THE STATISTICAL THEORY OF TURBULENCE AND THE PROBLEM OF DIFFUSION IN THE ATMOSPHERE

By Glenn W. Brier

U. S. Weather Bureau, Washington

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ABSTRACT

Developments in the statistical theory of turbulence during the past thirty years have been used by a number of investigators studying the problem of eddy diffusion. Taylor has derived a formula for the diffusion of particles from a point source. This formula and its extension by Sutton make use of a Lagrangian correlation coefficient R_ξ defined as the correlation between the velocity of a fluid particle at any instant and that of the same particle after a time interval ξ. A more general formula that takes account of the initial distance between the particles is presented here and it is shown that the Taylor formula can be derived as a special case of this formula.

An experiment is described where modern radar equipment was used to measure the scatter of a cluster of balloons released simultaneously in the free atmosphere. Questions are raised as to the usefulness of balloon data in obtaining estimates of the rate of diffusion, due to such factors as variable ascension rates of balloons, the change of wind speed and direction with height, and the failure of large balloons to respond to smaller-scale eddies that might contribute considerably to the diffusion of small particles such as smoke.

1. Introduction

Turbulent flow in the free atmosphere is of interest not only to the theoretical meteorologist, but also is of considerable importance to the practical worker concerned with such problems as forecasting smoke pollution or other phenomena where the diffusion process is operative. It seems to be generally agreed that the complexity of the problem makes impossible a complete physical theory of turbulence which will describe in detail the velocity and pressure fields of a fluid at every instant. However, in the past few decades considerable progress has been made in the development of statistical theories of turbulence[1] in which consideration is given to the frequency distributions, correlation coefficients, and other statistical properties of the irregular random motions and fluctuations that occur in a turbulent fluid. Satisfactory verification of some of these theories has been obtained in the laboratory, where it was possible to produce and control the turbulence in a wind tunnel or pipe and to measure the quantities required by the theory. However, relatively little is known regarding turbulence in the free atmosphere, since no control of the scale of the turbulence is possible and problems of measurement become more difficult. A few attempts have been made to study atmospheric turbulence by observing the scatter of balloons, and recent instrumental developments such as radar suggest the possibility of making more refined measurements of the quantities to be considered in any statistical theory of turbulence in the free atmosphere. This paper describes a recent experiment where measurements were made on the scatter of a cluster of balloons, using radar equipment, and points out some of the statistical concepts that should be considered in the analysis and interpretation of such data.

[1] An excellent review of the subject is given by Dryden (1943).

2. Mathematical theory

In the theory of turbulent flow, it is conventional to regard the flow as a mean motion with velocity components U, V, and W, on which are superposed fluctuations of the velocity with components of magnitude u, v, and w at any instant. The mean values of u, v, and w are zero, and the mean squares of the fluctuations are written as $[u^2]$, $[v^2]$, and $[w^2]$. For the sake of simplicity, let us consider only the movement in the x-direction of N particles in a field of uniform flow with mean velocity U. Let the velocity of the i-th particle during the r-th time interval be given by

$$U_{ri} = U + u_{ri}, \qquad (i = 1, 2, 3, \cdots N). \qquad (1)$$

The total distance X_{ni} traveled by the i-th particle in n intervals, relative to the mean flow, is given by

$$X_{ni} = u_{1i} + u_{2i} + u_{3i} + \cdots + u_{ni}, \qquad (2)$$

where, for convenience, *the time interval τ is considered as unity.* At the end of n unit time intervals, define the standard deviation σ_n of the N particles by

$$\sigma_n^2 = (N-1)^{-1} \sum_{i=1}^{N} (X_{ni} - [X_n])^2, \qquad (3)$$

where

$$[X_n] = N^{-1} \sum_{i=1}^{N} X_{ni}.$$

Now, it is easy to show that (3) is equivalent to

$$\sigma_n^2 = \tfrac{1}{2} N^{-1}(N-1)^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} (X_{ni} - X_{nj})^2, \qquad (4)$$

indicating that the square of the standard deviation of N measurements is proportional to the sum of the squares of all the possible differences. Thus, in the diffusion problem, if any two particles i and j are at a distance $l_n = X_{ni} - X_{nj}$ from each other at the end of n time intervals, the quantity l_n^2, defined as

$$l_n^2 = (X_{ni} - X_{nj})^2, \quad (5)$$

can be used to estimate σ_n^2. Study of the change of σ^2 with time is then identical with study of how the mean square distance between particles $[l^2]$ changes with time. Thus, if two particles are initially at distance l_0 from each other, the distance at the end of n time intervals will be given by

$$l_n = l_0 + (u_{1i} + u_{2i} + \cdots + u_{ni}) - (u_{1j} + u_{2j} + \cdots + u_{nj}). \quad (6)$$

