Nonlinear Feedback in a Five-Dimensional Lorenz Model

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ABSTRACT

In this study, based on the number of modes, the original three-dimensional Lorenz model (3DLM) is generalized with two additional modes [five-dimensional Lorenz model (5DLM)] to examine their role in the predictability of the numerical solutions and to understand the underlying processes that increase the solution stability. As a result of the simplicity of the 5DLM with respect to existing generalized Lorenz models (LMs), the author is able to obtain the analytical solutions of its critical points and identify the role of the major nonlinear term in the solution's stability, which have previously not been documented in the literature. The nonlinear Jacobian terms of the governing equations are analyzed to highlight the importance of selecting new modes for extending the nonlinear feedback loop of the 3DLM and thus effectively increasing the degree of nonlinearity (i.e., the nonlinear mode–mode interactions) in the 5DLM. It is then shown that numerical solutions in the 5DLM require a larger normalized Rayleigh number \( r \) for the onset of chaos and are more predictable than those in the 3DLM when \( r \) is between 25 and 40 and the Prandtl number \( \sigma \) is 10. The improved predictability is attributable to the negative nonlinear feedback enabled by the new modes. The role of the (negative) nonlinear feedback is further verified using a revised 3DLM with a parameterized nonlinear eddy dissipative term. The finding of the increased stability in the 5DLM and revised 3DLM with respect to the 3DLM is confirmed with the linear stability analysis and the analysis of the Lyapunov exponents using different values of \( r \) and \( \sigma \). To further understand the impact of an additional heating term, results from the 5DLM and a higher-dimensional LM [e.g., the six-dimensional LM (6DLM)] are analyzed and compared.

1. Introduction

Since the 1960s, when Lorenz presented the sensitive dependence of numerical solutions on initial conditions (ICs) with simplified governing equations describing a two-dimensional, forced, dissipative Rayleigh–Benard convection, it has been widely recognized that perfect deterministic weather predictions are impossible. In his breakthrough modeling study where three spatial Fourier modes were used to represent the streamfunction and temperature perturbations of the convection, Lorenz (1963a) showed that numerical results become chaotic, with sensitivities to ICs, when a normalized Rayleigh number \( r \) exceeds a critical number (e.g., \( r_c = 24.74 \) for a constant Prandtl number \( \sigma = 10 \)). This model is referred to as a three-dimensional Lorenz model (3DLM) in the present study. Lorenz associated the chaotic behavior with the inclusion of the nonlinearity. Subsequent to his follow-up presentation in 1972 (Lorenz 1972), the term “butterfly effect” was introduced to describe the sensitive dependence on ICs; later this became a metaphor (or symbol) for indicating that small-scale perturbations can make a huge impact on large-scale flows. In this study, the former and latter definitions are referred to as the butterfly effect of the first and second kind, respectively. The studies by Lorenz laid the foundation for chaos theory, which was viewed as the third scientific revolution of the twentieth century after relativity and quantum mechanics and is being applied in various fields including earth science, mathematics, philosophy, and physics (e.g., Gleick 1987; Anthes 2011).

Since the publications of Lorenz (1963a, 1972), views regarding the predictability of weather and climate have been significantly influenced by the butterfly effect (of the first and second kinds) or chaos theory (Solomon et al. 2007, 96–97; Pielke 2008). It is well accepted that weather is chaotic with only a finite predictability, and it
is believed that the source of chaos in the 3DLM is the nonlinearities. Based on this understanding, one might expect that solutions to the equations with more nonlinear modes would become more chaotic, equivalent to stating that the appearance of small-scale features and their nonlinear effects, resolved by the additional modes, may make the system more chaotic. Since high-resolution global modeling approaches (e.g., Atlas et al. 2005; Shen et al. 2006a,b), which require tremendous computing resources, have become a current trend for weather prediction and climate projection, it becomes important to understand the role of the increased resolutions in the solution’s stability (or predictability) of the models.

Three kinds of predictability that were proposed by Lorenz (1963b) include 1) intrinsic predictability that is dependent only on a flow itself, 2) attainable predictability that is limited by the imperfect initial conditions, and 3) practical predictability that shows dependence on (mathematical) formulas. The last type is discussed in this study by deriving a generalized Lorenz model (LM) and comparing its predictability with that of the 3DLM. In the literature, the term “a generalized LM” has been used to refer to the model that has modes more than the 3DLM.

Previous studies with the inclusion of additional Fourier modes have suggested that a larger \( r \) is required for the onset of chaos in generalized LMs [e.g., \( r_c \) of approximately 43.5 in the generalized LM with 14 modes (Curry 1978)]. As compared to the aforementioned studies, a more systematic study for examining the resolution dependence of chaotic solutions was conducted numerically by Curry et al. (1984). They observed an irregular change in the degree of chaos as the resolution increased from a low resolution (i.e., three Fourier modes) and obtained a steady-state solution with sufficiently high resolution. However, as the resolution of the numerical weather models is finite and has always been increasing incrementally, it is important to understand the role of the incremental degree of nonlinearity in the solution’s stability. The term “degree of nonlinearity” is loosely defined as the degree of mode–mode interactions and is introduced to emphasize that the nonlinearities in numerical models such as the 3DLM are truncated (or finite) as a result of mode truncation. A more specific definition is given in section 3a. In a recent study of the routes to chaos in generalized LMs, Roy and Musielak (2007a) emphasized the importance in selecting modes that can conserve the system’s energy in the dissipationless limit. Furthermore, by analyzing the onset of chaos in the 3DLM and different generalized LMs with five and up to nine modes, Roy and Musielak (2007a,b,c) reported that some generalized LMs required a larger \( r (r_c \sim 40) \) for the onset of chaos, but others displayed a comparable (e.g., \( r_c \sim 24.74 \) in one of their LMs with six modes) or even a smaller \( r_c \) (e.g., \( r_c \sim 22 \) in their LM with five modes).

