ENERGY TRANSFORMATION FUNCTIONS

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ABSTRACT

Separate energy equations for potential energy, kinetic energy of averaged motion, kinetic energy of eddy motion, and thermal energy are derived on the basis of four postulates concerning natural processes. The equations are transformed for use with space- and time-averaged values of the dependent variables in a general system whose boundaries have arbitrary shape and motion. The equations are written in a manner that makes apparent the significance of energy transformation functions, which are the mathematical expressions of the physical mechanisms by which one kind of energy is transformed into another.

1. The nature of energy

Energy equations for a fluid in motion have been derived and presented by numerous authors. Notable among these are Reynolds (1895) and, for atmospheric energy equations, Margules (1901; 1904) and Ertel (1943). Except for the fact that Reynolds considered only an incompressible fluid, his paper contains the essence of nearly everything that has been written since on atmospheric energy equations.

A study of the many papers led to the conclusions that the experimental foundation of the derived equations is far from complete, and that the various authors' points of view, on the same energy phenomena, may differ so much that they seem to be writing about quite different things. The purposes of the present paper are: to separate the experimental foundation from the purely mathematical analysis; to fill the gaps of the incomplete experimental foundation by means of postulates; to distinguish consistently between energy within a system, energy added to or taken from the system, and work done by the system; and to throw some light on the mechanisms of energy transformations by consideration of the corresponding mathematical functions. Accomplishment of these purposes has been greatly facilitated by discussions with other members of the Department of Meteorology, New York University, and especially with Mr. A. K. Blackadar, who has devoted much time to investigations of the same problem.

It is necessary first to discuss the terms work, system, and energy. When a body moves through a distance d, opposite to the direction of a force F that is acting on the body, it is said to have done work equal in amount to the product Fd. A portion of space with prescribed boundaries is a system. Energy is that property of a system which decreases when the system does work and increases when work is done on the system, the amount of decrease or increase being equivalent to the work done.

But a change of energy does not necessarily involve the performance of work, for energy may flow into or out of a system that is doing no work. Only when a system is insulated against energy flux across its boundaries can energy changes be recognized from the work that accompanies them. When, in systems so insulated, the changes of different forms of energy have been recognized and expressed in terms of changes of the parameters of state, it is assumed that such changes can again be recognized in more general processes by the associated changes of the state parameters. Further, when a system is prevented from doing work, the net flux of energy into or out of the system can be recognized from the change of the system's total energy, which is computed from the previously established relationships between changes of energy and of the state parameters. The energy flux can thereby be related to measurable quantities and computed in more general processes from these quantities.

The evidence of experimental investigations[2] indicates that the concept of energy, as defined, can always be applied consistently; that the necessary experimental conditions (insulation of boundaries against energy flux or restraint of systems from performing work) can be achieved, with a sufficient degree of approximation, for a wide variety of systems and processes; and that energy changes can always be expressed in terms of the parameters of the system's initial and final states.

The definitions lead to the energy equation, expressed here in terms of changes with time,

$$\begin{bmatrix}\text{Rate of change}\\ \text{of energy in the}\\ \text{system}\end{bmatrix} = \begin{bmatrix}\text{Rate at which energy}\\ \text{is added through the}\\ \text{boundaries of the system}\end{bmatrix} - \begin{bmatrix}\text{Rate at which}\\ \text{work is done}\\ \text{by the system}\end{bmatrix}. \quad (1)$$

It is not possible, *a priori*, to write this equation for any specific forms of energy. One must proceed from laws established experimentally, identifying energy change or energy flux by means of the associated rate at which work is done by the system; and one must

[1] Based on a report of a research project sponsored by the U. S. Weather Bureau and New York University.

[2] The classical examples are Joule's experiments which established the mechanical equivalent of heat.

make certain that the mathematical expression for energy change depends only upon the parameters of the system's initial and final states.

2. Equations of motion

When the Navier-Stokes equations (Lamb, 1932, p. 576) are referred to right-handed, rectangular axes fixed to the earth, whose rate of rotation is ω, they can be written:

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} + 2\omega_{ik} v_k = -\frac{\partial \Phi}{\partial x_i} + \alpha \frac{\partial p_{ki}}{\partial x_k}. \quad (2)$$

The index notation is adapted from that of tensor calculus, and it was apparently first used in meteorological equations by Ertel (1938).[3] The indices i and k have the values 1, 2, 3; the corresponding meanings of x_i, x_k, v_i, and v_k are given in table 1. ω_{ik} represents the components of the earth's angular velocity and has the values given in table 2. Φ is the geopotential, gx_3,

TABLE 1. Significance of subscripts for x and v.

