CORRESPONDENCE

Comments on "A Spectral Parameterization of Drag, Eddy Diffusion, and Wave Heating for a Three-Dimensional Flow Induced by Breaking Gravity Waves"

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(Manuscript received 8 September 2010, in final form 11 May 2011)

1. Introduction

In a recent paper Zhu et al. (2010, hereafter Z10) developed a single-column parameterization of the effects of unresolved gravity wave breaking on the grid-resolved flow of a climate model. They discuss a "breaking trinity" involving (i) the nonlocal wave-induced transport of momentum; (ii) the local wave-induced diffusion of momentum, heat, and tracers; and (iii) the local wave-induced transport of heat. While the authors' formulation captures some of the important energy and momentum transfers between the grid-resolved flow and the gravity waves, their formulation is essentially incomplete in that it does not conserve energy. The inconsistencies are clear when the Z10 formulation is compared to that of Becker (2004, hereafter B04) and Shaw and Shepherd (2009, hereafter SS09), which do conserve energy. A comparison between the formulations is performed here in order to highlight the cause of the non-conservation in the Z10 formulation.

2. Conservation of energy and momentum

Any parameterization of the effects of breaking gravity waves on the grid-resolved flow should consider the implications for the conservation laws inherent in the governing equations. In particular, the parameterization must preserve the conservative properties of the primitive equations, which govern the grid-resolved flow of climate models. The relevant conservation laws are conservation of angular momentum and energy, as discussed by Lorenz (1967). Z10 did not show that their formulation conserves energy. The energy equation corresponding to the Z10 grid-resolved flow [see Eqs. (1)–(3) in Z10] can be easily derived by adding $\rho_0\overline{u}$ times their zonal momentum Eq. (1), $\rho_0\overline{v}$ times their meridional momentum Eq. (2), and $\rho_0 H/\kappa$ times their thermal energy Eq. (3) after replacing N^2 by Eq. (6) in Z10:

$$\frac{\partial}{\partial t}\left[\rho_0\left(\frac{\overline{\mathbf{u}}^2}{2}+\frac{H}{\kappa}\overline{\Phi_z}\right)\right]+\nabla^H\cdot\left[\rho_0\overline{\mathbf{u}}\left(\frac{\overline{\mathbf{u}}^2}{2}+\frac{H}{\kappa}\overline{\Phi_z}\right)\right]+\nabla\cdot(\rho_0\overline{\mathbf{v}}\overline{\Phi})$$
$$+\frac{\overline{\mathbf{u}}^2}{2}\frac{\partial}{\partial z}(\rho_0\overline{w})=-\overline{u}\nabla\cdot\mathbf{F}_u-\overline{v}\nabla\cdot\mathbf{F}_v-\nabla\cdot\left(\frac{H}{\kappa}\mathbf{F}_{\Phi_z}\right)$$
$$+\overline{\mathbf{u}}\cdot\frac{\partial}{\partial z}\left(\rho_0 k_{zz-m}\frac{\partial\overline{\mathbf{u}}}{\partial z}\right)+\frac{\partial}{\partial z}\left(\rho_0 k_{zz-T}\frac{H}{\kappa}\frac{\partial\overline{\Phi_z}}{\partial z}\right),\qquad(1)$$

where $\overline{\mathbf{v}}=(\overline{u},\overline{v},\overline{w})$ and $\overline{\mathbf{u}}=(\overline{u},\overline{v})$ are the three- and two-dimensional wind vectors, respectively; $\nabla=(\partial/\partial x,\partial/\partial y,\partial/\partial z)$ and $\nabla^H=(\partial/\partial x,\partial/\partial y)$ are the three- and two-dimensional gradient operators, respectively; $H\overline{\Phi_z}/\kappa=c_p\overline{T}$ is the enthalpy of the grid-resolved flow; and $\mathbf{F}_u$, $\mathbf{F}_v$, and $H\mathbf{F}_{\Phi_z}/\kappa$ denote the three-dimensional subgrid-scale gravity wave fluxes of zonal momentum, meridional momentum, and enthalpy, respectively. Finally, k_{zz-m} and $\rho_0 k_{zz-T}$ are the eddy diffusion coefficients for momentum and heat, which are related to those in Z10 according to $K_{zz-m}=\rho_0 k_{zz-m}$ and $K_{zz-T}=\rho_0 k_{zz-T}$. When