The aforementioned studies give an inconclusive answer to the question of whether higher-dimensional LMs are more stable (predictable). A possible reason for this discrepancy among existing generalized LMs is presumably related to the various truncations of modes, leading to different degrees of nonlinearity. This is addressed using the following question in this study: under which conditions could the increased degree of nonlinearity improve solution stability? We will address this question with generalized LMs in this study.

In this study, we extend the 3DLM to the five-dimensional LM (5DLM) by including two additional Fourier modes with two additional vertical wavenumbers. In a companion paper (B.-W. Shen 2013, unpublished manuscript), we extend the 5DLM to the six-dimensional LM (6DLM) with an additional mode. Although the nonlinear mode–mode interactions in the 5DLM (6DLM) are still much less complicated than those in global weather models (e.g., Shen et al. 2006a,b) or the model used by Curry et al. (1984), they can be analyzed analytically to trace their impact on solution stability, illustrating the importance of proper selection in the new modes that can effectively increase the degree of nonlinearity. For example, we will discuss how additional nonlinear and damping terms, which are introduced in the 5DLM, can provide negative nonlinear feedback for improving the solution stability. The term “improvement of solution stability” is defined as the disappearance of a positive Lyapunov exponent (LE; e.g., Wolf et al. 1985) or appearance of a stable nontrivial critical point in a generalized LM that has the same system parameters (e.g., the normalized Rayleigh number) as the 3DLM. In a companion paper (B.-W. Shen 2013, unpublished manuscript) with the 6DLM, we further examine the competing impact of an additional heating term as compared to the dissipative and nonlinear terms that are first introduced in the 5DLM. In sections 2a and 2b of this study, we describe the governing equations and present the derivations of the 5DLM. In section 2c, we propose a revised 3D Lorenz model with a “parameterized” term (denoted 3DMP) that can effectively emulate the impact of the negative nonlinear feedback that is explicitly resolved in the higher-dimensional (5D) LMs. In section 2d, we present the analytical solutions of the critical points in the 5DLM, 3DLM, and 5DMP. Numerical approaches for the integrations of these models and the calculations of the Lyapunov exponents are discussed in section 2e. In sections 3a and 3b, we use mathematical equations to illustrate the nonlinear feedback loop in the 3DLM and discuss how the feedback loop can be extended with proper selection of new modes in the 5DLM. We then refer to the degree of the extension of the feedback loop, which depends on the number of modes and their hierarchical-scale interactions.
Table 1. The six modes and their derivatives. Here \( l \) and \( m \), representing the horizontal and vertical wavenumbers, are defined as \( \pi a/H \) and \( \pi l/H \), respectively. Also, \( H \) is the vertical scale of the convection, and \( a \) is a ratio of the vertical scale to the horizontal scale. The term \( \partial^2/\partial x^2 \) indicates that the outcome is expressed in terms of the selected modes. Mode \( M_4 \) is used only in the 6DLM of Shen (2013).

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \frac{\partial}{\partial x} )</th>
<th>( \frac{\partial^2}{\partial x^2} )</th>
<th>( \frac{\partial}{\partial z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 = \sqrt{2} \sin(lx) \cos(mz) )</td>
<td>( \sqrt{2} \cos(lx) \sin(mz) )</td>
<td>( lM_2 )</td>
<td>( \sqrt{2}m \cos(lx) \sin(mz) )</td>
</tr>
<tr>
<td>( M_2 = \sqrt{2} \cos(lx) \sin(mz) )</td>
<td>( -\sqrt{2} \sin(lx) \sin(mz) )</td>
<td>( -IM_1 )</td>
<td>( 2m \cos(2mx) )</td>
</tr>
<tr>
<td>( M_3 = \sin(2mx) )</td>
<td>( 0 )</td>
<td>( 2m \cos(2mx) )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( M_4 = \sqrt{2} \sin(lx) \sin(3mx) )</td>
<td>( \sqrt{2} \cos(lx) \sin(3mx) )</td>
<td>( lM_5 )</td>
<td>( 3 \sqrt{2}m \cos(lx) \sin(3mx) )</td>
</tr>
<tr>
<td>( M_5 = \sqrt{2} \cos(lx) \sin(3mx) )</td>
<td>( -\sqrt{2} \sin(lx) \sin(3mx) )</td>
<td>( -IM_4 )</td>
<td>( 3 \sqrt{2m} \cos(3mx) \sin(3mx) )</td>
</tr>
<tr>
<td>( M_6 = \sin(4mx) )</td>
<td>( \sqrt{2} \cos(lx) \sin(4mx) )</td>
<td>( 4m \cos(4mx) )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

(i.e., interconnectivity or interactions of interactions) in the LMs, as the degree of nonlinearity. Then, we present the numerical results of the 3DLM, 5DLM, and 3DLM in sections 3c and 3d. In section 3e, we discuss the dependence of the solution’s stability on the Prandtl number in the \((\sigma, r)\) space. Conclusions appear at the end.