Indices	i or k =	1	2	3
Coordinate axes	x_i or x_k =	x_1 (toward east)	x_2 (toward north)	x_3 (toward zenith)
Components of the velocity	v_i or v_k =	v_1	v_2	v_3

TABLE 2. Significance of subscripts for ω.

k \ i	1	2	3
1	0	$\omega \sin \varphi$	$-\omega \cos \varphi$
2	$-\omega \sin \varphi$	0	0
3	$\omega \cos \varphi$	0	0

where g (assumed independent of height) is the force of gravity per unit mass; $\partial\Phi/\partial x_i$ is therefore zero when $i = 1$ or 2, and is equal to g when $i = 3$. Time is represented by t and the specific volume of air by α. The quantity p_{ki} represents the stresses, acting in the x_i-direction on the air lying on the negative side of a plane normal to the x_k-axis. In the index notation these stresses are

$$p_{ki} = p_{ik} = -\delta_{ik}\left(p + \tfrac{2}{3}\mu \frac{\partial v_j}{\partial x_j}\right) + \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \quad (3)$$

Here j is an index which, like i and k, has the values 1, 2, 3; δ_{ik} is a unit tensor whose values are 1 when $i = k$ and zero when $i \neq k$; p is the air pressure; and μ is the coefficient of molecular viscosity, whose value for air at 20C is 1.818×10^{-4} gm sec^{-1} cm^{-1} (Montgomery, 1947).

Equations (2) and (3) and subsequent equations are written with the aid of the summation convention: whenever an index, i, j, or k, is repeated in a term, that term is understood to be summed over all values of the repeated index. Thus, for example,

$$v_k \frac{\partial v_i}{\partial x_k} = \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k} = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}, \quad \text{for} \quad i = 1,$$

or,

$$v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}, \quad \text{for} \quad i = 2,$$

and so on for $i = 3$.

The three component equations of motion can be obtained from (2) by giving i the successive values 1, 2, 3.

Although the Navier-Stokes equations are generally assumed to be correct for laminar flow, no complete test of them has ever been made, at least up until 1934 (Prandtl and Tietjens, 1934, p. 260). Reynolds (1895) suggested that the equations are valid for velocities representing mean values over spaces and time intervals that are large compared to the scale of molecular motions, but small compared to the lower limits of the scale of turbulent motions. Thus, Reynolds postulated the existence of a scale domain in which the flow may be treated as laminar and the Navier-Stokes equations are applicable. There is some experimental evidence that, if the kinetic energy of air motions is separated into energies of mean and of relative motion, the kinetic energy of the relative motion approaches negligible values as the scale is reduced (Dryden, 1943). Much more experimental evidence would be required to establish Reynolds' postulate as fact. Nevertheless, his postulate has been adopted for the present analysis and is restated in the following form:

Postulate 1. *Equation (2) is a complete and correct statement of the relationship between acceleration and force when applied to motions of the atmosphere on the rotating earth, if the velocity, stresses, and specific volume represent space- and time-means in a certain domain of small scale where the means are independent of the scale.*

Reynolds' postulate has been extended to include pressure and specific volume, whose values are also subject to fluctuations with varying periods in space and time.

3. First law of thermodynamics

The energy equation, when applied to a perfect gas and limited to a form of energy that will be called thermal energy,[4] becomes the first law of thermo-

[3] The distinction between contravariant and covariant tensors will not enter into the following analysis; all indices will, therefore, be written as subscripts.

[4] *Thermal energy* is a euphemism for *internal energy*, which is sometimes intended to include all the energy, of whatever form, contained within a system.

dynamics

$$d(c_vT)/dt = Q + \alpha p_{ki}\, \partial v_i/\partial x_k. \tag{4}$$

The operator d/dt represents rate of change following the motion; c_v is the specific heat of the gas at constant volume; T, the temperature; $d(c_vT)/dt$, the rate of change of thermal energy per unit mass; and Q, the rate at which thermal energy is added by radiation, molecular conduction, or condensation of water vapor.

The last term is usually written in the less general form, $-p\, d\alpha/dt$, which is correct only for a non-viscous gas. That the less general form is contained in the complete form is shown by expanding the latter, in part,

$$\alpha p_{ki}\frac{\partial v_i}{\partial x_k} = -\alpha p\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) + \left(\begin{matrix}\text{additional terms}\\ \text{involving } \mu\end{matrix}\right),$$

and comparing it with the equation of continuity, multiplied by p,

$$-p\frac{d\alpha}{dt} = -\alpha p\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right). \tag{5}$$

Omission of the terms involving μ does not cause serious error in many meteorological problems, but the terms are essential in discussions of energy transformations. Their omission from energy equations in a number of textbooks has resulted in total energy equations that violate the law of conservation of energy (Stewart, 1942), as will become apparent in the last section.