The square of l_n is given by

$$l_n^2 = l_0^2 + \sum_{r=1}^{n}\sum_{t=1}^{n} u_{ri}u_{ti} + \sum_{r=1}^{n}\sum_{t=1}^{n} u_{rj}u_{tj} - 2\sum_{r=1}^{n}\sum_{t=1}^{n} u_{ri}u_{tj} + 2l_0\{\sum_{r=1}^{n} u_{ri} - \sum_{r=1}^{n} u_{rj}\}. \quad (7)$$

For a large number of repeated trials, the expected value of l_n^2 is given by

$$[l_n^2] = l_0^2 + 2[u^2]\{\sum_{r=1}^{n}\sum_{t=1}^{n} R_{rt} - \sum_{r=1}^{n}\sum_{t=1}^{n} {}_rR_t\}, \quad (8)$$[2]

where

$$[u_{ri}u_{ti}] = [u_{rj}u_{tj}] = [u^2]R_{rt},$$

and

$$[u_{ri}u_{tj}] = [u_{ti}u_{rj}] = [u^2]\,{}_tR_r = [u^2]\,{}_rR_t.$$

R_{rt} is defined as the correlation coefficient between the movement of a particle during interval r and the *same* particle during the interval t. ${}_rR_t$ is defined as the correlation coefficient between the movement of the i-th particle during the r-th interval and the j-th particle during the t-th interval. The mean value of the term in brackets in (7) is zero, since the mean values of u_{ri} and u_{rj} are zero.

It would be interesting to know whether a relationship can be established between the Lagrangian correlations R_{rt} and the ${}_rR_t$ intercorrelations between the particles. Apparently relatively little is known on this point, either from theoretical considerations or from actual observation. Therefore, it may be appropriate to speculate on a *possible* connection between these correlations. Consider fig. 1, a symbolic diagram that presents the values $u_{1i}, u_{2i}, \cdots; u_{1j}, u_{2j}, \cdots$, *etc.* in a systematic arrangement to facilitate further study.

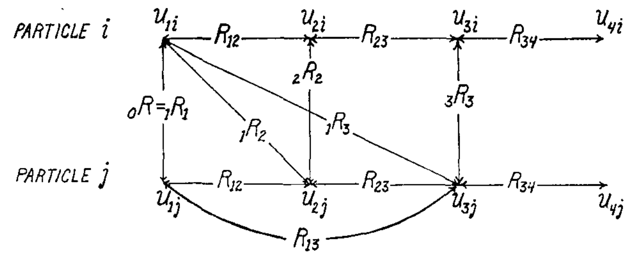

FIG. 1. Symbolic diagram illustrating the various correlations involving the movements of two particles.

The figure is not intended to represent the movement of particles in a plane, and the geometry of the pattern has no particular significance. It would appear quite reasonable to assume that u_{1i} and u_{2j} are only correlated because they are both correlated with u_{1j}, or, in other words, to assume that the partial correlation ${}_1P_2$ between variables u_{1i} and u_{2j} is zero when the effect of the variable u_{1j} is removed. The usual formula for the partial correlation coefficient gives

$${}_1P_2 = (1 - {}_1R_1^2)^{-\frac{1}{2}}(1 - R_{12}^2)^{-\frac{1}{2}}({}_1R_2 - {}_1R_1\,R_{12}),$$

so that ${}_1P_2 = 0$ when

$${}_1R_2 = {}_1R_1\,R_{12}, \quad (9)$$

providing, of course, neither ${}_1R_1^2$ nor R_{12}^2 is exactly unity. In the trivial case of ${}_1R_1 = 1$, where the particles i and j are infinitely close together, there is no diffusion (by definition) and ${}_1P_2$ is indeterminate.

By a similar argument, one would expect that

$${}_2R_2 = {}_1R_1(R_{12})^2,$$
$${}_1R_3 = {}_1R_1\,R_{13},$$
$${}_2R_3 = {}_1R_1\,R_{12}\,R_{13},\ etc.,$$

or, in general,

$${}_rR_t = {}_1R_1\,R_{1r}\,R_{1t} = {}_0R\,R_{1r}\,R_{1t}, \quad (10)$$

remembering that the trivial cases of ${}_1R_1 = 1$ and $R_{12} = 1$ are excluded by virtue of being of no particular interest.