2. The generalized Lorenz models and numerical methods

a. The governing equations

By assuming 2D \((x, z)\) Boussinesq flow, the following equations were used in Saltzman (1962) and Lorenz (1963a):

\[
\begin{align*}
\frac{\partial}{\partial t} \nabla^2 \psi &= -J(\psi, \nabla^2 \psi) + \nu \nabla^4 \psi + \frac{g \alpha}{H} \frac{\partial \theta}{\partial x}, \\
\frac{\partial \theta}{\partial t} &= -J(\psi, \theta) + \frac{\Delta T}{H} \frac{\partial \psi}{\partial x} + \kappa \nabla^2 \theta,
\end{align*}
\]

(1)

(2)

where \( \psi \) is the streamfunction that gives \( u = -\psi_z \) and \( w = \psi_x \), which represent the horizontal and vertical velocity perturbations, respectively, \( \theta \) is the temperature perturbation, and \( \Delta T \) is the difference in temperature between the top and bottom boundaries. The constants \( g \), \( \alpha \), \( \nu \), and \( \kappa \) denote the acceleration of gravity, the coefficient of thermal expansion, the kinematic viscosity, and the thermal diffusivity (or thermal conductivity), respectively. The Jacobian of two arbitrary functions is defined as

\[
J(A, B) = \frac{\partial A}{\partial x} \frac{\partial B}{\partial z} - \frac{\partial A}{\partial z} \frac{\partial B}{\partial x},
\]

(3)

\[
\nabla^4 \psi = \frac{\partial}{\partial x} \left( \nabla^2 \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial z} \left( \nabla^2 \frac{\partial \psi}{\partial z} \right).
\]

(4)

For the reader’s convenience, we use the same symbols as those in Lorenz (1963a).

b. The 5D Lorenz model

To derive the 5DLM, we use the following five Fourier modes (which are also listed in Table 1):

\[
\begin{align*}
M_1 &= \sqrt{2} \sin(lx) \sin(mz), \\
M_2 &= \sqrt{2} \cos(lx) \sin(mz), \\
M_3 &= \sin(2mx), \\
M_4 &= \sqrt{2} \sin(lx) \sin(3mx), \\
M_5 &= \sqrt{2} \cos(lx) \sin(3mx), \\
M_6 &= \sin(4mx).
\end{align*}
\]

(5)

(6)

Here \( l \) and \( m \) are defined as \( \pi a/H \) and \( \pi l/H \), representing the horizontal and vertical wavenumbers, respectively, and \( a \) is a ratio of the vertical scale of the convection cell to its horizontal scale, i.e., \( a = 1/lm \). The term \( H \) is the domain height, and \( 2H/\alpha \) represents the domain width. An additional mode \( M_4 = \sqrt{2} \sin(lx) \sin(3mx) \) is included to derive the 6DLM, and a comparison of the 6DLM with the 5DLM will be made to examine the impact of an additional heating term (B.-W. Shen 2013, unpublished manuscript). With the five modes in Eqs. (5) and (6), \( \psi \) and \( \theta \) can be represented as

\[
\begin{align*}
\psi &= C_1 (XM_1), \\
\theta &= C_2 (YM_2 - ZM_3 + Y_1M_5 - Z_1M_6),
\end{align*}
\]

(7)

(8)

\[
C_1 = \kappa \frac{1 + a^2}{a}, \quad C_2 = \frac{\Delta T}{\pi} \frac{R_e}{R_a},
\]

\[
R_e = \frac{\pi^4}{a^2}(1 + a^2)^3, \quad R_a^{-1} = \frac{\nu \kappa}{g \alpha H^2 \Delta T},
\]

(9)

where \( C_1 \) and \( C_2 \) are constants, \( R_a \) is the Rayleigh number, and \( R_e \) is its critical value for the free-slip Rayleigh–Benard problem. With Eqs. (7) and (8), solutions in the 5DLM are represented by the five spatial modes \( M_1 - M_3 \) and \( M_5 - M_6 \) and their corresponding time-varying amplitudes \( (X, Y, Z, Y_1, Z_1) \), respectively. In the original 3DLM, only three modes \( (M_1, M_2, M_3) \) with their amplitudes \( (X, Y, Z) \) were used. While the 3DLM and 5DLM have one horizontal wavenumber, they have two and four vertical wavenumbers, respectively. With these modes, the partial differential equations [Eqs. (1) and (2)] can be transformed into ordinary differential equations with only \( \partial^2/\partial t^2 \) retained. Note
that an implicit limitation of this approach is that the nonlinear interactions among the selected modes cannot generate (impact) any new (other) modes that are not preselected, suggesting limited (spatial)-scale interactions. In other words, nonlinear mode–mode interactions are predetermined and limited by the finite number of selected modes. The impact of additional modes \((Y_1, Z_1)\) on the improvement of nonlinear interactions and numerical solutions is discussed in section 3.

To transform Eqs. (1) and (2) into the phase space, a major step is to calculate the nonlinear Jacobian functions. As a result of \(J(M_1, M_1) = 0\), there is no explicit term associated with \(J(\psi, \nabla^2 \psi)\) in the 3DLM or 5DLM. In contrast, the Jacobian term of Eq. (2) can be approximated by four Jacobian terms with selected Fourier number or the heating parameter, \(M_1\).