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DOI: 10.1175/2011JAS3663.1

integrated over a control volume, the energy Eq. (1) does not reduce to boundary terms, which is required for energy conservation. The nonconservation of energy is clear because the right-hand side of Eq. (1) cannot be written in flux-divergent form. The first four terms on the right-hand side of Eq. (1) represent conversions between the grid-resolved and subgrid-scale energies, which are incomplete and violate energy conservation. Note that the left-hand side of Eq. (1) is also not in flux-divergent form because the horizontal momentum equations in Z10 do not include vertical advection.[1]

Z10 include a short discussion of energy conservation. In particular, in section 2a (page 2523, first paragraph) they note that parameterizations of the effects of subgrid-scale gravity waves have been developed using an energy equation and cite B04. However, Z10 fail to acknowledge that in order to conserve energy, the conversion terms on the right-hand side of Eq. (1) must appear in flux-divergent form when deriving the grid-resolved energy equation from the equations of motion. Instead, Z10 argue that because their parameterized subgrid-scale gravity wave momentum and thermal energy tendencies are in flux-divergent form, their formulation is conservative (see page 2523, first paragraph, lines 1–10 in Z10). The right-hand side of the momentum equation in Z10 is in flux-divergent form and thus conservation of angular momentum is satisfied. However, the flux-divergent form of the thermal energy equation is irrelevant because conservation of thermal energy is not a physical principle.[2] Z10 state that the kinetic and available potential energy are the relevant quantities to consider (see their page 2523) although they do not derive budgets for these quantities. Note that the sum of available potential energy and kinetic energy is not subject to a conservation law because the complete energy cycle includes the generation of unavailable potential energy due to the turbulent dissipation of kinetic energy (Lorenz 1967, chapter 5).

The cause of the nonconservation of grid-resolved energy in the Z10 formulation can be determined by comparing it to previously derived parameterizations for the energetic effects of subgrid-scale gravity wave breaking on the grid-resolved flow that are conservative, such as those of B04 and SS09. In B04 and SS09 the grid-resolved energy equation was derived and energy was shown to be conserved. While Z10 cite both B04 and SS09, they fail to acknowledge the inconsistencies between their formulation and those of B04 and SS09. Here we will review B04 and SS09 in order to highlight the inconsistencies in Z10.

The grid-resolved energy equation in B04 is

$$\frac{\partial}{\partial t}\left[\rho_0\left(\frac{\overline{\mathbf{u}}^2}{2}+\frac{H}{\kappa}\overline{\Phi_z}\right)\right]+\nabla\cdot\left[\rho_0\overline{\mathbf{v}}\left(\frac{\overline{\mathbf{u}}^2}{2}+\frac{H}{\kappa}\overline{\Phi_z}\right)\right] + \nabla\cdot(\rho_0\overline{\mathbf{v}}\overline{\Phi}) = -\frac{\partial}{\partial z}(\mathbf{F}\cdot\overline{\mathbf{u}}) - \frac{\partial \mathbf{F}_\Phi^{(z)}}{\partial z} - \frac{\partial}{\partial z}\left(\frac{H}{\kappa}F_{\Phi_z}^{(z)}\right) + \frac{\partial}{\partial z}\left(\rho_0 k_{zz-T}\frac{H}{\kappa}\frac{\partial\overline{\Phi_z}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho_0 k_{zz-m}\overline{\mathbf{u}}\cdot\frac{\partial\overline{\mathbf{u}}_z}{\partial z}\right) + \rho_0 c_p Q, \quad (2)$$

