} \tag{A3}$$

where $\delta(*)$ represents the usual functional derivative and (20) has been used to derive the first variation of the Hamiltonian. It can be easily verified that

$$\frac{\delta \mathscr{H}}{\delta \mathbf{v}} = \rho \mathbf{v}, \quad \frac{\delta \mathscr{H}}{\delta \theta} = \frac{1}{M^2}\frac{1}{\kappa}\rho\pi,$$
$$\frac{\delta \mathscr{H}}{\delta \rho} = \frac{1}{2}|\mathbf{v}|^2 + \frac{1}{\mathrm{Fr}^2}z + \frac{1}{M^2}\frac{1}{\kappa}\pi\theta, \tag{A4a}$$

$$\frac{\delta \mathscr{M}}{\delta \mathbf{u}} = \rho, \quad \frac{\delta \mathscr{M}}{\delta \boldsymbol{\omega}} = 0, \quad \frac{\delta \mathscr{M}}{\delta \theta} = 0, \quad \frac{\delta \mathscr{M}}{\delta \rho} = \mathbf{u}, \tag{A4b}$$

$$\frac{\delta \mathscr{C}}{\delta \mathbf{v}} = \boldsymbol{\nabla} \times (C_q \boldsymbol{\nabla}\theta), \quad \frac{\delta \mathscr{C}}{\delta \theta} = \rho C_\theta - \boldsymbol{\nabla} C_q \cdot \boldsymbol{\omega},$$
$$\frac{\delta \mathscr{C}}{\delta \rho} = C - qC_q. \tag{A4c}$$

The pseudoenergy functional is defined by

$$\mathscr{A}^{\mathcal{E}} = \mathscr{H}(\boldsymbol{\xi}) + \mathscr{C}^{\mathcal{E}}(\boldsymbol{\xi}) - \mathscr{H}(\mathbf{X}) - \mathscr{C}^{\mathcal{E}}(\mathbf{X}), \tag{A5}$$

where $\boldsymbol{\xi}$ is the state vector (a vector of the dependent variables) and $\mathbf{X}$ is the vertically dependent planetary-scale background state, subject to the condition that the first variation of $\mathscr{A}^{\mathcal{E}}$ vanish when $\boldsymbol{\xi} = \mathbf{X}$:

$$\left.\frac{\delta \mathscr{H}}{\delta \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\mathbf{X}} = -\left.\frac{\delta \mathscr{C}^{\mathcal{E}}}{\delta \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\mathbf{X}}. \tag{A6}$$

This extremal condition is used to define the pseudoenergy Casmir $\mathscr{C}^{\mathcal{E}}$. Using the first variations above, the extremal conditions for the pseudoenergy are

$$\rho_p \mathbf{u}_p = -\boldsymbol{\nabla} \times (C_q^{\mathcal{E}} \boldsymbol{\nabla}\theta_p), \tag{A7a}$$

$$\rho_p C_\theta^{\mathcal{E}} = -\frac{1}{M^2}\frac{1}{\kappa}\rho_p \pi_p + \boldsymbol{\nabla} C_q^{\mathcal{E}} \cdot (\boldsymbol{\nabla} \times \mathbf{u}_p), \tag{A7b}$$

$$C^{\mathcal{E}} - q_p C_q^{\mathcal{E}} = -\frac{1}{2}|\mathbf{u}_p|^2 - \frac{1}{\mathrm{Fr}^2}z - \frac{1}{M^2}\frac{1}{\kappa}\pi_p\theta_p. \tag{A7c}$$