The 3DLM contains only two nonlinear terms \((-XZ\) and \(XY\)). These two terms form a nonlinear feedback loop, which is discussed in section 3a. A comparison between the 3DLM and Eqs. (10)–(12) of 5DLM shows that the only difference between them is the nonlinear feedback term \(-XY_1\), where \(Y_1\) is missing in the 3DLM. One may wonder if it is possible to emulate the impact of \(-XY_1\) by representing \(Y_1\) with the existing modes (i.e., resolved modes) of the 3DLM. Indeed, this can be achieved by comparing the solutions of the lower- and higher-dimensional LMs (i.e., the 3DLM and 5DLM), namely, the coarser- and finer-resolution models. Based on the analysis of the critical point solutions in the next section and the calculation of the ratio of \(Y_1\) to \(X\) from numerical results (not shown), we assume \(Y_1\) to be linearly proportional to \(X\). As shown in appendix A, that potential energy may cascade from mode \(M_3\) to mode \(M_5\) and dissipate subsequently [e.g., Eqs. (A7) and (A8)], we therefore propose to emulate (or approximate) the feedback processes associated with \(-XY_1\) using an eddy dissipation term \(-qX^2\), where \(q\) is a tunable nonnegative parameter \((q \geq 0\)). Mathematically, we can express \(q\) as a function of time. However, to illustrate the nature of the negative feedback without the loss of generality, we simply assume \(q\) to be a constant. The procedure of emulating the unresolved term \(-XY_1\) using the resolved term \(X^2\) is indeed parameterization, per se. Therefore, Eq. (17) [or Eq. (12)] is modified to become

\[
\frac{dX}{d\tau} = -\sigma X + \sigma Y, \\
\frac{dY}{d\tau} = -XZ + rX - Y, \\
\frac{dZ}{d\tau} = XY - bZ, \\
\frac{dY_1}{d\tau} = XZ - 2XZ_1 - d_0 Y_1, \\
\frac{dZ_1}{d\tau} = 2XY_1 - 4bZ_1. 
\]
\[
\frac{dZ}{dt} = XY - qX^2 - bZ. \tag{18}
\]

Equations (15), (16), and (18) form the revised 3DLM with the parameterized feedback term. To facilitate the discussions below, the revised 3DLM with a reasonable value of \( q \) is referred to as revised 3DLM, revised 3DLMP, or 3DLMP. The choice of \( q \) is to improve the solution’s stability as well as to produce “reasonable” results, which are discussed in section 3d.

d. Analytical solutions of critical points

Critical points are defined as the solutions to the set of the simultaneous algebraic equations derived from Eqs. (10)–(14) or (15)–(17) with no time-dependent terms (Bender and Orszag 1978). Critical points are also called equilibrium points or fixed points in the literature. Three kinds of critical points are categorized, including a stable node, an unstable node, and a saddle node. The solution of an LM is called a trajectory in the phase space. For a stable (unstable) node, all trajectories converge toward (diverge away from) the critical point. For a saddle node, some trajectories may move toward the critical point, while others may move away from it. A nonlinear system with an unstable (nontrivial) critical point may show a sensitive dependence of solutions on initial conditions and thus are less predictable than the system with only stable critical points. As a trajectory approaches a stable critical point, the solutions, which are normalized or rescaled by the values at a critical point, should eventually become positive (or negative) one. Therefore, the evolution of the differences between the normalized solution and the unity can be a good indicator of whether the solution reaches a steady state. This criterion is potentially useful for examining the time evolution of a system over a wide range of values in the \( r \), as to be illustrated in section 3. To calculate the normalized solution, we need first to solve for the critical point(s) analytically or numerically. It has been challenging to achieve this because of the following. First, in general, it is not easy to obtain the analytical solutions of the critical points in a nonlinear generalized LM containing more equations than the original 3DLM. Second, critical points are usually a function of multiple parameters (such as \( r \) and \( b \)) and thus cannot be determined numerically prior to time integration. In other words, given a specific set of parameters, (stable) critical points can be obtained only after the completion of the integration that eventually leads to steady-state solutions. However, it still remains challenging to obtain the solutions of unstable critical points numerically. Although the 5DLM contains two additional modes and is indeed more complicated than the 3DLM, its mathematical simplicity with respect to the existing generalized LMs makes it easier to obtain the analytical solutions of the critical points as follows:

\[
Z_c = r - 1, \tag{19a}
\]

\[
Z_{1c} = \frac{-d_o \pm \sqrt{d_o^2 + 4Z_c^2}}{4}, \tag{19b}
\]

\[
X_c = Y_c = \pm \sqrt{\frac{2bd_o Z_{1c}}{Z_c - 2Z_{1c}}} = \pm \sqrt{b(Z_c + 2Z_{1c})}, \tag{19c}
\]

\[
Y_{1c} = \frac{X_c}{d_o}(Z_c - 2Z_{1c}), \tag{19d}
\]

which can be used to normalize solutions. Note that the sign of \( Z_{1c} \) determines whether the solution of \( X_c \) is real or imaginary. In the above, only positive \( Z_{1c} \) in Eq. (19b) is chosen to have two real roots for \( X_c \). In cases with larger \( r \) and comparable domain height and width [namely, \( a = O(1) \)], we can assume \( d_o^2 + 4Z_c^2 \sim 4Z_c^2 \), leading to

\[
Z_{1c} \sim \frac{-d_o}{4} + \frac{Z_c}{2}, \tag{20a}
\]

\[
X_c \sim Y_c \sim \pm \sqrt{4bZ_{1c}} \sim \pm \sqrt{2bZ_c}, \tag{20b}
\]

\[
Y_{1c} \sim X_c/2. \tag{20c}
\]

The last approximation was used in a simple parameterization scheme in section 2c.