where $\mathbf{F}$ is the gravity wave vertical flux of horizontal momentum, $\mathbf{F}_\Phi^{(z)} = \rho_0\overline{w'\Phi'}$ is the gravity wave vertical flux of potential energy, and $F_{\Phi_z}^{(z)} = \rho_0\overline{w'\Phi_z'}$ is the vertical component of $\mathbf{F}_{\Phi_z}$. The external diabatic heating rates due to radiation, latent heating, etc. are combined into Q. The B04 energy equation [see Eq. (20) in B04] can be derived by adding $\rho\overline{\mathbf{u}}$ times Eq. (9), $\overline{w}$ times Eq. (10), $\overline{\mathbf{u}}^2$ times Eq. (11), and ρ times Eq. (12) in B04. Note, however, that the energy equation in B04 was derived in geometric height coordinates and was here transformed to Eq. (2), which is in log-pressure coordinates, so that it could be directly compared to the energy Eq. (1) from Z10. Details of the transformation can be found in the appendix. Note also that a horizontal average was applied to Eq. (2), which is consistent with the horizontal scale separation assumption applied in all current gravity wave drag parameterizations. Z10 apply a horizontal average later [see Eq. (16) in Z10]. An energy conservation law identical to Eq. (2) can be derived from SS09 if we combine Eqs. (38) and (46) in SS09, recalling that S_θ^p includes not only the turbulent dissipation of kinetic energy but also diffusive mixing, and apply the log-pressure coordinate transformation.

The Z10 grid-resolved energy Eq. (1) is clearly different from Eq. (2). The cause of the difference is not the result of differences in the momentum equation. The momentum equation in Z10 [see Eq. (1) in Z10] is identical to the momentum equations in B04 [see Eq. (9) in B04] and SS09 [see Eq. (26) in SS09] after a horizontal average is applied to the subgrid-scale terms on the right-hand side. Since the momentum equations are identical, the kinetic energy equations are also identical. The cause of the difference between the energy Eqs. (1) and (2) is differences in the respective thermal energy equations. The thermal energy equation in Z10 [see Eq. (3) in Z10] is

$$\frac{\partial}{\partial t}\left(\rho_0\frac{H}{\kappa}\frac{\partial\overline{\Phi}}{\partial z}\right)+\nabla\cdot\left(\rho_0\frac{H}{\kappa}\frac{\partial\overline{\Phi}}{\partial z}\overline{\mathbf{v}}\right)+\rho_0\overline{w}\frac{\partial\overline{\Phi}}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{H}{\kappa}F_{\Phi_z}^{(z)}\right)+\frac{\partial}{\partial z}\left(\rho_0 k_{zz-T}\frac{H}{\kappa}\frac{\partial\overline{\Phi_z}}{\partial z}\right) \quad (3)$$

[1] We suspect that the neglect of the vertical advection in Eq. (5) and the missing factor of ρ_0^{-1} in front of the diffusion terms in Eqs. (1)–(3) of Z10 are typographic errors and instead choose to focus on the nonconservation resulting from the subgrid-scale terms.

[2] The equation for internal energy can only be written in conservative form for an isentropic incompressible fluid. Likewise, enthalpy is only conserved for an isentropic isobaric flow. None of these idealizations apply to the primitive equations.

after replacing N^2 by Eq. (6) in Z10. The thermal energy equation in B04 [see Eq. (12) in B04] in log-pressure coordinates is

$$\frac{\partial}{\partial t}\left(\rho_0\frac{H}{\kappa}\frac{\partial\overline{\Phi}}{\partial z}\right)+\nabla\cdot\left(\rho_0\frac{H}{\kappa}\frac{\partial\overline{\Phi}}{\partial z}\overline{\mathbf{v}}\right)+\rho_0\overline{w}\frac{\partial\overline{\Phi}}{\partial z}=-\frac{\partial}{\partial z}\left(\frac{H}{\kappa}F^{(z)}_{\Phi_z}\right)+\frac{\partial}{\partial z}\left(\rho_0 k_{zz-T}\frac{H}{\kappa}\frac{\partial\overline{\Phi}_z}{\partial z}\right)-\rho_0\frac{R}{H}\overline{T'w'}+\rho_0 k_{zz-m}\overline{\left(\frac{\partial\mathbf{u}'}{\partial z}\right)^2}+\rho_0 k_{zz-m}\left(\frac{\partial\overline{\mathbf{u}}}{\partial z}\right)^2+\rho_0 c_p Q. \quad (4)$$

The third and fourth terms on the right-hand side of Eq. (4) represent buoyancy production and the turbulent dissipation of gravity wave kinetic energy, respectively. The fifth term on the right-hand side represents the turbulent dissipation of grid-resolved kinetic energy. A thermal energy equation identical to Eq. (4) can be derived from SS09 if we combine Eqs. (32) and (34) in SS09.