Given that only the first two conditions are required to define the Casimirs, we can appeal to the results of SS08, who solved for Casimirs satisfying the first two conditions in the context of their derivation of wave-activity conservation laws for the anelastic equations. In their derivation for the conservative equations, they noted that for a strictly vertically varying background state there is no background potential vorticity, and the wave-activity conservation laws must be considered in the limit of vanishing perturbation potential vorticity. (Indeed, q_p is a higher-order term because it involves gradients on the long horizontal planetary scale and because Coriolis forces are higher-order terms.) In the more general case, we need to consider finite perturbation potential vorticity. In the general case the quadratic form of the pseudoenergy density is

$$A^{\mathcal{E}} = \frac{1}{2}\bigg[\rho|\delta\mathbf{v}|^2 + 2\mathbf{u}\cdot\delta\mathbf{u}\delta\rho + (\rho C_{\theta\theta}^{\mathcal{E}} - \boldsymbol{\nabla} C_{\theta q}^{\mathcal{E}}\cdot\boldsymbol{\omega})(\delta\theta)^2 - 2C_{qq}^{\mathcal{E}}\boldsymbol{\nabla} q\cdot\delta\boldsymbol{\omega}\delta\theta + \rho C_{qq}^{\mathcal{E}}(\delta q)^2$$
$$+ 2\left(C_\theta^{\mathcal{E}} + \frac{1}{M^2}\frac{1}{\kappa}\pi - qC_{\theta q}^{\mathcal{E}}\right)\delta\rho\delta\theta + \frac{1}{M^2}\frac{\rho\theta}{\kappa^2\gamma\pi}(\delta\pi)^2\bigg], \tag{A8}$$

where $\delta\pi/\kappa\gamma\pi = \delta\theta/\theta + \delta\rho/\rho$ according to (23). Upon making use of the Casimir coefficients given in SS08 and the ansatz (11), and equating the Mach and Froude numbers as done in section 3, the pseudoenergy density becomes

$$A^{\mathcal{E}} = \frac{\rho_p}{2}|\mathbf{v}_m|^2 + \mathrm{M}^2\mathbf{u}_p\cdot(\mathbf{u}_m\rho_m) + \mathrm{M}^4\frac{\rho_p}{2}\left[\frac{1}{M^2}\frac{2}{(\theta_p^2)_z} - \frac{\rho_p}{(\theta_{p_z})^2}\mathbf{u}_p\cdot\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z\right](\theta_m)^2 - M^2\frac{\rho_p}{\theta_{p_z}}\mathbf{u}_p\cdot\boldsymbol{\omega}_m^\perp\theta_m$$
$$- M^4\frac{\rho_p^2 W}{(\theta_{p_z})^2}\frac{q_{p_z}}{q_{p_n}}\theta_m q_m + M^4\frac{\rho_p^2 W}{\theta_{p_z}q_{p_n}}(q_m)^2 - M^4\frac{1}{2}\left[\frac{(\mathbf{u}_p^2)_z}{\theta_{p_z}}\rho_m\theta_m - \frac{1}{M^2}\frac{\rho_p\theta_p}{\kappa^2\gamma\pi_p}\pi_m^2\right], \tag{A9}$$

where $\boldsymbol{\omega}_m^\perp = -\omega_{m_{(y)}}\hat{\mathbf{x}} + \omega_{m_{(x)}}\hat{\mathbf{y}}, q_m = \theta_{p_z}\omega_{m_{(z)}} + \mathbf{u}_{p_z}^\perp\cdot\boldsymbol{\nabla}_m^H\theta_m$, $W = \sqrt{(u_p)^2 + (v_p)^2}$, and q_{p_n} is the background potential vorticity gradient in the normal direction defined by $\hat{\mathbf{n}} = \sin\alpha\hat{\mathbf{x}} - \cos\alpha\hat{\mathbf{y}}$, where α is the angle between the background (planetary-scale) velocity and the $\hat{\mathbf{x}}$ direction. The effects of compressibility are associated with the second, sixth, and seventh terms. The pseudoenergy wave-activity conservation law, calculated by taking a time derivative of the pseudoenergy density and making use of the mesoscale dynamics (17), (18), and (22), is