Used as normalization scales in section 3, the nontrivial critical points of the original 3DLM (Lorenz 1963a) are

\[
Z_{c}^{3d} = r - 1, \tag{21a}
\]

\[
X_{c}^{3d} = Y_{c}^{3d} = \pm \sqrt{bZ_c^{3d}}. \tag{21b}
\]

When the parameterized feedback term \(-qX^2\) is included, the critical points in the revised 3DLM (i.e., 3DLMP) are changed to

\[
Z_{c}^{3d} = r - 1, \tag{22a}
\]

\[
X_{c}^{3d} = Y_{c}^{3d} = \pm \sqrt{\frac{bZ_c^{3d}}{1 - q}}. \tag{22b}
\]

Note that while a critical point for \( Z \) in the 3DLM, revised 3DLM, and 5DLM has the same mathematical form, the critical points for \( X \) and \( Y \) are different. A choice of \( q \) between 0 and 0.5 leads to \( X_{c}^{3d} \leq X_{c}^{3d} \leq X_{c}^{3d} \).
as shown in Eqs. (19c), (21b), and (22b). The differences between \(X^\text{3d}\) and \(X^\text{2d}\) can be small when a small \(q\) is used, which will be discussed in section 3d.

Based on the previous discussions, one may wonder if a four-dimensional LM (4DLM) can be obtained with additional simplifications. By ignoring \(Z_1\) (i.e., \(XZ_1\)) in Eq. (13), the nontrivial critical point solution of \(X_c\) for Eqs. (10)–(13) can be obtained: \(X^2_c = -b d_o (r - 1)/(r - d_o - 1)\). The term \(X^2_c\) becomes negative when \(d_o + 1 < r\), and thus \(X_c\) has no real root. In addition, the domain-averaged total energy of the 4DLM is not conserved, which is discussed in appendix A. Therefore, the 4DLM is not discussed in this study as a result of the choice with \(d_o = 19/3\) and \(r > 20\).

e. Numerical approaches

The 3DLM, 3DLMP, and 5DLM are integrated forward in time with the fourth-order Runge–Kutta scheme. Since our main goal is to understand the impact of the additional modes on the representation of the advection of the temperature perturbation and the subsequent nonlinear interaction, we vary the value of \(r\) with other parameters kept constant, including \(\sigma = 10\), \(a = 1/\sqrt{2}\), and \(b = 8/3\), as commonly used in previous studies with the 3DLM. The choice of \(a = 1/\sqrt{2}\) gives a minimum value for \(R_c = 27\pi^4/4\) and \(d_o = 19/3\). Note that \(d\) only appears in the 5DLM. The dependence of solution stability on \(\sigma\) will be discussed in section 3e. A dimensionless time interval \(\Delta t\) of 0.0001 is used, and a total number of time steps is 500 000, giving a total dimensionless time \(\tau\) of 50. A larger \(\tau\) is used to measure the chaotic behavior of the numerical solutions, as discussed in the next paragraph. [In Figs. 3–6 and 9, the initial value of \(Y\) is one \((Y = 1)\) and the initial conditions for the rest of the modes \((X, Z, Y_1, Z_1)\) are set to zero.] All of the solutions for the 5DLM, 3DLM, and 3DLMP, unless stated otherwise, are rescaled (or normalized) using the solutions of the critical points in Eqs. (19), (21), and (21), respectively.

To quantitatively evaluate whether the system is chaotic or not, we calculate LE, which is a measure of the average separation speed of nearby trajectories on the critical point. The mathematical definition of the LE (\(\lambda_{\text{LE}}\)) is defined as

\[
\lambda_{\text{LE}} = \text{LE} = \lim_{T \to \infty} \frac{1}{T} \log \left| \frac{\delta s(T)}{\delta s(0)} \right|
\]

for a system

\[
\frac{ds_i}{d\tau} = f_i(s_1, s_2, \ldots, s_n), \quad i = 1, 2, 3, \ldots, n,
\]

where \(n\) represents the dimensions of the LM (i.e., number of variables), the integration time \(T = N\Delta \tau\), \(s\) is a column vector representing the solution [e.g., \(s = (s_1, s_2, \ldots, s_n)\) and \(s = (X, Y, Z)\) for the 3DLM], and \(f\) is a vector consisting of the terms on the right-hand side in each of the LMs. The value \(|\delta s|\) represents the distance between the perturbed and unperturbed trajectories, and \(|\delta s(0)|\) is an initial distance. Note that to examine the predictability in weather prediction models, a finite-time (FT) LE is calculated (e.g., Nese 1989; Eckhardt and Yao 1993; Kazantsev 1999; Ding and Li 2007; Li and Ding 2011), and it is defined as

\[
\lambda_{\text{FTLE}}[s(k\Delta \tau)] = \frac{1}{\Delta \tau} \log \left| \frac{\delta s[(k + 1)\Delta \tau]}{\delta s(k\Delta \tau)} \right|
\]

which depends on \(\Delta \tau\), a starting point (or an initial point), and an initial perturbation. The relationship between the LE and FT LE is shown as follows:

\[
\lambda_{\text{LE}} = \lim_{N \to \infty} \sum_{k=0}^{N-1} \lambda_{\text{FTLE}}[s(k\Delta \tau)].
\]

Over the past few decades, different numerical schemes have been proposed to calculate \(\delta s\) and thus the LE (e.g., Froyland and Alfsen 1984; Wolf et al. 1985). For example, \(\delta s\) can be calculated by solving the 5DLM and the following equation (e.g., Nese 1989; Eckhardt and Yao 1993):

\[
\frac{d\delta s}{d\tau} = \frac{\partial f}{\partial s^j} \delta s, \quad i, j = 1, 2, 3 \ldots n.
\]