Whether (4) can be applied to complex processes involving, in addition to thermal energy, other forms of energy such as kinetic and potential, is an open question. Textbooks on thermodynamics usually restrict the first law to stationary systems, whose changes of state proceed at an infinitesimal rate; but on the whole, they are rather non-committal concerning the limits of applicability of the law.

In the absence of adequate experimental evidence, it will be necessary to adopt the following postulate:

Postulate 2. *Equation (4) is a complete and correct statement of the energy equation for the rate of change of thermal energy per unit mass,* $d(c_vT)/dt$, *in the atmosphere, regardless of associated changes of other forms of energy, if the velocity, temperature, stresses, and specific volume represent space- and time-means in a certain domain of small scale where the means are independent of the scale.*

Present experimental knowledge is sufficient to show that the required scales of space and time are usually very small, often less than 1 cm^3 and 1 sec, respectively. Hence, a system for which the term $d(c_vT)/dt$ of (4) represents the rate of change of thermal energy per unit mass must, in general, be very small. For a larger system, (4) must be integrated over the volume of the system.

4. Kinetic-energy equation

Another energy equation can be derived by multiplying (2) by v_i:

$$v_i\frac{\partial v_i}{\partial t} + v_iv_k\frac{\partial v_i}{\partial x_k} + 2\omega_{ik}v_iv_k = -v_i\frac{\partial \Phi}{\partial x_i} + \alpha v_i\frac{\partial p_{ki}}{\partial x_k}. \tag{6}$$

All terms of the Navier-Stokes equation (2) contain the index i, which is not repeated in any term. Thus, (2) represents three component equations corresponding to the three values of i. The derived equation (6) has i repeated in all terms and k repeated in some; neither index appears singly in any term. Therefore, (6) is a single equation, each of whose terms represents a summation.

When expanded, the term $2\omega_{ik}v_iv_k$ becomes

$$2\omega(-v_1v_2\sin\varphi + v_1v_3\cos\varphi + v_2v_1\sin\varphi - v_3v_1\cos\varphi) = 0.$$

The expansion of $v_i\,\partial\Phi/\partial x_i$ is simply v_3g. Finally,

$$v_i\frac{\partial v_i}{\partial t} + v_iv_k\frac{\partial v_i}{\partial x_k} = \frac{\partial}{\partial t}\left(\frac{v_i^2}{2}\right) + v_k\frac{\partial}{\partial x_k}\left(\frac{v_i^2}{2}\right) = d(\tfrac{1}{2}v_i^2)/dt.$$

where v_i^2 represents v_iv_i and, consequently, by the summation convention, is equivalent to $v_1^2 + v_2^2 + v_3^2$. Therefore, (6) can be written as

$$d(\tfrac{1}{2}v_i^2)/dt = -(v_3g - \alpha v_i\,\partial p_{ki}/\partial x_k). \tag{7a}$$

The term v_3g is clearly the rate at which work is done by the air against the force of gravity. The term $-\alpha v_i\,\partial p_{ki}/\partial x_k$ is the rate at which work is done by the air against forces resulting from variations of the stresses; for example, if the air pressure is considered alone, the term becomes

$$\alpha(v_1\,\partial p/\partial x_1 + v_2\,\partial p/\partial x_2 + v_3\,\partial p/\partial x_3).$$

Comparison of (7a) with (1) shows that (7a) defines a form of energy per unit mass, $\frac{1}{2}v_i^2$; this is called kinetic energy. Since, in (7a), there is no term corresponding to the energy-added term of (1), it must be presumed that the only way in which kinetic energy can be transferred from one small mass to another is through the mechanism of work. It will be shown later that an energy-added term does appear in the integrated form of (7a).

Sometimes the term v_3g is expressed (assuming g constant) in the equivalent form $d(gx_3)/dt$, which is transposed to the left side,

$$d(\tfrac{1}{2}v_i^2 + gx_3)/dt = \alpha v_i\,\partial p_{ki}/\partial x_k, \tag{7b}$$

where gx_3 is defined as the potential energy per unit mass.

If Postulate (1) is correct, (7a) and (7b) are valid for space- and time-means of velocity, stresses, and specific volume, in a scale domain where the means are independent of the scale. Since the experimental evidence suggests that the means are independent of the scale only when the scale is very small, these energy equations can be applied in general only to a very small system; for a larger system they must be integrated.