$$
\begin{aligned}
\frac{\partial A^{\mathcal{E}}}{\partial t_m} = &-\frac{\partial}{\partial x_m}\left(u_p A^{\mathcal{E}} - \frac{\rho_p}{2}u_p|\mathbf{v}_m|^2 + \rho_p\mathbf{u}_p\cdot\mathbf{u}_m u_m - \frac{\rho_p}{\theta_{p_z}}\upsilon_{p_z}\mathbf{u}_p\cdot\mathbf{u}_m^{\perp}\theta_m + \frac{\rho_p\theta_p}{\kappa}\pi_m u_m\right)\\
&-\frac{\partial}{\partial x_m}\left\{M^2\frac{u_p}{2}\left[\frac{2\rho_p}{(\theta_p^2)_z}\theta_m^2 + \frac{\rho_p\theta_p}{\kappa^2\gamma\pi_p}\pi_m^2\right]\right\}\\
&-\frac{\partial}{\partial y_m}\left(\upsilon_p A^{\mathcal{E}} - \frac{\rho_p}{2}\upsilon_p|\mathbf{v}_m|^2 + \rho_p\mathbf{u}_p\cdot\mathbf{u}_m v_m - \frac{\rho_p}{\theta_{p_z}}u_{p_z}\mathbf{u}_p^{\perp}\cdot\mathbf{u}_m\theta_m + \frac{\rho_p\theta_p}{\kappa}\pi_m\upsilon_m\right)\\
&-\frac{\partial}{\partial y_m}\left\{M^2\frac{\upsilon_p}{2}\left[\frac{2\rho_p}{(\theta_p^2)_z}\theta_m^2 + \frac{\rho_p\theta_p}{\kappa^2\gamma\pi_p}\pi_m^2\right]\right\} - \frac{\partial}{\partial z}\left(\rho_p\mathbf{u}_p\cdot\mathbf{u}_m w_m + \frac{\rho_p\theta_p}{\kappa}\pi_m w_m\right) + D^{\mathcal{E}},
\end{aligned}
\tag{A10}
$$

where the pseudoenergy source–sink term is

$$
\begin{aligned}
\overline{D^{\mathcal{E}}} = &\overline{\mathbf{v}_m\cdot\mathbf{S}_{\mathbf{v}}^m} + M^2\frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S_\theta^m} + M^2\frac{1}{\pi_p}\overline{\pi_m S_\theta^m} + \mathbf{u}_p\cdot\Bigg[M^2\frac{1}{\rho_p}\overline{\rho_m\mathbf{S}_{\mathbf{u}}^m}\\
&- M^4\frac{\kappa}{\pi_p}\frac{\rho_p}{(\theta_{p_z})^2}\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z\overline{\theta_m S_\theta^m} - M^4\frac{\kappa}{\rho_p\pi_p}\frac{\mathbf{u}_{p_z}}{\theta_{p_z}}\overline{\rho_m S_\theta^m} - M^2\frac{\kappa}{\pi_p\theta_{p_z}}\overline{\boldsymbol{\omega}_m^{\perp}S_\theta^m}\\
&+ M^2\frac{\rho_p}{\theta_{p_z}}\overline{\theta_m\left[\nabla_m\times\left(\frac{\mathbf{S}_{\mathbf{v}}^m}{\rho_p}\right)\right]^{\perp}} + M^2\frac{\rho_p^2}{\theta_{p_z}}\overline{\mathbf{u}_m^{\perp}q_m} + M^4\frac{\rho_p^2}{(\theta_{p_z})^2}\mathbf{u}_p\alpha_z\overline{\theta_m q_m}\Bigg].
\end{aligned}
\tag{A11}
$$

The term in (A11) involving the mesoscale potential vorticity is associated with the mesoscale source/sink terms because the mesoscale potential vorticity itself is only generated by source/sink terms:

$$
\begin{aligned}
\frac{\partial q_m}{\partial t_m} + \nabla_m^H\cdot(\mathbf{u}_p q_m) = &\,\theta_{p_z}(\nabla_m^H)^{\perp}\cdot\mathbf{S}_{\mathbf{u}}^m\\
&+ M^2\kappa\mathbf{u}_{p_z}^{\perp}\cdot\nabla_m^H S_\theta^m/\rho_p\theta_p.
\end{aligned}
\tag{A12}
$$

Because the background potential-vorticity gradient is higher order, there is no generation of q_m through the interaction with the background. In regions where the source–sink terms are zero, as is the case between the source and sink regions of the nonlocal transfer of energy and momentum, the mesoscale potential vorticity is zero. Thus, all terms in (A11) can be related to mesoscale source–sink terms.