The Z10 and B04 thermal energy equations agree with regard to the terms on the left-hand side and the first two terms on the right-hand side. However, the Z10 thermal energy equation only has two subgrid-scale gravity wave terms on the right-hand side whereas the B04 equation has three additional terms (two subgrid-scale terms and one grid-resolved term). The additional subgrid-scale gravity wave terms in the B04 thermal energy equation also appear in the gravity wave kinetic energy equation; that is,

$$-\rho_0\frac{R}{H}\overline{T'w'}+\rho_0 k_{zz-m}\overline{\left(\frac{\partial\mathbf{u}'}{\partial z}\right)^2}=-\frac{\partial\mathbf{F}^{(z)}_{\Phi}}{\partial z}-\mathbf{F}\cdot\frac{\partial\overline{\mathbf{u}}}{\partial z} \quad (5)$$

[see Eq. (16) in B04]. Note that Eq. (5) assumes statistical stationarity, which is an assumption made by all current gravity wave parameterizations. The turbulent dissipation terms in Eqs. (4) and (5) represent the shear production of turbulent kinetic energy as determined by a diffusive closure. Assuming small-scale Kolmogorov turbulence, this shear production equals the molecular frictional heating due to the turbulent energy cascade. It is well known that this heating is substantial in the upper mesosphere (Lübken 1997).

The Z10 parameterization fails to conserve grid-resolved energy because it does not account for the following terms on the right-hand side of the thermal energy Eq. (3):

$$-\rho_0\frac{R}{H}\overline{T'w'}+\rho_0 k_{zz-m}\overline{\left(\frac{\partial\mathbf{u}'}{\partial z}\right)^2}+\rho_0 k_{zz-m}\left(\frac{\partial\overline{\mathbf{u}}}{\partial z}\right)^2=-\frac{\partial\mathbf{F}^{(z)}_{\Phi}}{\partial z}-\mathbf{F}\cdot\frac{\partial\overline{\mathbf{u}}}{\partial z}+\rho_0 k_{zz-m}\left(\frac{\partial\overline{\mathbf{u}}}{\partial z}\right)^2, \quad (6)$$

where the equality follows from Eq. (5). The terms on the right-hand side of Eq. (6) represent the conversion between grid-resolved energy and gravity wave kinetic energy, including the turbulent dissipation of kinetic energy. When Eqs. (4) and (6) are combined, the thermal energy equation becomes

$$\frac{\partial}{\partial t}\left(\rho_0\frac{H}{\kappa}\frac{\partial\overline{\Phi}}{\partial z}\right)+\nabla\cdot\left(\rho_0\frac{H}{\kappa}\frac{\partial\overline{\Phi}}{\partial z}\overline{\mathbf{v}}\right)+\rho_0\overline{w}\frac{\partial\overline{\Phi}}{\partial z}=-\frac{\partial}{\partial z}\left(\frac{H}{\kappa}F^{(z)}_{\Phi_z}\right)+\frac{\partial}{\partial z}\left(\rho_0 k_{zz-T}\frac{H}{\kappa}\frac{\partial\overline{\Phi}_z}{\partial z}\right)-\frac{\partial\mathbf{F}^{(z)}_{\Phi}}{\partial z}-\mathbf{F}\cdot\frac{\partial\overline{\mathbf{u}}}{\partial z}+\rho_0 k_{zz-m}\left(\frac{\partial\overline{\mathbf{u}}}{\partial z}\right)^2+\rho_0 c_p Q. \quad (7)$$

If the Z10 energy equation is subsequently recalculated using Eq. (7) instead of Eq. (3), the third, fourth, and fifth terms on the right-hand side of Eq. (7) then appear on the right-hand side of the Z10 energy Eq. (1) and would combine with the first, second, and fourth terms on the right-hand side of Eq. (1) to produce the same flux divergent form as in Eq. (2), which satisfies energy conservation.