If the value of ${}_rR_t$ given by (10) is substituted in (8), the result is

$$[l_n^2] = l_0^2 + 2[u^2]\{\sum_{r=1}^{n}\sum_{t=1}^{n} R_{rt} - {}_0R\sum_{r=1}^{n}\sum_{t=1}^{n} R_{1r}\,R_{1t}\}. \quad (11)$$

In the case where r and t vary continuously from time $t = 0$ to $t = T$, this equation can be represented as

$$[l_T^2] = l_0^2 + 2[u^2]\left\{\int_0^T\int_0^T R_{rt}\,dr\,dt - {}_0R\int_0^T\int_0^T R_{0r}\,R_{0t}\,dr\,dt\right\}. \quad (12)$$

If the transformation $\xi = t - r$ is made, $\partial\xi/\partial r = -1$

[2] The method of discrete motion used in the development of this equation is not intended to imply that the mechanism of turbulence is not continuous. An analogous formula can also be derived directly from considerations of the continuous motion of the particles.

and $R_{rt} = R_{r,r+\xi} = R_\xi$, where R_ξ is the correlation between the value of u for a particle at any instant and the value of u for the same particle after an interval ξ. Equation (12) can then be written

$$[l_T^2] = l_0^2 + 2[u^2]\left\{ 2\int_0^T \int_0^t R_\xi \, d\xi \, dt - {}_0R\left(\int_0^T R_\xi \, d\xi\right)^2 \right\}. \quad (13)$$

If the particles starting from a point source ($l_0 = 0$) are considered and it is *assumed* that ${}_0R = 0$, the result is that

$$\sigma_T^2 = \tfrac{1}{2}[l_T^2] = 2[u^2]\int_0^T \int_0^t R_\xi \, d\xi \, dt, \quad (14)$$

which is the formula derived by Taylor (1921). In addition to the mean square distance between the particles, it is also of interest to consider the quantity $S_n = l_0 + X_{ni} + X_{nj}$, which will be related to the center of gravity of the pair of particles at the end of n time intervals. The expected value of the square of S_n is given by

$$[S_n^2] = l_0^2 + 2[u^2]\{\sum_{r=1}^{n}\sum_{t=1}^{n} R_{rt} + \sum_{r=1}^{n}\sum_{t=1}^{n} {}_rR_t\}, \quad (15)$$

so that it follows from (8) that

$$[S_n^2] + [l_n^2] = 2l_0^2 + 4[u^2]\sum_{r=1}^{n}\sum_{t=1}^{n} R_{rt}. \quad (16)$$

Thus, the intercorrelation terms ${}_rR_t$ drop out. This is very interesting if considered from the point of view of the analysis of variance. For if we consider a large number of repeated experiments with sample pairs of particles, $[l_n^2]$ is proportional to the variance *within* samples while $[S_n^2]$ is proportional to the variation *between* the means of samples.

Equations (8), (15), and (16) thus suggest an indirect method of learning something about the R_{rt} and ${}_rR_t$ correlation if measurements are available regarding the *simultaneous* scatter of particles about their center of gravity, as well as measurements of the variation in the center of gravity from time to time. It appears that the Taylor formula (14) is more appropriate when the total variation $l_n^2 + S_n^2$ is being considered, but should not be used to express the *simultaneous* scatter of particles from their *mean* position at time T. For equation (8) clearly shows that consideration must be given to the additional terms that involve the correlations between particles i and j, and the initial distance l_0. If the particles start from a point,[3] these correlations cannot be zero since it would be expected that particles close together will tend to be affected by the same eddies while particles that are farther apart will show less correlation in their movements. Sutton (1932) has pointed out that "the outstanding weakness of the older theories of eddy diffusion lay in their inability to express the fact that, owing to the various sizes of eddies encountered, diffusivity in a turbulent fluid such as the atmosphere must be regarded as a function of the distance apart of the particles."

[3] In connection with using a diffusing dot as a "natural standard," Richardson (1926) states (italics his): "So we must conclude that *in the atmosphere a spreading dot will not serve as an element from which general distributions can be built up.*"

Sutton attempted to overcome this weakness by developing a theory which made use of the idea of an "effective eddy," which governed the rate of diffusion at each instant. However, he made use of Taylor's formula (14) as a basic starting point. Later, Taylor (1935) suggested a method of describing the scale of turbulence in the Eulerian system, based on the variation of the correlation coefficient R_y between the values of the component u at two points separated by the distance y in the direction of the y-coordinate, as y varied. A generalization of this concept was introduced by von Kármán's (1936) correlation tensor function. In this theory, the correlation coefficients between any component of the speed fluctuation at a given point and any component of the speed fluctuation at another point form a tensor. If one point is held fixed and the other varied, the tensor varies as a function of the coordinates of the variable point with respect to the fixed point.

These concepts, expressing the scale of turbulence in the Eulerian system, suggest a method of taking account of the distance apart of the particles. It will now be shown how an expression for the diffusion of particles might be obtained without need for assumption that the movements of the individual particles are independent or that they start from a point source. The object will be to obtain a general formula for the standard deviation σ_T of particles at time T, when the initial standard deviation is σ_0. Actually it is more convenient to consider the mean square distance between particles $[l_T^2]$, which is proportional to σ_T^2. The case of discontinuous motion will be treated, and for the sake of simplicity only diffusion in the x-direction will be considered. But there is no particular difficulty in generalization to three-dimensional isotropic turbulence or even to non-isotropic turbulence.