In the case of the pseudomomentum, we define streamwise and normal pseudomomentum components. The streamwise pseudomomentum functional is defined by

$$
\mathscr{A}^{\mathcal{P}_s} = \mathscr{M}_s(\boldsymbol{\xi}) + \mathscr{C}^{\mathcal{P}_s}(\boldsymbol{\xi}) - \mathscr{M}_s(\mathbf{X}) - \mathscr{C}^{\mathcal{P}_s}(\mathbf{X}),
\tag{A13}
$$

where the Casimir $\mathscr{C}^{\mathcal{P}_s}$ must satisfy

$$
\left.\frac{\delta\mathscr{M}_s}{\delta\boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\mathbf{X}} = -\left.\frac{\delta\mathscr{C}^{\mathcal{P}_s}}{\delta\boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\mathbf{X}}
\tag{A14}
$$

to ensure the background state is an extremal. The normal component of the pseudomomentum is analogously defined. The extremal conditions for the streamwise pseudomomentum are

$$
\rho_p\hat{\mathbf{s}} = -\nabla\times(C_q^{\mathcal{P}_s}\nabla\theta_p),
\tag{A15a}
$$

$$
\rho_p C_\theta^{\mathcal{P}_s} = \nabla C_q^{\mathcal{P}_s}\cdot(\nabla\times\mathbf{u}_p),
\tag{A15b}
$$

$$
C^{\mathcal{P}_s} - q_p C_q^{\mathcal{P}_s} = -\mathbf{u}_p\cdot\hat{\mathbf{s}},
\tag{A15c}
$$

where $\hat{\mathbf{s}} = \cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{y}}$. As for the pseudoenergy, the streamwise pseudomomentum Casimir is defined by the first two conditions and we can make use of the results of SS08. The general quadratic form of the streamwise pseudomomentum is

$$
\begin{aligned}
A^{\mathcal{P}_s} = \frac{1}{2}[&2\delta\mathbf{u}\delta\rho\cdot\hat{\mathbf{s}} + (\rho C_{\theta\theta}^{\mathcal{P}_s} - \nabla C_{\theta q}^{\mathcal{P}_s}\cdot\boldsymbol{\omega})(\delta\theta)^2\\
&- 2C_{qq}^{\mathcal{P}_s}\nabla q\cdot\delta\boldsymbol{\omega}\,\delta\theta + \rho C_{qq}^{\mathcal{P}_s}(\delta q)^2\\
&+ 2(C_\theta^{\mathcal{P}_s} - qC_{\theta q}^{\mathcal{P}_s})\,\delta\rho\,\delta\theta].
\end{aligned}
\tag{A16}
$$

Upon making use of the Casimir coefficients given in SS08 and the ansatz (11), the streamwise pseudomomentum becomes

$$A^{\mathcal{P}_s} = \hat{\mathbf{s}} \cdot \left[M^2 \mathbf{u}_m \rho_m - M^4 \frac{1}{2} \frac{(\rho_p)^2}{(\theta_{p_z})^2} \left(\frac{\mathbf{u}_{p_z}}{\rho_p} \right)_z (\theta_m)^2 - M^2 \frac{\rho_p}{\theta_{p_z}} \boldsymbol{\omega}_m^{\perp} \theta_m - M^4 \frac{\mathbf{u}_{p_z}}{\theta_{p_z}} \rho_m \theta_m \right] - M^4 \frac{\rho_p^2}{(\theta_{p_z})^2} \frac{q_{p_z}}{q_{p_n}} \theta_m q_m + M^4 \frac{\rho_p^2}{\theta_{p_z} q_{p_n}} (q_m)^2. \tag{A17}$$