B04 quantified all of the parameterized gravity wave terms in the thermal energy Eq. (4) and the gravity wave kinetic energy Eq. (5) using a modified version of the Lindzen (1981) gravity wave drag parameterization. In B04 the energy deposition [first two terms on the right-hand side of Eq. (6)] is denoted by the symbol E while the turbulent dissipation of grid-resolved kinetic energy [last term on the right-hand side of Eq. (6)] is labeled ϵ_m. Figures 2a and 2b of B04 show that $E + \epsilon_m$ is on the order of 10 K day^{-1} in the mesosphere during summer, which is in agreement with observational estimates (Lübken 1997). The sum of the terms on the right-hand side of the Z10 thermal energy Eq. (3), labeled μ by B04, is of a similar magnitude (see Fig. 2c in B04). Thus, quite apart from issues of energetic consistency, $E + \epsilon_m$ is nonnegligible and hence the terms in Eq. (6) must be included in the grid-resolved thermal energy Eq. (3) for an accurate treatment of the energy budget. We also note that several studies have shown that the global energy budget of a climate model can be significantly biased if dissipative heating is not treated properly (Boville and Bretherton 2003; Becker 2003; Burkhardt and Becker 2006).

3. Conclusions

Parameterizations included in a climate model to represent the effects of subgrid-scale processes should respect the conservation laws inherent in the governing equations. In the case of climate models the governing equations are the primitive equations. The recent parameterization by Z10, which claims to account for the effects of subgrid-scale gravity wave breaking on the grid-resolved flow, does not conserve energy. The nonconservation results from missing terms in the formulation of the thermal effects of gravity wave breaking. The missing terms were identified after the Z10 formulation was compared to formulations by B04 and SS09, which preserve energy conservation.

If the Z10 parameterization as it is currently formulated were incorporated into a climate model, it would lead to nonconservation of energy. The quantitative estimates from B04 suggest that the error due to nonconservation of energy in the Z10 formulation corresponds to a missing heating rate on the order of 10 K day^{-1} above 0.1 hPa. Assuming a radiative relaxation rate of 5 days, the error would result in a 50-K temperature bias in the summer mesosphere, which would, in practice, require further (and completely unnecessary) tuning of gravity wave drag parameters to compensate for the bias.[3] In general, model biases due to nonconservation are very difficult to diagnose and attribute because they can have subsequent effects on the grid-resolved dynamics, which can lead to feedbacks. For example, nonconservation of momentum in gravity wave drag parameterization leads to large biases in the propagation and breaking of grid-resolved planetary-scale waves, particularly during winter in the Northern Hemisphere (Shaw et al. 2009; Shaw and Perlwitz 2010).

The formulation of Z10 could certainly be extended to include the missing terms discussed in section 2. The Z10 parameterization is based on the Alexander and Dunkerton (1999) scheme, which invokes the saturation hypothesis of Lindzen (1981) as a wave dissipation mechanism. B04 showed how the terms missing in the Z10 parameterization can be included in the Lindzen (1981) parameterization. Becker and McLandress (2009) showed how the Doppler-spread parameterization of Hines (1997) could also be extended to include missing thermodynamic effects. It is only after accounting for these missing terms that the Z10 parameterization can be said to consistently include all effects of gravity wave breaking on the grid-resolved flow.

Acknowledgments. TAS acknowledges support from the Natural Sciences and Engineering Research Council of Canada through a postdoctoral fellowship. The authors are grateful to Dr. Rolando Garcia and two anonymous reviewers for their helpful comments.

APPENDIX

Conversion between Geometric Height and Log-Pressure Coordinates

In section 2 we presented the total energy and thermal energy budgets of B04 and SS09 in log-pressure coordinates. The original B04 and SS09 formulations used geometric height as a vertical coordinate. Here we show how the geometric height equations can be transformed to log-pressure coordinates. We will use z^* and w^* to denote geometric height and its vertical velocity and z and w to denote log-pressure height and its vertical velocity, which is consistent with the notation used by Z10. Note that the transformation is performed for both the grid-resolved variables (defined with an overbar) and the subgrid-scale variables (defined with a prime). Some general definitions and transformation rules are as follows:

$$z = -H \ln p, \tag{A1}$$

$$\begin{aligned} w^* &= \frac{Dz^*}{Dt} - \frac{1}{\rho g}\frac{Dp}{Dt} = -\frac{p}{\rho g}\frac{D\ln p}{Dt} \\ &= \frac{p}{\rho g H} w = \frac{RT}{gH} w, \end{aligned} \tag{A2}$$

$$\frac{\partial(\cdot)}{\partial z^*} = -\frac{\rho g}{p}\frac{\partial(\cdot)}{\partial \ln p} = \frac{gH}{RT}\frac{\partial(\cdot)}{\partial z}, \tag{A3}$$

$$\frac{\partial}{\partial z^*}[w^*(\cdot)] = \frac{gH}{RT}\frac{\partial}{\partial z}\left[w\frac{RT}{gH}(\cdot)\right] = \frac{\partial}{\partial z}[w(\cdot)]. \tag{A4}$$