5. The general system

So that the energy equations may be integrated over a large volume, the elementary system must be changed from a unit mass to a unit volume. Hence, the equations must be multiplied through by the mass element $dm = \rho\, dV$, where ρ is the density and V the volume. The total derivatives are rendered more suitable for integration as follows. Let E be any property of the system; then

$$\rho \frac{dE}{dt} = \rho \frac{\partial E}{\partial t} + \rho v_k \frac{\partial E}{\partial x_k}. \quad (8)$$

The equation of continuity (5) can be written

$$0 = \partial\rho/\partial t + \partial(\rho v_k)/\partial x_k. \quad (9)$$

Multiplication of (9) by E and addition to (8) gives

$$\rho\, dE/dt = \partial(\rho E)/\partial t + \partial(\rho v_k E)/\partial x_k. \quad (10)$$

Then (4) and (7a), after multiplication through by ρ, become with the aid of (10),

$$\frac{\partial}{\partial t}(\rho c_v T) + \frac{\partial}{\partial x_k}(\rho v_k c_v T) = \rho Q + p_{ki}\frac{\partial v_i}{\partial x_k}, \quad (11)$$

$$\frac{\partial}{\partial t}\left(\rho \frac{v_i^2}{2}\right) + \frac{\partial}{\partial x_k}\left(\rho v_k \frac{v_i^2}{2}\right) = -\rho v_3 g + v_i \frac{\partial p_{ki}}{\partial x_k}. \quad (12)$$

For the sake of brevity, the energies will be represented by

$$\left.\begin{array}{l} I = \rho c_v T \\ K = \tfrac{1}{2}\rho v_i^2 \end{array}\right\} \text{per unit volume}$$

$$\left.\begin{array}{l} I^* = \int_V \rho c_v T\, dV \\ K^* = \int_V \tfrac{1}{2}\rho v_i^2\, dV \end{array}\right\} \text{for the total volume.} \quad (13)$$

By virtue of (13) and the divergence theorem, the energy equations become, after integration over the volume V,

$$\partial I^*/\partial t = -\int_S v_n I\, dS + \int_V \rho Q\, dV + \int_V p_{ki}\, \partial v_i/\partial x_k\, dV, \quad (14)$$

$$\partial K^*/\partial t = -\int_S v_n K\, dS - \int_V \rho v_3 g\, dV + \int_V v_i\, \partial p_{ki}/\partial x_k\, dV, \quad (15)$$

where dS is an element of the boundary surface of the system, and v_n is the component of v_k normal to the boundary surface, directed outward from the system.

Equations (14) and (15) give the local rates of change of energy in a large system whose boundaries are stationary with respect to the chosen coordinate system. The first term on the right of each equation evidently represents the mass flux of energy into the system through the stationary boundaries. The most general system may be defined so that its boundary surfaces move with an arbitrary, variable velocity, V_n, normal to the surface and directed outward. The motion of the boundaries will result in net fluxes $\int_S V_n I\, dS$, $\int_S V_n K\, dS$, and $\int_S V_n P\, dS$, of energy into the system. The sum of the local change of energy and the net energy flux due to motion of the boundaries gives the total change for the general system. For example, for thermal energy

$$DI^*/Dt = \partial I^*/\partial t + \int_S V_n I\, dS,$$

where DI^*/Dt is the rate of change of thermal energy in the system. Thus, for the general system the energy equations become:

$$DI^*/Dt = \int_S (V_n - v_n) I\, dS + \int_V \rho Q\, dV + \int_V p_{ki}\, \partial v_i/\partial x_k\, dV, \quad (16)$$

$$DK^*/Dt = \int_S (V_n - v_n) K\, dS - \int_V \rho v_3 g\, dV + \int_V v_i\, \partial p_{ki}/\partial x_k\, dV. \quad (17)$$

The relation between gravity work and potential energy similarly can be written as

$$DP^*/Dt = \int_S (V_n - v_n) P\, dS + \int_V \rho v_3 g\, dV, \quad (18)$$

where

$$P = \rho g x_3, \quad \text{and} \quad P^* = \int_V \rho g x_3\, dV. \quad (19)$$

6. The molecular transformation function

If the last term of (16) is expanded with the aid of (3) and the terms are collected in a certain way, there is obtained

$$\int_V p_{ki}\frac{\partial v_i}{\partial x_k}dV = -\int_V p\frac{\partial v_i}{\partial x_i}dV + \int_V \mu\left[\frac{2}{3}\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)^2 + \frac{2}{3}\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_3}{\partial x_3}\right)^2 + \frac{2}{3}\left(\frac{\partial v_2}{\partial x_2}-\frac{\partial v_3}{\partial x_3}\right)^2 + \left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_3}{\partial x_1}+\frac{\partial v_1}{\partial x_3}\right)^2 + \left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right)^2\right]dV \quad (20)$$

or,

$$\int_V p_{ki}\frac{\partial v_i}{\partial x_k}dV = M_p + M_\mu, \quad (21)$$

where M_p represents $-\int_V p(\partial v_i/\partial x_i)\,dV$, and M_μ is the sum of the remaining terms on the right of (20).

Since μ is always positive, M_μ is also always positive. Therefore, the frictional work represented by M_μ always contributes to an increase of the thermal energy of the system.