The pseudoenergy and streamwise pseudomomentum densities satisfy the relation

$$A^{\mathcal{E}} = \frac{\rho_p}{2} \left[|\mathbf{v}_m|^2 + M^2 \frac{2}{(\theta_p^2)_z} (\theta_m)^2 + M^2 \frac{\theta_p}{\kappa^2 \gamma \pi_p} (\pi_m)^2 \right] + W A^{\mathcal{P}_s}. \tag{A18}$$

The extremal conditions and the general quadratic form of the normal component of the pseudomomentum are the same as those for the streamwise component with $\hat{\mathbf{s}}$ replaced by $\hat{\mathbf{n}}$, such that the density is

$$A^{\mathcal{P}_n} = \hat{\mathbf{n}} \cdot \left[M^2 \mathbf{u}_m \rho_m - M^4 \frac{1}{2} \frac{(\rho_p)^2}{(\theta_{p_z})^2} \left(\frac{\mathbf{u}_{p_z}}{\rho_p} \right)_z (\theta_m)^2 - M^2 \frac{\rho_p}{\theta_{p_z}} \boldsymbol{\omega}_m^{\perp} \theta_m - M^4 \frac{\mathbf{u}_{p_z}}{\theta_{p_z}} \rho_m \theta_m \right] + M^4 \frac{\rho_p^2}{(\theta_{p_z})^2} \frac{q_{p_z}}{q_{p_s}} \theta_m q_m - M^4 \frac{\rho_p^2}{\theta_{p_z} q_{p_s}} (q_m)^2, \tag{A19}$$

where q_{p_s} is the background potential-vorticity gradient in the streamwise direction. The streamwise pseudomomentum wave-activity conservation law, calculated by taking the time derivative of the streamwise pseudomomentum density and making use of the mesoscale dynamics, is

$$\begin{aligned} \frac{\partial A^{\mathcal{P}_s}}{\partial t_m} = &- \frac{\partial}{\partial x_m} \left[u_p A^{\mathcal{P}_s} - \cos\alpha \frac{\rho_p}{2} |\mathbf{v}_m|^2 + \rho_p \hat{\mathbf{s}} \cdot \mathbf{u}_m u_m - \frac{\rho_p}{\theta_{p_z}} v_{p_z} \hat{\mathbf{s}} \cdot \mathbf{u}_m^{\perp} \theta_m \right] \\ &- \frac{\partial}{\partial x_m} \left\{ M^2 \cos\alpha \frac{1}{2} \left[\frac{2\rho_p}{(\theta_p^2)_z} \theta_m^2 + \frac{\rho_p \theta_p}{\kappa^2 \gamma \pi_p} \pi_m^2 \right] \right\} \\ &- \frac{\partial}{\partial y_m} \left(v_p A^{\mathcal{P}_s} - \sin\alpha \frac{\rho_p}{2} |\mathbf{v}_m|^2 + \rho_p \hat{\mathbf{s}} \cdot \mathbf{u}_m v_m + \frac{\rho_p}{\theta_{p_z}} u_{p_z} \hat{\mathbf{n}} \cdot \mathbf{u}_m \theta_m \right) \\ &- \frac{\partial}{\partial y_m} \left\{ M^2 \sin\alpha \frac{1}{2} \left[\frac{2\rho_p}{(\theta_p^2)_z} \theta_m^2 + \frac{\rho_p \theta_p}{\kappa^2 \gamma \pi_p} \pi_m^2 \right] \right\} - \hat{\mathbf{s}} \cdot \frac{\partial}{\partial z} (\rho_p \mathbf{u}_m w_m) + D^{\mathcal{P}_s}. \end{aligned} \tag{A20}$$