The total energy equation in B04 [see Eq. (20) in B04] can be transformed to log-pressure coordinates [i.e., to Eq. (2)], using the following relations:

$$e = h - \frac{p}{\rho} = (c_p - R)T = \frac{H}{\kappa\gamma}\Phi_z, \tag{A5}$$

$$\nabla \cdot (p\mathbf{u}) = \nabla \cdot (\rho_0 \mathbf{u}\Phi), \tag{A6}$$

$$-\frac{\partial}{\partial z^*}(\mathbf{F}\cdot\mathbf{u}) = -\frac{\partial}{\partial z^*}(\overline{\rho\mathbf{u}'w'^*}\cdot\mathbf{u}) = -\frac{\partial}{\partial z}(\overline{\mathbf{u}'w'}\cdot\mathbf{u}), \tag{A7}$$

$$\begin{aligned} -\frac{\partial}{\partial z^*}\left(c_p\rho\frac{T}{\Theta}\mathbf{F}_\Theta\right) &= -\frac{\partial}{\partial z^*}\left(c_p\rho\frac{T}{\Theta}\overline{\theta' w'^*}\right) \\ &= -\frac{\partial}{\partial z}\left(\rho_0\frac{H}{\kappa}\overline{\Phi_z' w'}\right), \end{aligned} \tag{A8}$$

[3] Z10 add a parameterized cooling term to their thermal energy equation when they formulate the numerical representation of their parameterization, which is based on Alexander and Dunkerton's (1999) dissipation mechanism [see Eqs. (3) and (30) in Z10]. The cooling term is needed to properly account for the vertical flux of sensible heat [the second term on the right-hand side of Eq. (3)] between the breaking level and the critical level. In the Alexander and Dunkerton (1999) parameterization the breaking and critical levels are the same and thus the ad hoc cooling is a numerical fix, which preserves the formulation of the right-hand side of Eq. (3). Here we are focused on the analytical formulation of the thermal energy Eq. (3) and how it compares to the B04 and SS09 formulations.

$$-\frac{\partial \mathbf{F}_p}{\partial z^*} = -\frac{\partial}{\partial z^*}(\overline{p'w'^*}) = -\frac{\partial}{\partial z}(\rho_0 \overline{\Phi' w'}), \quad \text{(A9)}$$

$$\frac{\partial}{\partial z^*}\left(\rho K_z \frac{\partial \mathbf{u}}{\partial z^*}\right) = \frac{\partial}{\partial z}\left(\rho_0 k_{zz-m} \frac{\partial \mathbf{u}}{\partial z}\right), \quad \text{(A10)}$$

$$\frac{\partial}{\partial z^*}\left(\rho \frac{T}{\Theta} K_z \frac{\partial \Theta}{\partial z^*}\right) = \frac{\partial}{\partial z}\left(\rho_0 k_{zz-T} \frac{\partial \overline{\Phi}_z}{\partial z}\right). \quad \text{(A11)}$$

Equations (A6) and (A9) result from the transformation between ∇p in geometric height coordinates to $\nabla\Phi$ in log-pressure coordinates in the momentum equation. In Eqs. (A10) and (A11), the diffusion coefficients have been redefined. Note that the second term and the vertical component of the third term on the right-hand side of Eq. (20) in B04 cancel out using the hydrostatic relation. In addition to the above relations, the thermal energy equation requires the following relations:

$$h = c_p T = \frac{H}{\kappa}\overline{\Phi_z}, \quad \text{(A12)}$$

$$\frac{1}{\overline{p}}\frac{D\overline{p}}{Dt} = \frac{R}{H}\overline{T}H\frac{D}{Dt}\ln\overline{p} = -\frac{R}{H}\overline{T}\,\overline{w}. \quad \text{(A13)}$$

The same transformations can be applied to the total energy and thermal energy equations in SS09 [see Eq. (38) combined with Eqs. (42) and (32) in SS09] to obtain Eqs. (2) and (4).

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