Equation (17) will be transformed so that M_p and M_μ appear in it. The last term of (17) can be written

$$\int_V v_i\frac{\partial p_{ki}}{\partial x_k}dV = -\int_V p_{ki}\frac{\partial v_i}{\partial x_k}dV + \int_V \frac{\partial}{\partial x_k}(v_i p_{ki})\,dV = -M_p - M_\mu + \int_S v_i p_{ni}\,dS, \quad (22)$$

where p_{ni} represents the stress components in a plane tangent to the boundary surface.

When the relation between gravity work and potential energy, (18), is written together with (16) and (17), transformed by means of (21) and (22), a significant parallel becomes apparent:

$$DP^*/Dt = \left[\int_S (V_n - v_n)P\,dS\right]_{\mathrm{A}} - \left[-\int_V \rho v_3 g\,dV\right]_{\mathrm{W}}, \quad (23)$$

$$DK^*/Dt = \left[\int_S (V_n - v_n)K\,dS\right]_{\mathrm{A}} - [M_p + M_\mu]_{\mathrm{W}} - \left[\int_V \rho v_3 g\,dV\right]_{\mathrm{W}} - \left[-\int_S v_i p_{ni}\,dS\right]_{\mathrm{W}}, \quad (24)$$

$$DI^*/Dt = \left[\int_S (V_n - v_n)I\,dS + \int_V \rho Q\,dV\right]_{\mathrm{A}} - [-M_p - M_\mu]_{\mathrm{W}}. \quad (25)$$

The subscript A indicates that a term represents energy added to the system; the subscript W, that a term represents work done by the system.

Equation (23) merely defines potential energy, and hence its validity does not depend upon experimental evidence. The term $\int_V \rho v_3 g\,dV$, which appears in (23) and (24) with opposite signs, clearly represents the work by means of which transformations between potential and kinetic energy take place. In the absence of other effects, when that term is negative (downward motion) the potential energy is decreased and the kinetic energy simultaneously increased by an equal amount.

Equations (24) and (25) both contain M_p and M_μ with opposite signs. If M_p is positive, it decreases the kinetic energy and increases the thermal energy; this process is compressional heating, and its converse is expansional cooling. M_μ, however, is always positive, so it always represents a transformation of kinetic into thermal energy; this process may be called frictional heating. The term M_μ is known as Stokes' *dissipation function* (Lamb, 1932, p. 580), in the case of an incompressible fluid.

The quantity $M_p + M_\mu$ will be called the molecular transformation function, because it is the mathematical representation of the physical mechanism, presumably associated with molecular motions, which transforms kinetic into thermal energy or thermal into kinetic energy.

The terms $\int_V \rho v_3 g\,dV$, M_p, and M_μ express the rate at which work is performed within the system, while the term $\int_S v_i p_{ni}\,dS$ expresses work performed only at the boundary surfaces. When (23), (24), and (25) are added together, to form a total energy equation, the internal work terms vanish and

$$D(P^* + K^* + I^*)/Dt = \left[\int_S (V_n - v_n)(P + K + I)\,dS + \int_V \rho Q\,dV\right]_{\mathrm{A}} - \left[-\int_S v_i p_{ni}\,dS\right]_{\mathrm{W}}. \quad (26)$$

Thus, the total energy equation permits an evaluation of energy changes without consideration of the mechanisms. If one wishes to investigate the mechanisms, the individual equations must be used. For example, if the potential energy increases as a result of a flux of thermal energy into the system, nothing about the nature of the process could be learned from (26); but examination of (23), (24), and (25) shows that the added thermal energy is transformed first into kinetic energy (the transformation function M_p in (24) and (25)) and then into potential energy (the transformation function $\int_V \rho v_3 g\,dV$ in (23) and (24)).

7. Effects of the averaging process

Observed values of velocity, pressure, specific volume, density, and temperature are determined in meteorological practice by instruments whose size and period of response are generally larger than the size and period of the smallest eddies or fluctuations. Therefore, according to Postulates 1 and 2, the commonly observed values cannot be substituted into the energy equations that have been discussed up to this point. The averaging process of instruments, of numerical averaging, or of both, can be duplicated by a mathematical process as follows. Let $\bar{v}_i$, $\bar{p}$, *etc.*, represent values of the dependent variables determined by instrumental observations or numerical averaging of such observations. Then, for any dependent variable q,