The streamwise and normal pseudomomentum components can be combined to give the usual vertical fluxes of horizontal momentum:

$$\frac{\partial}{\partial t_m} (\cos\alpha A^{\mathcal{P}_s} + \sin\alpha A^{\mathcal{P}_n}) + \boldsymbol{\nabla}_m \cdot \mathbf{F}^{\mathcal{P}_x} = D^{\mathcal{P}_x}, \tag{A21a}$$

$$\frac{\partial}{\partial t_m} (\sin\alpha A^{\mathcal{P}_s} - \cos\alpha A^{\mathcal{P}_n}) + \boldsymbol{\nabla}_m \cdot \mathbf{F}^{\mathcal{P}_y} = D^{\mathcal{P}_y}, \tag{A21b}$$

with $\overline{\mathrm{F}_{(z)}^{\mathcal{P}_x}} \hat{\mathbf{x}} + \overline{\mathrm{F}_{(z)}^{\mathcal{P}_y}} \hat{\mathbf{y}} = \rho_p \overline{\mathbf{u}_m w_m}$ (see SS08). The pseudomomentum source–sink terms are

$$\begin{aligned} \overline{D^{\mathcal{P}_x}} \hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}} \hat{\mathbf{y}} = &M^2 \frac{1}{\rho_p} \overline{\rho_m \mathbf{S}_{\mathbf{u}}^m} - M^4 \frac{\kappa}{\pi_p} \frac{\rho_p}{(\theta_{p_z})^2} \left(\frac{\mathbf{u}_{p_z}}{\rho_p} \right)_z \overline{\theta_m S_\theta^m} - M^4 \frac{\kappa}{\rho_p \pi_p} \frac{\mathbf{u}_{p_z}}{\theta_{p_z}} \overline{\rho_m S_\theta^m} \\ &- M^2 \frac{\kappa}{\pi_p \theta_{p_z}} \overline{\boldsymbol{\omega}_m^{\perp} S_\theta^m} + M^2 \frac{\rho_p}{\theta_{p_z}} \overline{\theta_m \left[\boldsymbol{\nabla}_m \times \left(\frac{\mathbf{S}_{\mathbf{v}}^m}{\rho_p} \right) \right]^{\perp}} + M^2 \frac{\rho_p^2}{\theta_{p_z}} \overline{\mathbf{u}_m^{\perp} q_m} + M^4 \frac{\rho_p^2}{(\theta_{p_z})^2} \mathbf{u}_p \alpha_z \overline{\theta_m q_m}. \end{aligned} \tag{A22}$$

APPENDIX B

Anelastic Wave-Activity Conservation Laws

The anelastic pseudoenergy and pseudomomentum wave-activity densities are obtained by applying the anelastic limit described in section 5 to (A9), (A17), and (A19) to obtain

$$A^{\mathcal{E}} = \frac{\rho_p}{2}|\mathbf{v}_m|^2 + M^4\frac{\rho_p}{2}\left[\frac{1}{M^2}\frac{2}{(\theta_p^2)_z} - \frac{\rho_p}{(\theta_{p_z})^2}\mathbf{u}_p\cdot\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z\right](\theta_m)^2 - M^2\frac{\rho_p}{\theta_{p_z}}\mathbf{u}_p\cdot\boldsymbol{\omega}_m^{\perp}\theta_m$$
$$- M^4\frac{\rho_p^2 W}{(\theta_{p_z})^2}\frac{q_{p_z}}{q_{p_n}}\theta_m q_m + M^4\frac{\rho_p^2 W}{\theta_{p_z}q_{p_n}}(q_m)^2, \tag{B1a}$$

$$A^{\mathcal{P}_s} = \hat{\mathbf{s}}\cdot\left[-M^4\frac{1}{2}\frac{(\rho_p)^2}{(\theta_{p_z})^2}\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z(\theta_m)^2 - M^2\frac{\rho_p}{\theta_{p_z}}\boldsymbol{\omega}_m^{\perp}\theta_m\right] - M^4\frac{\rho_p^2}{(\theta_{p_z})^2}\frac{q_{p_z}}{q_{p_n}}\theta_m q_m + M^4\frac{\rho_p^2}{\theta_{p_z}q_{p_n}}(q_m)^2, \tag{B1b}$$

$$A^{\mathcal{P}_n} = \hat{\mathbf{n}}\cdot\left[-M^4\frac{1}{2}\frac{(\rho_p)^2}{(\theta_{p_z})^2}\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z(\theta_m)^2 - M^2\frac{\rho_p}{\theta_{p_z}}\boldsymbol{\omega}_m^{\perp}\theta_m\right] + M^4\frac{\rho_p^2}{(\theta_{p_z})^2}\frac{q_{p_z}}{q_{p_s}}\theta_m q_m - M^4\frac{\rho_p^2}{\theta_{p_z}q_{p_s}}(q_m)^2. \tag{B1c}$$