Equation (27) is called the variational equation with additional \(n^2\) equations. In this study, the following two methods are used: 1) the trajectory separation (TS or orbit separation) method (e.g., Sprott 2003) and 2) the Gram–Schmidt reorthonormalization (GSR) procedure (e.g., Wolf et al. 1985; Christiansen and Rugh 1997). The differences between these two schemes are briefly discussed as follows. The TS scheme determines an LE by directly solving Eq. (24) to measure the distance of two trajectories with tiny differences (i.e., \(10^{-9}\) in this study) at the location of the starting points, while the GSR method calculates an LE by simultaneously solving Eqs. (24) and (27), that is, the LM and its variational equation. In both schemes, renormalizations are required during the time integrations. Using the given ICs and a set of parameters in the LMs, the TS scheme calculates the largest LE, and the GSR scheme produces \(n\) LEs. Since our interest is to understand whether the system is chaotic with a positive LE, we only analyze the leading (largest) LE with \(\Delta \tau = 0.0001\) and \(N = 10 000 000\),...
Table 2. The Jacobian functions for the nonlinear interactions of the six modes. Coef indicates the coefficient corresponding to the specific Jacobian function. The crossed-out symbol indicates the negligence of a term that involves the crossed-out term. For example, $\cos(2\tau)$ means that any multiplications of the $\cos(2\tau)$ are neglected as a result of mode truncation. Mode $M_5$ is used only in the 6DLM of Shen (2013).

<table>
<thead>
<tr>
<th>Jacobian</th>
<th>Outcome</th>
<th>Coef</th>
<th>6DLM</th>
<th>5DLM</th>
<th>3DLM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J(M_1,M_2)$</td>
<td>$mlM_3$</td>
<td>$XY$</td>
<td>As in 6DLM</td>
<td>As in 5DLM</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_3)$</td>
<td>$ml(M_5 - M_2)$</td>
<td>$XZ$</td>
<td>As in 6DLM</td>
<td>$M_5$ is missing</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_6)$</td>
<td>$2mlM_6 - mlM_3$</td>
<td>$XY_1$</td>
<td>$\cos(2\tau)$</td>
<td>As in 6DLM</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_6)$</td>
<td>$-2mlM_5$</td>
<td>$XZ_1$</td>
<td>$\sin(5m\tau)$</td>
<td>As in 6DLM</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_6)$</td>
<td>$2mlM_6 - mlM_3$</td>
<td>$X_1Y$</td>
<td>$\cos(2\tau)$</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_6)$</td>
<td>$mlM_2$</td>
<td>$X_1Z$</td>
<td>$\sin(5m\tau)$</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_6)$</td>
<td>$3ml\sin(6m\tau)$</td>
<td>$X_1Y_1$</td>
<td>All missing</td>
<td>N/A</td>
<td>N/A</td>
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<tr>
<td>$J(M_1,M_6)$</td>
<td>$-2mlM_2$</td>
<td>$X_1Z_1$</td>
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<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>$J(M_1,M_6)$</td>
<td>$\geq \sin(2\tau)$...</td>
<td>$XX_1$</td>
<td>$\sin(2\tau)$</td>
<td>N/A</td>
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3. Nonlinear feedback of additional modes and their impacts on system stability

In the following sections, we discuss the impact of additional modes on the degree of nonlinearity and solution stability. We analyze the Jacobian term $J(\psi, \theta)$ to illustrate the nonlinear feedback loop of the 3DLM in section 3a and discuss how the feedback loop is extended by the proper selection of the $M_5$ and $M_6$ modes in section 3b. Then, we present numerical results from the 5DLM to examine the impact of the nonlinear feedback processes enabled by the two new modes in section 3c. The results of the revised 3DLM with a parameterized term are analyzed in section 3d to verify the role of the negative feedback in improving solution stability.

a. The nonlinear feedback loop in the 3DLM

In this section, we first discuss the characteristics of nonlinearity in the partial differential equation [Eq. (2)], which can be written in terms of Jacobian terms or Fourier models as follows:

$$J(\psi, \theta) = C_1C_2[XXYJ(M_1, M_2) - XZJ(M_1, M_3) + XY_1J(M_1, M_5) - XZ_1J(M_1, M_6)]$$

or

$$J(\psi, \theta) = mlC_1C_2[XZM_2 + XYM_3 - XY_1M_3 - XZM_5 + 2XZ_1M_5 + 2XY_1M_6].$$

The outcome for each of the four Jacobian terms on the right-hand side of Eq. (28) is listed in Table 2. Only the first two Jacobian terms are included in the 3DLM. The four nonlinear terms $J(M_1, M_j)$, where $j = 2, 3, 5$, and 6, may involve downscale and/or upscale transfer processes, as described by Eqs. (B2) and (B3) in appendix B, which are briefly summarized. The nonlinear interaction of two wave modes via the Jacobian term can
generate or impact a third wave mode through a downscale (or upscale) transfer process; its subsequent upscale (or downscale) transfer process can provide feedback to the incipient wave mode(s). The downscale and upscale transfer processes form a nonlinear feedback loop, which can be continuously extended as long as new modes could be continuously generated. This suggests that a numerical model should include an infinite number of Fourier modes. However, practically, all of the available numerical models have a finite number of modes and thus the extension of their nonlinear feedback loop is finite (and incomplete). The degree of nonlinearity was previously defined as the degree of mode–mode interactions in the introduction, and it indeed refers to the degree of the extension of the nonlinear feedback loop. We discuss the nonlinear feedback loop of the 3DLM below and its extension in the 5DLM in section 3b.