$$\bar{q} = \frac{\int_{x_1-\delta x_1}^{x_1+\delta x_1} \int_{x_2-\delta x_2}^{x_2+\delta x_2} \int_{x_3-\delta x_3}^{x_3+\delta x_3} \int_{t-\delta t}^{t+\delta t} q \, dx_1 \, dx_2 \, dx_3 \, dt}{\int_{x_1-\delta x_1}^{x_1+\delta x_1} \int_{x_2-\delta x_2}^{x_2+\delta x_2} \int_{x_3-\delta x_3}^{x_3+\delta x_3} \int_{t-\delta t}^{t+\delta t} dx_1 \, dx_2 \, dx_3 \, dt}, \quad (27)$$

where δx_1, δx_2, δx_3, and δt are variable limits of integration corresponding to the size, shape, and response of the instrument, and to the arbitrary limits that may be set in numerical averaging. The value of $\bar{q}$ thus obtained depends upon the position of the instrument and the time; or,

$$\bar{q} = \bar{q}(x_1, x_2, x_3, t). \quad (28)$$

Since every dependent variable must be a continuous function of the coordinates and time (or the basic equations, (2) and (4), could not be valid), $\bar{q}$ must also be a continuous function of the coordinates and time.

The value of $\bar{q}$ will, in general, differ from the value of q at the same point and time. The deviations, v_i', v_k', p_{ki}', $\cdots$, are defined by

$$\begin{aligned} v_i &= \bar{v}_i + v_i' \\ v_k &= \bar{v}_k + v_k' \\ p_{ki} &= \bar{p}_{ki} + p_{ki}' \\ &\cdots\cdots\cdots \end{aligned} \quad (29)$$

The effects of the averaging process on velocity and stresses only will be considered here. The quantities v_i, v_k, and p_{ki} will be replaced in the energy equations (11) and (12) by their equivalent expressions from (29). For the sake of simplicity in the analysis, the following postulate will be adopted:

Postulate 3. *The deviations ρ' and α' of density and specific volume from their average values are negligible.* This postulate is not necessarily valid; but if it is not adopted, the resulting equations become very complex (Miller, 1949).[5] A final postulate concerning the character of the motion will be adopted:

Postulate 4. *The average values $\bar{v}_i$, $\bar{v}_k$, and $\bar{p}_{ki}$, are not changed when averaged again over space and time by (27); and the deviations v_i', v_k', and p_{ki}' vanish when similarly averaged.*

The validity of Postulate 4 is discussed in detail by Reynolds (1895), whose arguments have been summarized by Miller (1949).

Formulas (29) are substituted into (11) and (12), each term of which is then averaged by (27). By virtue of Postulates 3 and 4, (11) and (12) finally become

$$\frac{\partial}{\partial t}(\overline{\rho c_v T}) + \frac{\partial}{\partial x_k}(\bar{v}_k \, \overline{\rho c_v T} + \overline{v_k' \rho c_v T}) = \overline{\rho Q} + \bar{p}_{ki} \frac{\partial \bar{v}_i}{\partial x_k} + \overline{p_{ki}' \frac{\partial v_i'}{\partial x_k}}, \quad (30)$$

$$\frac{\partial}{\partial t}\left(\bar{\rho} \frac{\bar{v}_i^2}{2} + \overline{\rho \frac{v_i'^2}{2}}\right) + \frac{\partial}{\partial x_k}\left(\bar{v}_k \bar{\rho} \frac{\bar{v}_i^2}{2} + \bar{v}_k \overline{\rho \frac{v_i'^2}{2}} + \bar{v}_i \overline{\rho v_i' v_k'} + \overline{v_k' \rho \frac{v_i'^2}{2}}\right) = -\overline{\rho v_3}\, g + \bar{v}_i \frac{\partial \bar{p}_{ki}}{\partial x_k} + \overline{v_i' \frac{\partial p_{ki}'}{\partial x_k}}. \quad (31)$$

Next, (30) and (31) are transformed to apply to the general system, by the procedure of section 5, with the results

$$D\bar{I}^*/Dt = \left[\int_S \left\{(V_n - \bar{v}_n)\bar{I} - \overline{v_n' I}\right\} dS + \int_V \overline{\rho Q}\, dV\right]_A - [-\bar{M}_p - \bar{M}_\mu - \overline{M_p'} - \overline{M_\mu'}]_W; \quad (32)$$

$$D(K_m^* + \overline{K'}^*)/Dt = \left[\int_S \{(V_n - \bar{v}_n)(K_m + \overline{K'}) - \overline{v_n' K'}\}\, dS\right]_A - \left[\bar{M}_p + \bar{M}_\mu + \overline{M_p'} + \overline{M_\mu'} - \int_S (\bar{v}_i T_{ni} + \bar{v}_i \bar{p}_{ni} + \overline{v_i' p_{ni}'})\, dS + \int_V \overline{\rho v_3}\, g\, dV\right]_W, \quad (33)$$

where the divergence theorem has again been employed. The relation between gravity work and potential energy becomes, similarly,

$$D\bar{P}^*/Dt = \left[\int_S (V_n - \bar{v}_n)\bar{P}\, dS\right]_A - \left[-\int_V \overline{\rho v_3}\, g\, dV\right]_W. \quad (34)$$

[5] Without this postulate there would also be some doubt whether the instrumental averaging process for velocity should not be duplicated mathematically by applying (27) to ρv_i instead of to v_i, because a wind-measuring instrument may be affected more by the momentum of the air than by the velocity.