From (A9), (A17), and (A19), the only terms involving M that are retained are those containing M^2/θ_{p_z}, which is assumed to be $O(1)$ under the weak stratification assumption discussed in section 5. The other terms involving M are higher order under the anelastic approximation. In the anelastic case, the pseudoenergy and streamwise pseudomomentum densities satisfy the relation

$$A^{\mathcal{E}} = \frac{\rho_p}{2}\left[|\mathbf{v}_m|^2 + M^2\frac{2}{(\theta_p^2)_z}(\theta_m)^2\right] + WA^{\mathcal{P}_s}. \tag{B2}$$

In the limit of vanishing perturbation potential vorticity, the wave activity densities (B1) correspond exactly to (4.3)–(4.6) in SS08.

The anelastic pseudoenergy wave-activity conservation law, calculated by taking a time derivative of the pseudoenergy density and making use of the mesoscale dynamics (17), (18), and $\nabla\cdot(\rho_p\mathbf{v}_m) = 0$, is

$$\frac{\partial A^{\mathcal{E}}}{\partial t_m} = -\frac{\partial}{\partial x_m}\left[u_pA^{\mathcal{E}} - \frac{\rho_p}{2}u_p|\mathbf{v}_m|^2 + \rho_p\mathbf{u}_p\cdot\mathbf{u}_m u_m - \frac{\rho_p}{\theta_{p_z}}\upsilon_{p_z}\mathbf{u}_p\cdot\mathbf{u}_m^{\perp}\theta_m + \frac{\rho_p\theta_p}{\kappa}\pi_m u_m\right]$$
$$-\frac{\partial}{\partial x_m}\left[M^2u_p\frac{\rho_p}{(\theta_p^2)_z}\theta_m^2\right] - \frac{\partial}{\partial y_m}\left[\upsilon_pA^{\mathcal{E}} - \frac{\rho_p}{2}\upsilon_p|\mathbf{v}_m|^2 + \rho_p\mathbf{u}_p\cdot\mathbf{u}_m\upsilon_m - \frac{\rho_p}{\theta_{p_z}}u_{p_z}\mathbf{u}_p^{\perp}\cdot\mathbf{u}_m\theta_m + \frac{\rho_p\theta_p}{\kappa}\pi_m\upsilon_m\right]$$
$$-\frac{\partial}{\partial y_m}\left[M^2\upsilon_p\frac{\rho_p}{(\theta_p^2)_z}\theta_m^2\right] - \frac{\partial}{\partial z}\left[\rho_p\mathbf{u}_p\cdot\mathbf{u}_m w_m + \frac{\rho_p\theta_p}{\kappa}\pi_m w_m\right] + D^{\mathcal{E}}, \tag{B3}$$

with

$$\overline{D^{\mathcal{E}}} = \overline{\mathbf{v}_m\cdot\mathbf{S}_{\mathbf{v}}^m} + M^2\frac{\kappa}{\pi_p}\frac{2}{(\theta_p^2)_z}\overline{\theta_m S_\theta^m} + \mathbf{u}_p\cdot\left[-M^4\frac{\kappa}{\pi_p}\frac{\rho_p}{(\theta_{p_z})^2}\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z\overline{\theta_m S_\theta^m} - M^2\frac{\kappa}{\pi_p\theta_{p_z}}\overline{\boldsymbol{\omega}_m^{\perp}S_\theta^m}\right.$$
$$\left. + M^2\frac{\rho_p}{\theta_{p_z}}\overline{\theta_m\left[\nabla_m\times\left(\frac{\mathbf{S}_{\mathbf{v}}^m}{\rho_p}\right)\right]^{\perp}} + M^2\frac{\rho_p^2}{\theta_{p_z}}\overline{\mathbf{u}_m^{\perp}q_m} + M^4\frac{\rho_p^2}{(\theta_{p_z})^2}\mathbf{u}_p\alpha_z\overline{\theta_m q_m}\right]. \tag{B4}$$