The degree of nonlinearity is discussed below and is briefly summarized in Fig. 1 where for a given \( J(M_1, M_j) \) term, the associated downscale and upscale transfer processes are indicated by a downward arrow and an upward arrow, respectively. In the 3DLM, the nonlinear terms \( J(M_1, M_2) \) and \( J(M_1, M_3) \) form a feedback loop as a result of the following:

\[
J(M_1, M_2) = mlM_3, \quad (30)
\]

\[
J(M_1, M_3) = -mlM_2. \quad (31)
\]

The loop with \( M_2 \rightarrow M_3 \rightarrow M_2 \) is enabled by the inclusion of the coefficients \( M_j \) in Fig. 1. We now illustrate the role of the nonlinear feedback loop in the 3DLM. Without the inclusion of the nonlinear terms \(-XZ\) and \(XY\), the 3DLM, that is, Eqs. (15)–(17), reduces to

\[
\frac{dX}{dr} = -aX + aY, \quad (32)
\]

\[
\frac{dY}{dr} = rX - Y, \quad (33)
\]

\[
\frac{dZ}{dr} = -bZ. \quad (34)
\]

Equations (32) and (33), which are decoupled with Eq. (34), form a forced dissipative system with only linear terms. Equations (32)–(34) also represent the original Lorenz system linearized at the trivial critical point. The system has only a trivial critical point \((X = Y = 0)\) and produces unstable normal-mode solutions (i.e., exponentially growing with time) as \( r > 1 \). While Eq. (33) contains one heating term \((rX)\) and one dissipative term \((-Y)\), Eq. (34) has one dissipative term \((-bZ)\). Therefore, our analysis indicates that the inclusion of \( M_3 \) introduces Eq. (34) and the enabled feedback loop [i.e., Eqs. (30) and (31)] couples Eq. (34) with Eqs. (32) and (33) to form the (nonlinear) 3DLM [Eqs. (15)–(17)] that enables the appearance of convection solutions. From a perspective of total energy conservation, the inclusion of the \( M_3 \) mode can help conserve the total energy in the dissipationless limit, which is discussed in appendix A. Mathematically, the feedback loop with the nonlinear terms in Eqs. (16) and (17) leads to the change in the behavior of the system’s solutions; the (nonlinear) 3DLM system produces nontrivial critical points, which may be stable (e.g., for \(1 < r < 24.74\)) or “unstable” (chaotic) (e.g., for
which is proportional to Eq. (B2) with \((M, 0) = (m, 2m)\). Similarly, Eqs. (13) and (14) in the 5DLM, which are introduced by the new modes, contain only nonlinear and dissipative terms (e.g., with no additional heating terms). Their collective impact on the solution stability is examined in Sections 3c and 3e. Next, we discuss how the new modes are selected in the 5DLM to extend the feedback loop of the 3DLM.

b. The extended nonlinear feedback loop in the 5DLM

Physically, the two modes \((M, 0) = (m, 2m)\) with higher vertical wavenumbers are added to improve the presentation of vertical temperature gradients and, therefore, the accuracy of the vertical advection of temperature. From the nonlinear perspective, the inclusion of \(M, 0\) is to improve the representation of \(J(M_1, M_3)\) in the original 3DLM. This is illustrated as follows. \(J(M_1, M_3)\) in the 5DLM is written as

\[
J(M_1, M_3) = 2ml \cos(\lambda x)[\sin(3mz) + \sin(-mz)] = ml(M_5 - M_2),
\]

(35)

which is proportional to Eq. (B2) with \((p, q) = (m, 2m)\). Equation (35) indicates the route of the energy transfer to both the \(M_5\) and \(M_2\) modes from \(J(M_1, M_3)\), leading to the appearance of \(-XZ\) and \(XZ\) in Eqs. (11) and (13), respectively, in the 5DLM. More importantly, the interaction of the \(M_1\) and \(M_5\) modes provides feedback to the \(M_3\) mode through

\[
J(M_1, M_5) = 2mlM_6 - mlM_3.
\]

(36)

The above equation, which shares the similarity with Eq. (B3) as \((p, q) = (m, 2m)\), adds the \(-XY_1\) term into \(dZ/dr\) in Eq. (12) for the 5DLM. The processes in Eqs. (35) and (36) extend the (existing) feedback loop (e.g., \(M_2 \rightarrow M_5 \rightarrow M_3\)) of the 3DLM with a new loop (e.g., \(M_3 \rightarrow M_5 \rightarrow M_3\)). The former and latter may be viewed as the main trunk and branch, respectively. Note that the term “extension of the nonlinear feedback loop” indicates the linkage between the existing loop and the new loop and thus suggests the importance in the proper selection of new modes. It was reported that the inclusion of new modes could produce additional equations that are not coupled with the 3DLM, leading to a generalized LM with the same stability as the 3DLM [e.g., Eqs. (11)–(16) of Roy and Mysiak (2007a)]. In this case, the original nonlinear feedback loop of the 3DLM is not extended with the new modes.

Equations (31) and (35) indicate the differences in the representation of the nonlinear \(J(M_1, M_3)\) for the 3DLM and 5DLM. The missing \(M_5\) in Eq. (31) is equivalent to replacing the \([\sin(3mz) + \sin(-mz)]\) by \(\sin(-mz)\). As indicated by a simple comparison between the two terms in Fig. 2, the inclusion of the new mode leads to finer presentation of \(J(\psi, \theta)\) near the top and bottom boundaries. Specifically, the solutions in Eqs. (31) and (35) have different signs in layers of \((0 < z < H/4)\) and \((3H/4 < z < H)\), suggesting opposite phases. The differences are presumably related to the rapid changes in the sign of the solutions in the presence of chaos, which, however, is beyond the scope of the present study.