In these equations,

$$\begin{aligned}
\bar{I} &= \overline{\rho c_v T} & \bar{I}^* &= \int_V \bar{I}\, dV \\
K_m &= \tfrac{1}{2}\bar{\rho}\bar{v}_i^2 & K_m^* &= \int_V K_m\, dV \\
\overline{K'} &= \tfrac{1}{2}\overline{\rho v_i'^2} & \overline{K'}^* &= \int_V \overline{K'}\, dV \\
\bar{M}_p + \bar{M}_\mu &= \int_V \bar{p}_{ki}\, \partial\bar{v}_i/\partial x_k\, dV & \overline{M_p'} + \overline{M_\mu'} &= \int_V \overline{p_{ki}'\, \partial v_i'/\partial x_k}\, dV \\
T_{ki} &= T_{ik} = -\overline{\rho v_i' v_k'} & T_{ni} &= -\overline{\rho v_i' v_n'} \\
\bar{P} &= \overline{\rho x_3}\, g = \rho x_3 g = P & \bar{P}^* &= \int_V \bar{P}\, dV = P^*.
\end{aligned} \tag{35}$$

The term T_{ki} is usually regarded as a stress, against the average motion, resulting from eddy motions; it is called the eddy stress or Reynolds stress. The term $-\int_S \overline{v_n' I}\, dS$ in (32) is the net flux of thermal energy into the system as a result of eddy motions; and the term $-\int_S \overline{v_n' K'}\, dS$ in (33) is the net flux of eddy kinetic energy into the system as a result of eddy motions. The work terms that are expressed as surface integrals represent work at the boundary surfaces only; the other work terms represent work within the system.

Two forms of kinetic energy are represented in (33): the kinetic energy of the averaged motion, K_m; and the average kinetic energy of the eddy motion, $\overline{K'}$. Their sum is the average total kinetic energy, $\bar{K}$. In order that the analysis of energy transformation functions be complete, it will be necessary to separate (33) into two equations, one for K_m and one for $\overline{K'}$.

8. The eddy transformation function

An equation of the kinetic energy of mean motion alone can be derived by applying the averaging process at an earlier stage in the analysis than was done in the preceding section. If (29) is substituted into the equations of motion (2), if each term is then averaged by (27), and if the equation is finally multiplied by $\bar{v}_i$, there is obtained

$$\partial(\tfrac{1}{2}\bar{v}_i^2)/\partial t + \bar{v}_k\, \partial(\tfrac{1}{2}\bar{v}_i^2)/\partial x_k + \bar{v}_i \overline{v_k'\, \partial v_i'/\partial x_k} = -\bar{v}_3\, g + \alpha\bar{v}_i\, \alpha\, \partial\bar{p}_{ki}/\partial x_k. \tag{36}$$

The last term on the left side of (36) will be transformed as follows. From the equation of continuity (9), because of (29),

$$\partial\rho/\partial t + \partial(\rho\bar{v}_k + \rho v_k')/\partial x_k = 0.$$

After multiplication by v_i' and averaging by (27), this becomes, according to Postulates 3 and 4,

$$\overline{v_i'\, \partial(\rho v_k')/\partial x_k} = 0. \tag{37}$$

The last term on the left of (36) is

$$\bar{v}_i \overline{v_k' \frac{\partial v_i'}{\partial x_k}} = \alpha\bar{v}_i \frac{\partial}{\partial x_k}(\overline{\rho v_i' v_k'}) - \alpha\bar{v}_i \overline{v_i' \frac{\partial(\rho v_k')}{\partial x_k}} = \alpha\bar{v}_i \frac{\partial}{\partial x_k}(\overline{\rho v_i' v_k'}),$$

because of (37). Finally, this can be written

$$\bar{v}_i \overline{v_k' \frac{\partial v_i'}{\partial x_k}} = \alpha T_{ki} \frac{\partial\bar{v}_i}{\partial x_k} - \alpha \frac{\partial}{\partial x_k}(\bar{v}_i T_{ki}). \tag{38}$$