Consider the initial coordinates at time $t = 0$ of two particles, i and j, as x_{0i} and x_{0j}, respectively, as shown in fig. 2. The square of the distance between the particles is given by

$$l_0^2 = (x_{0j} - x_{0i})^2. \quad (17)$$

Suppose that the two particles are subjected to turbulent disturbance such that after a small *unit* in-

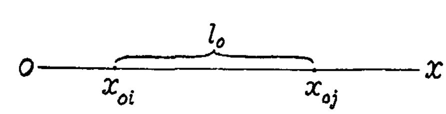

FIG. 2. Position of two particles i and j at time $t = 0$.

FIG. 3. Position of two particles i and j at time $t = 1$.

terval of time their new positions are given by

$$x_{1i} = x_{0i} + u_{1i},$$
$$x_{1j} = x_{0j} + u_{1j},$$

as shown in fig. 3. At the end of the first interval of time, $[l_1^2]$, the mean square of the distance between the particles, is given by

$$[l_1^2] = l_0^2 + [u_{1i}^2 + u_{1j}^2 - 2u_{1i}u_{1j}] \qquad (18)$$

$$= l_0^2 + 2[u^2](1 - {}_0R), \qquad (19)$$

where

$$[u_{1i}^2] = [u_{1j}^2] = [u^2], \qquad [u_{1i}] = [u_{1j}] = 0,$$

and

$$[u_{1i}u_{1j}] = [u^2]\, {}_1R_1 = [u^2]\, {}_0R. \qquad (20)$$

${}_0R$ is defined as the correlation between the movement of the two particles during the first interval of time. The subscript is placed in front to distinguish this correlation from R_ξ, which refers to the correlation between the movement of the *same* particle at two different times, t and $t + \xi$.

At the end of the second interval of time, $[l_2^2]$, the mean square of the distance between the particles, is given by

$$[l_2^2] = [l_1^2] + 2[u^2](1 - {}_1R) + 4[u^2](R_{12} - {}_1R_2), \qquad (21)$$

where

$${}_1R = {}_2R_2 = [u_{2i}u_{2j}][u^2]^{-1}.$$

${}_1R$ is a correlation coefficient analogous to ${}_0R$, except that it refers to the second interval of time.

In a similar manner, it can be shown that the mean square distance of the particles at the end of the third interval is

$$[l_3^2] = [l_2^2] + 2[u^2](1 - {}_2R) + 4[u^2](R_{23} - {}_2R_3) + 4[u^2](R_{13} - {}_1R_3), \qquad (22)$$

where ${}_2R = {}_3R_3$ is the correlation between the movement of the two particles during the third time interval.

Now, three different types of correlations have been discussed up to this point. The correlations R_ξ refer to the movement of the same particles at times t and $t + \xi$, and thus might be called the "Lagrangian" correlations. On the other hand, the correlations ${}_tR_t = {}_{t-1}R$ refer to the simultaneous movements of two particles at some distance l apart so they might be considered as related to the "Eulerian" correlations. The correlations ${}_rR_t$ are somewhat of a mixture, involving both space and time. In the special case where these correlations are connected by the relationship suggested by (10), the following may be written:

$${}_2R_2 = {}_1R = {}_0R\, R_{12}^2,$$
$${}_2R = {}_0R\, R_{13}^2,$$
$${}_1R_2 = {}_0R\, R_{12},$$
$${}_1R_3 = {}_0R\, R_{13},$$
$${}_2R_3 = {}_1R\, R_{23}, \textit{ etc.},$$

from which it can be deduced that

$$R_{12} = ({}_1R/{}_0R)^{\frac{1}{2}}, \qquad {}_1R_2 = {}_0R({}_1R/{}_0R)^{\frac{1}{2}},$$
$$R_{23} = R_{12} = ({}_1R/{}_0R)^{\frac{1}{2}}, \qquad {}_2R_3 = {}_1R({}_1R/{}_0R)^{\frac{1}{2}},$$
$$R_{13} = ({}_2R/{}_0R)^{\frac{1}{2}}, \qquad {}_1R_3 = {}_0R({}_2R/{}_0R)^{\frac{1}{2}}.$$