The anelastic streamwise pseudomomentum wave-activity conservation law is

$$\frac{\partial A^{\mathcal{P}_s}}{\partial t_m} = -\frac{\partial}{\partial x_m}\left[u_p A^{\mathcal{P}_s} - \cos\alpha\frac{\rho_p}{2}|\mathbf{v}_m|^2 + \rho_p\hat{\mathbf{s}}\cdot\mathbf{u}_m u_m - \frac{\rho_p}{\theta_{p_z}}\upsilon_{p_z}\hat{\mathbf{s}}\cdot\mathbf{u}_m^{\perp}\theta_m\right]$$
$$-\frac{\partial}{\partial x_m}\left[M^2\cos\alpha\frac{\rho_p}{(\theta_p^2)_z}\theta_m^2\right] - \frac{\partial}{\partial y_m}\left[\upsilon_p A^{\mathcal{P}_s} - \sin\alpha\frac{\rho_p}{2}|\mathbf{v}_m|^2 + \rho_p\hat{\mathbf{s}}\cdot\mathbf{u}_m\upsilon_m + \frac{\rho_p}{\theta_{p_z}}u_{p_z}\hat{\mathbf{n}}\cdot\mathbf{u}_m\theta_m\right]$$
$$-\frac{\partial}{\partial y_m}\left[M^2\sin\alpha\frac{\rho_p}{(\theta_p^2)_z}\theta_m^2\right] - \hat{\mathbf{s}}\cdot\frac{\partial}{\partial z}[\rho_p\mathbf{u}_m w_m] + D^{\mathcal{P}_s}. \tag{B5}$$

As with the compressible wave activities, we combine the streamwise and normal pseudomomentum to obtain

$$\frac{\partial}{\partial t_m}(\cos\alpha A^{\mathcal{P}_s} + \sin\alpha A^{\mathcal{P}_n}) + \nabla_m\cdot\mathbf{F}^{\mathcal{P}_x} = D^{\mathcal{P}_x}, \tag{B6a}$$

$$\frac{\partial}{\partial t_m}(\sin\alpha A^{\mathcal{P}_s} - \cos\alpha A^{\mathcal{P}_n}) + \nabla_m\cdot\mathbf{F}^{\mathcal{P}_y} = D^{\mathcal{P}_y}, \tag{B6b}$$

with $\overline{\mathrm{F}_{(z)}^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{\mathrm{F}_{(z)}^{\mathcal{P}_y}}\hat{\mathbf{y}} = \rho_p\overline{\mathbf{u}_m w_m}$. In the anelastic case, the pseudomomentum source–sink terms become

$$\overline{D^{\mathcal{P}_x}}\hat{\mathbf{x}} + \overline{D^{\mathcal{P}_y}}\hat{\mathbf{y}} = -M^4\frac{\kappa}{\pi_p}\frac{\rho_p}{(\theta_{p_z})^2}\left(\frac{\mathbf{u}_{p_z}}{\rho_p}\right)_z\overline{\theta_m S_\theta^m} - M^2\frac{\kappa}{\pi_p\theta_{p_z}}\overline{\boldsymbol{\omega}_m^{\perp}S_\theta^m}$$
$$+ M^2\frac{\rho_p}{\theta_{p_z}}\overline{\theta_m\left[\nabla_m\times\left(\frac{\mathbf{S}_\mathbf{v}^m}{\rho_p}\right)\right]^{\perp}} + M^2\frac{\rho_p^2}{\theta_{p_z}}\overline{\mathbf{u}_m^{\perp}q_m} + M^4\frac{\rho_p^2}{(\theta_{p_z})^2}\mathbf{u}_p\alpha_z\overline{\theta_m q_m}. \tag{B7}$$

In effect, the anelastic constraint eliminates the exchange of elastic energy between the two scales as represented by those terms in (A9), (A17), (A19), (A10), and (A20) proportional to ρ_m and π_m.

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