Substitution of (38) in (36) and integration over the volume of the general system gives the energy equation for kinetic energy of averaged motion alone

$$DK_m^*/Dt = \left[\int_S (V_n - \bar{v}_n) K_m\, dS\right]_A - \left[\bar{M}_p + \bar{M}_\mu - \int_S (\bar{v}_i T_{ni} + \bar{v}_i \bar{p}_{ni})\, dS + \int_V T_{ki} \frac{\partial\bar{v}_i}{\partial x_k} dV + \int_V \bar{\rho}\bar{v}_3\, g\, dV\right]_W. \tag{39}$$

Subtraction of (39) from (33) gives the equation of average eddy kinetic energy alone,

$$D\overline{K'}^*/Dt = \left[\int_S \{(V_n - \bar{v}_n)\overline{K'} - \overline{v_n' K'}\}\, dS\right]_A - \left[\overline{M_p'} + \overline{M_\mu'} - \int_S \overline{v_i' p_{ni}'}\, dS - \int_V T_{ki} \frac{\partial\bar{v}_i}{\partial x_k} dV\right]_W. \tag{40}$$ [6]

Equations (39) and (40) both contain the term $\int_V T_{ki}\, \partial\bar{v}_i/\partial x_k\, dV$ with opposite signs. When the

[6] Equation (40) is the basis of the turbulence criterion originated by Reynolds (1895) for an incompressible fluid and applied, in a much altered form, to the compressible atmosphere by Richardson (1920).

term is positive, it contributes to an increase of eddy energy and to an equal decrease of the energy of averaged motion. Thus, since it is the mathematical equivalent of the mechanism, operating through eddy motions, which produces transformations between the kinetic energies of averaged and eddy motions, it will be called the eddy transformation function.

9. Summary

Energy equations for potential energy, kinetic energy of averaged motion, kinetic energy of eddy motion, and thermal energy have been derived by a purely mathematical process, on the basis of four postulates whose validity can be determined only by extended experimental investigations. These equations have been written for a general open system, of any size or shape, whose boundary surfaces move in any manner; and the equations have been referred to values of the various dependent variables averaged over any space and any time, either by instrumental effects, by numerical process, or by a combination of the two.

The ultimate purpose of the analysis is to show the significance of the energy transformation functions. In order to accomplish this purpose in the simplest manner, the energy equations are collected in the following array, where the energy-added terms and boundary-surface work terms are indicated only schematically. The arrows connecting the transformation functions show the possible directions of energy transformations according to present knowledge.

$$
\begin{aligned}
D\bar{P}^*/Dt &= \begin{bmatrix}\text{Energy added}\\ \text{through}\\ \text{boundaries}\end{bmatrix} + \int_V \overline{\rho v_3}\, g\, dV \\
DK_m^*/Dt &= \begin{bmatrix}\text{Energy added}\\ \text{through}\\ \text{boundaries}\end{bmatrix} - \bar{M}_p - \bar{M}_\mu - \int_V T_{ki}\, \partial\bar{v}_i/\partial x_k\, dV - \int_V \overline{\rho v_3}\, g\, dV - \begin{bmatrix}\text{Boundary}\\ \text{work}\end{bmatrix} \\
D\overline{K'}^*/Dt &= \begin{bmatrix}\text{Energy added}\\ \text{through}\\ \text{boundaries}\end{bmatrix} - \overline{M_p'} - \overline{M_\mu'} + \int_V T_{ki}\, \partial\bar{v}_i/\partial x_k\, dV - \begin{bmatrix}\text{Boundary}\\ \text{work}\end{bmatrix} \\
D\bar{I}^*/Dt &= \begin{bmatrix}\text{Energy added}\\ \text{through}\\ \text{boundaries}\end{bmatrix} + \bar{M}_p + \bar{M}_\mu + \overline{M_p'} + \overline{M_\mu'} - \begin{bmatrix}\text{Boundary}\\ \text{work}\end{bmatrix}
\end{aligned}
$$

The sum of these four equations is the total energy equation, in which there are no transformation functions,

$$
\frac{D}{Dt}(\bar{P}^* + K_m^* + \overline{K'}^* + \bar{I}^*) = \begin{bmatrix}\text{Energy added}\\ \text{through}\\ \text{boundaries}\end{bmatrix} - \begin{bmatrix}\text{Boundary}\\ \text{work}\end{bmatrix}.
$$

If there is no energy added and no work done at the boundaries, the total energy equation expresses the conservation of energy within the system,[7]

$$
\frac{D}{Dt}(\bar{P}^* + K_m^* + \overline{K'}^* + \bar{I}^*) = 0.
$$

The validity of these equations rests only upon the validity of Postulates 1 to 4 and the accuracy of the mathematical analysis. The postulates should be tested by experimental investigations; there is no other way to test them.

[7] If frictional work is omitted from the thermal-energy equation, as is often done (see Section 3), the right side of the equation of energy conservation contains the term $-\overline{M_\mu'}$, implying that energy is continuously lost in a closed system even in the absence of work at the boundaries.

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