By use of these relationships, the formulae for $[l_1^2]$, $[l_2^2]$, $[l_3^2]$, $\cdots$ $[l_n^2]$ become

$$[l_1^2] = l_0^2 + 2[u^2](1 - {}_0R), \qquad (23)$$

$$[l_2^2] = [l_1^2] + 2[u^2](1 - {}_1R) + 4[u^2]({}_1R/{}_0R)^{\frac{1}{2}}(1 - {}_0R), \qquad (24)$$

$$[l_3^2] = [l_2^2] + 2[u^2](1 - {}_2R) + 4[u^2]({}_1R/{}_0R)^{\frac{1}{2}}(1 - {}_1R) + 4[u^2]({}_2R/{}_0R)^{\frac{1}{2}}(1 - {}_0R), \qquad (25)$$

$$[l_n^2] = [l_{n-1}^2] + 2[u^2](1 - {}_{n-1}R) + 4[u^2]\, {}_0R^{-\frac{1}{2}}\{{}_1R^{\frac{1}{2}}(1 - {}_{n-2}R) + {}_2R^{\frac{1}{2}}(1 - {}_{n-3}R) + \cdots + {}_{n-1}R^{\frac{1}{2}}(1 - {}_0R)\}. \qquad (26)$$

Now, assume that the correlation between the movements of particles i and j during a small interval of time is a function of the distance apart of the particles. Then

$${}_0R = R(l_0) = C, \qquad (27)$$

since the initial distance l_0 is considered fixed. The value of $[l_1^2]$ is thus determined by (23). It would be expected that the correlation ${}_1R$ between the movements of the two particles during the second time interval will in general be different from ${}_0R$ and will be a function of $[l_1^2]$, so if

$${}_1R = R([l_1^2]^{\frac{1}{2}}), \qquad (28)$$

the value of $[l_2^2]$ is determined by (24). Thus, if the Eulerian correlation function $R([l^2]^{\frac{1}{2}})$ were known, the mean square distance between particles at the end of n time intervals could be determined by the summation equation (26). If more than two particles were involved, as would usually be the case, it would be necessary to perform the additional summation over all i and j indicated by (4) in order to express the mean square distance in terms of the standard deviation of the particles.

Thus, a possibility appears for expressing the scatter of particles as a function of Eulerian coefficients of correlation if measurements can be made relating the simultaneous motions of two particles of air at different distances apart. Such measurements appear to be more feasible than measuring the correlation between motions of an individual particle of air at successive instants of time.[4] An investigation now under way suggests that the function $R([l^2]^{\frac{1}{2}})$ can be determined, at least approximately, from the Eulerian correlation function $R(l)$. This will not be discussed here, but will form the basis for a later paper.

Since the diffusion of particles depends upon $[u^2]$, the correlation function R_ξ, and the intercorrelations $_rR_t$ between the particles, it would appear that the problem resolves itself into one of measuring these quantities either directly or indirectly. Direct measurement of these correlations by following the paths of individual particles would be difficult or impossible in the atmosphere, especially if small particles or large distances were involved. It might be possible to follow the trajectory of larger objects, such as soap bubbles or balloons, for a considerable distance and thus obtain estimates of the quantities necessary in the statistical theory. This would raise the question as to the effect of the size of the balloons on the measurements of quantities such as $[u^2]$. Intuitively, it would seem that a free balloon would be responsive only to eddies that were of the order of magnitude of the balloon size or larger. Smaller eddies, that might be very effective in diffusing small particles such as smoke, might be quite ineffective in producing scattering in objects as large as two or three meters in diameter. In order to obtain some rough quantitative estimate of this effect, consider the following mathematical model.

Suppose that the distribution of the u components along the y-axis for some small unit interval of time is represented by a curve such as that shown in fig. 4. Thus, for every position $y_i (i = 1, 2, 3, \cdots)$ there is a corresponding value of u_i. During the unit interval of time, the i-th particle will have moved in the direction of the x-axis a distance u_i, so that the scatter of a large number of particles after the small time interval has elapsed will be given by $\sigma^2 = [u^2]$ by virtue of the original definition of $[u^2]$.

[4] In this connection, the following comments by Sheppard (1947) are quite pertinent: "But Taylor deals with a correlation coefficient, R_ξ, which may be termed Lagrangian, in that it deals with the correlation between motions of an individual particle of air at successive instants of time and, as such, is almost impossible to measure directly. One looks forward to the time when a relation may be established between R_ξ and an Eulerian coefficient of correlation, *i.e.*, of the correlation between the motions of air at a fixed point at successive instants, for the experimentalist could then establish a bridge between theory and actual behavior at the most important stage of development."

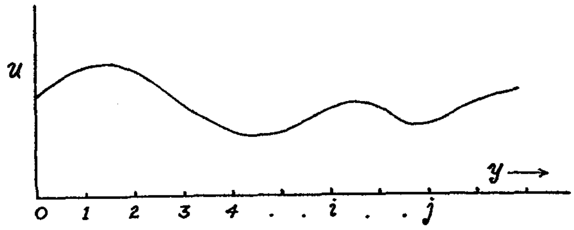

FIG. 4. Illustration of distribution of u velocity component along y-axis for some arbitrary time.

Suppose there is an object of length OY being acted upon by N particles with individual velocity components $u_1, u_2, u_3, \cdots u_N$ in such a way that the effective or average u component acting on this object is given by

$$\tilde{u} = N^{-1}(u_1 + u_2 + u_3 + \cdots + u_N). \qquad (29)$$

The standard deviation ($\sigma_{\tilde{u}}$) of a large number of such objects is then defined by

$$\begin{aligned}\sigma_{\tilde{u}}^2 &= N^{-2}\{[u_1^2] + [u_2^2] + \cdots \\ &\qquad + [u_N^2] + \cdots + 2[u_1u_2] + \cdots\} \\ &= N^{-2}\sum_{i=1}^{N}\sum_{j=1}^{N}[u_iu_j] = N^{-2}[u^2]\sum_{i=1}^{N}\sum_{j=1}^{N}{}_{ij}R,\end{aligned} \qquad (30)$$

since $[u_iu_j] = [u^2]\,{}_{ij}R$, where ${}_{ij}R$ is the correlation coefficient between u_i and u_j. The quantity $\sigma_{\tilde{u}}^2$ will be less than $[u^2]$, for it is well known that the variability of a mean is, in general, less than the variability of the individual components making up the mean. Thus, it would be expected that objects large enough to be acted upon by a number of particles with variance $\sigma^2 = [u^2]$ will show a variance less than $[u^2]$.

Following the procedure used earlier, (30) may be represented as

$$\sigma_{\tilde{u}}^2 = 2Y^{-2}[u^2]\int_0^Y\int_0^y R_y\,dy\,dY, \qquad (31)$$

where Y represents the length of the object on the y-axis and R_y is the correlation coefficient between the values of u at two points distance y apart in the direction of the y-coordinate. Taylor (1935) has shown how the variation of R_y against y might be used to describe the scale of turbulence in the Eulerian system. If the average diameters of the eddies are large, for example, a high degree of correlation must exist between velocities at two points which are close together when compared with this diameter. On the other hand, the correlation is likely to be small between the velocity at two points situated many eddy-diameters apart. Thus, when R_y approaches zero rapidly, the "average size of the eddies" might be considered as small compared with the case where R_y approaches zero slowly.

If the correlation function R_y can be represented by

$$R_y = \exp(-y/B), \qquad (32)$$

where B is a constant, (31) can be integrated and becomes

$$\sigma_{\tilde{u}}^2 = 2[u^2]\{BY^{-1} - (BY^{-1})^2(1 - \exp[-YB^{-1}])\}. \qquad (33)$$

This becomes

$$\sigma_{\tilde{u}}^2 = 2[u^2]BY^{-1}, \qquad (34)$$

when B is small relative to Y, *i.e.*, when the size of the object is large compared to the "average size of the eddies." Frenkiel (1946) has obtained this same result under more general conditions than those indicated by

(32). Under these assumptions, doubling the length of the object would reduce the standard deviation by 30 per cent. No doubt the aerodynamics of floating balloons is more complicated than this, but it would appear, at least qualitatively, that the effect of using a group of balloons of larger size would be to decrease the apparent scattering.

3. A diffusion experiment

To study the diffusion of particles in the free atmosphere, a series of experiments using meteorological balloons was carried out in November 1947 at the U. S. Weather Bureau's Cloud Physics Project near Wilmington, Ohio.[5] A number of 350-g balloons with attached radar targets were available, and the original plan called for the simultaneous release of groups of about 10 balloons arranged in various geometric patterns over a relatively small area. It was expected that a very large and powerful "V-Beam" search-radar would permit determination of the positions of the balloons at 10-sec intervals by photography of the scope. After some initial experimentation, it was concluded that the balloons should be released about $\frac{1}{2}$ mile apart in order to obtain satisfactory images of individual balloons on the radar scope. At 1500 EST 20 November, ten balloons arranged in a circle, with one balloon in the center, were released simultaneously from an altitude of 100 ft above the ground or about 1170 ft above sea level. The diameter of the circle of balloons was approximately three miles, and a rawinsonde balloon was released from the center of the circle at the same time. The targets were initially about 22 miles from the radar, and it was not until after the balloons had risen a few thousand feet in the first five minutes that the targets became visible on the radar scope. The first photograph showing all ten balloons was taken at 1504, and the next, at 1505, is shown in fig. 5. During the next 55 min, the balloons traveled a distance of about 50 miles, reaching an elevation of approximately 35,000 ft. During this time it was possible to determine the horizontal positions of the balloons at 10-sec intervals, except for a few occasions when a balloon target was turned in such a way as to produce no visible image on the scope. Fig. 6 shows the cluster of balloons at 1554. Two of the ten balloons were apparently defective, one lagging behind the others both in vertical and horizontal distance covered, and one bursting about 20 minutes after release.

From the series of photographs of the scope, it was possible to take measurements and plot the path of each balloon in the horizontal plane. Since direct measurements of the altitudes of each balloon were not available, it was necessary to estimate the altitude of a balloon at a particular time by comparing the variations of the path in the horizontal plane with the winds as given by the rawinsonde. The rawinsonde indicated winds varying from approximately 10 to 110 mi hr^{-1} in the first 20,000 feet above the surface, with directions ranging between north-northeast and west. The general movement of the balloon cluster was toward the east-southeast. From a careful examination of the horizontal trajectories and the wind field, it was possible to obtain satisfactory estimates of the altitude of each balloon at specified times, and thus determine the individual ascension rates.

On the basis of this analysis, it was found that there was considerable variation between the balloons in their ascension rates. This resulted in some of the balloons reaching the fast westerly winds sooner than others. The question then arises as to whether the changes in the positions of the balloons in the horizontal plane relative to their center of gravity, as indicated by figs. 5 and 6, were produced primarily by diffusion, divergence, or some other cause, such as the effect of vertical wind-shear. Since the elevation of each balloon had been determined as a function of time, it was possible to make a prediction as to where the balloon should be after an elapsed time T if it were acted upon by the wind field as given by the rawinsonde observation. The computed positions in the horizontal plane were in close agreement with the positions observed on the radar scope, indicating that practically the entire variation of the positions of the balloons from their initial positions can be explained by the effect of wind shear combined with the non-uniform ascension rates of the balloons. The differences between the observed and computed positions of the balloons are of the order of magnitude of the errors introduced by the measurements on the photographs and by the numerical integration of the time $\times$ height $\times$ speed curves used.

These results certainly suggest that the effect of wind shear on the differential ascension rates of a cluster of balloons makes it difficult, if not impossible, to obtain measurements of diffusion under such circumstances as reported here. Some of these difficulties might be overcome by the use of "constant-level" balloons, but even here there is the problem that the random turbulent components in the z-direction would produce some scattering in the z-direction, and, if there is a change of wind speed with height, some balloons would move farther in the x-direction than others. The standard deviation of the balloons along the x-axis would then not only be a function of $[u^2]$, but a function of $[u^2]$, $[w^2]$, and dU/dz. Furthermore, there is the complicating factor that the size of the standard weather balloon changes with height, making it possible for the balloon to respond to different size eddies at various elevations, as suggested by the previous

[5] A more complete report describing this experiment and presenting data in greater detail will be published elsewhere at a later date.

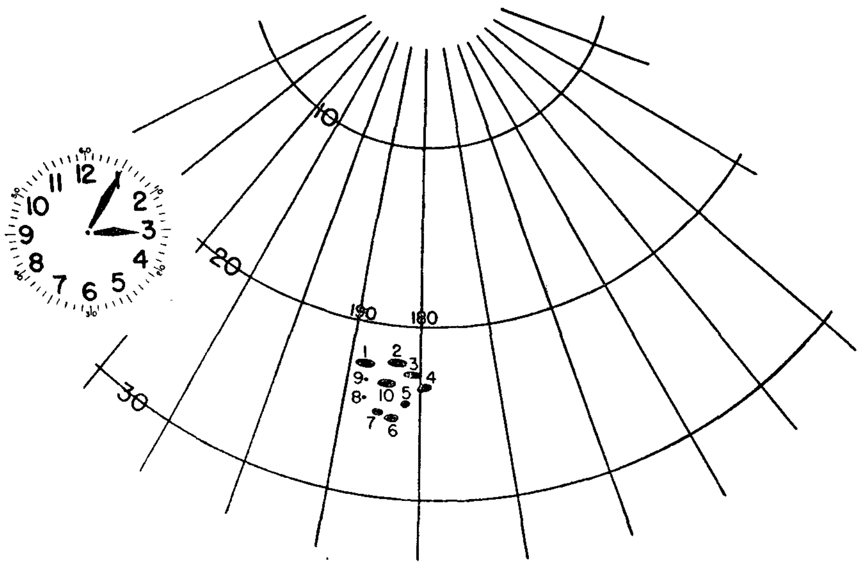

Fig. 5. Radar scope picture showing position of balloons at 1505.

mathematical analysis. This difficulty might be overcome to some extent by the use of constant-level and constant-volume balloons, but it would seem desirable to use several sizes of balloons in order to study the effect of balloon size.

4. Summary

To summarize briefly the arguments of the preceding sections, it is suggested here that the diffusion of particles in the free atmosphere depends upon the initial distance apart of the particles, and that Taylor's formula,

$$\sigma_T^2 = 2[u^2] \int_0^T \int_0^t R_\xi \, d\xi \, dt,$$

can be generalized to make allowance for this initial separation of the particles if information regarding the Eulerian correlation function $R(l)$ is available. Furthermore, it is suggested that the use of the Lagrangian correlations R_ξ might be avoided by using a function which depends explicitly on the Eulerian correlation function $R(l)$. The question is also raised as to the usefulness of balloons in measuring eddy diffusion in the free atmosphere, and a description is given of an unsuccessful experimental attempt to study diffusion by means of following target balloons with radar equipment.

In connection with the application of the statistical theory of turbulence to diffusion in the free atmos-

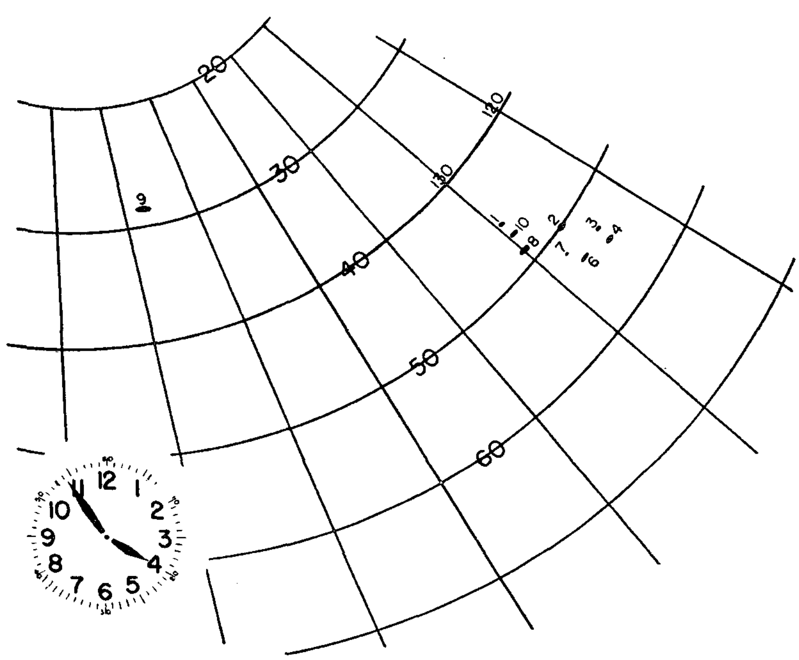

Fig. 6. Radar scope picture showing position of balloons at 1555.

phere, it seems to the writer that the greatest need is for observations that will shed some light on the Eulerian correlations $R(l)$ and how these correlations are related to the Lagrangian correlations R_ξ. Until such information is available, it would appear that questions regarding the rate of diffusion of particles in the free atmosphere must remain largely unanswered.

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TABLE OF SYMBOLS

U, V, W	The velocity components of the mean motion of the turbulent fluid in the direction of the x-, y-, and z-axes, respectively
u, v, w	At any arbitrary point, the instantaneous values of the velocity components superposed on the mean flow and taken in the direction of the x-, y-, and z-axes, respectively
u_{ri}, u_{rj}	For discrete motion, the velocity components in the x-direction during the r-th interval of time for particles i and j, respectively
$\tilde{u}$	The mean value of the velocity component u taken over some small specified region
$[u^2], [v^2], [w^2]$	The mean value of the square of the three superposed velocity components
x_{ri}, x_{rj}	The x-coordinate of the particles i and j, respectively, at the end of the r-th time interval
X_{ni}, X_{nj}	The total distance traveled, relative to the mean flow and in the direction of the x-coordinate, during n elementary time intervals by particles i and j respectively
l_0	The initial distance between two particles
l_n, l_T	The distance between two particles after n time intervals, or after time T has elapsed
σ_n^2, σ_T^2	The square of the standard deviation of N particles after n time intervals, or after time T has elapsed
R_{rt}	The (Lagrangian) correlation coefficient between the movement of a particle during interval r and the same particle during the interval t
R_ξ	The (Lagrangian) correlation between the velocity of a fluid particle at any instant and that of the same particle after a time interval ξ
R_y	The correlation between the values of u at two points distance y apart in the direction of the y-coordinate
${}_rR_t$	The correlation coefficient between the movement of one particle during the r-th interval and another particle during the t-th interval
${}_rP_t$	The partial correlation coefficient between the movement of one particle during the r-th interval and another particle during the t-th interval
${}_tR_t$	$= {}_{t-1}R$. The correlation coefficient between the movement of one particle during the t-th interval and another particle (at unspecified distance) during the same interval
$R(l)$	The (Eulerian) correlation coefficient between the velocity of two particles separated by distance l
[ ]	The mean value of the quantity within the bracket, either taken over space, time, or considered as the mathematical expectation over a large number of trials.

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