Mathematically, Eqs. (35) and (36) collectively represent a “forcing” term, \(J(M_1, J(M_1, M_3))\), in \(d^2Z/dr^2\) that
can be derived by taking the time derivative of Eq. (2). With the inclusion of $M_5$, $J(M_1, M_3)$ provides not only upscaling feedback to the $M_3$ mode but also a downscale energy transfer to a smaller-scale wave mode that, in turn, requires the inclusion of the $M_6$ mode (Fig. 1; Table 2). As discussed in appendix A, the $M_6$ mode is required to conserve the total energy in the dissipationless limit. The term $XY_1$ is responsible for the transfer of the domain-averaged potential energy at different scales (between the $M_3$ and $M_6$ modes). The feedback loop is further extended to $M_5 \rightarrow M_6 \rightarrow M_5$ through $J(M_1, M_5)$ and $J(M_1, M_6)$. In summary, while the inclusion of $M_5$ forms a feedback loop by introducing Eq. (17) in the 3DLM, the inclusion of $M_5$ and $M_6$ extends the feedback loop by introducing Eqs. (13) and (14) where additional dissipative terms are included. In the next section, we examine whether the feedback of the aforementioned nonlinear processes is positive or negative and show that the $-X Y_1$ term can provide the negative feedback to stabilize solutions.

c. Numerical results of the 5DLM

Figure 3 shows the normalized solutions of $(Y, Z)$ and $(Y_1, Z_1)$ using the 3DLM and 5DLM with three different values of $r$. The scales for normalization are the critical points (e.g., $Y_c$ and $Z_c$) as defined in Eqs. (19) and (21) for the 5DLM and 3DLM, respectively. As first shown in Lorenz (1963a), when the 3D system (in the 3DLM) becomes chaotic at a large $r$ ($r > r_c$, $r_c = 24.74$), the solution never reaches a steady state but oscillates irregularly with time around the nontrivial critical points. This feature can be seen in Fig. 3a from the 3DLM with $r = 25$. It has been shown that the solution is sensitive to initial conditions, which are referred to as the butterfly effect (of the first kind). As compared to the 3DLM, the 5DLM with the same $r$ value of 25 produces a steady-state solution, as indicated by the converged trajectory that approaches a critical point at $(Y/Y_c, Z/Z_c) = (0.21, 1)$ in Fig. 3b. The 5DLM continues to generate steady-state solutions until $r$ is beyond 43 (which will be discussed in Fig. 7). For an $r$ value of 43.5, the 5DLM produces a chaotic solution with a butterfly pattern in the $Y-Z$ space (Fig. 3c). The corresponding solutions for $Y_1$ and $Z_1$ are shown in Fig. 3d and have low values when $Y$ rapidly changes its signs.

Numerical results that display temporal fluctuations near the critical points are analyzed in Figs. 4 and 5. For
stable cases in both the 3DLM and 5DLM (Figs. 4a,c), the solutions oscillate at small time scales and their envelopes decay at large time scales. The decay rate that leads to steady states is larger in the 5DLM than in the 3DLM. For chaotic cases shown in Figs. 4b and 4d, from the 3DLM with $r_5 = 25$ to the 5DLM with $r_5 = 43.5$, respectively, the solutions oscillate in the beginning and gradually grow with time. Chaos appears subsequently; its onset can be identified by rapid changes in the signs of $X$ (or $Y$).

By calculating the numerical solutions of the 5DLM over a wide range of $r$ and normalizing them using the corresponding critical points in Eq. (19), we show that the $r$–time diagram of the normalized solutions is useful in displaying stable and chaotic regions, providing a qualitative method of determining the $r_c$ for the onset of chaos. In Fig. 5 where $(Z/Z_c, Z_1/Z_{1c}, -Y/Y_c, and -Y_1/Y_{1c})$ are shown, white areas display the normalized values of $1 \pm 0.01$. For $r = 25–43$, the appearance of stable critical points is indicated by the white areas with a sufficient long period of time. In contrast, a chaotic regime can be identified as $r > r_c$ (where $r_c \sim 43$) by rapid changes in both the sign and magnitude of the normalized solutions. This critical value is consistent with the analysis of the Lyapunov exponent (discussed later with Fig. 7b). Other than the above, this figure is able to monitor the transient processes and suggests a longer time for solutions to become steady (chaotic) when $r$ gets closer to $r_c$, consistent with the analysis of the eLEs that are close to zero as $r \sim r_c$.

To examine the improved stability of the solutions in the 5DLM, we analyze the time evolution of each term on the right-hand side of Eqs. (10)–(12). Results from Eq. (12) are compared with those from Eq. (17) to illustrate the major difference between the 5DLM and
For a stable case in the 3DLM (e.g., \( r = 20 \) in Fig. 6a), a steady-state solution exists in association with a balance between the nonlinear term \( XY \) and the dissipative term \( bZ \). However, at a large \( r \) (e.g., \( r = 25 \) in Fig. 6b), both of the terms evolve with time at a different growth rate and the solutions appear chaotic. The analysis seemingly supports the understanding that the source of chaos is the nonlinearity, as \( XY \) appears as a forcing term with respect to the other term \( bZ \) for the \( M_3 \) mode in Eq. (17). However, by contrast, the 5DLM using the same normalized Rayleigh number (\( r \) = 25) produces a steady-state solution that corresponds to the balanced state achieved by the three terms \( XY, bZ, \) and \( XY_1 \) (Fig. 6c). The second nonlinear term \( XY_1 \) has a magnitude comparable to \( bZ \) but is missing in the 3DLM. A similar balanced state can be found in the case with \( r = 35 \) (Fig. 6d). The comparison between Figs. 6b and 6c suggests the importance of \( XY_1 \) in stabilizing the solution with \( r = 25 \), indicating the importance of an increased degree of nonlinearity. As discussed earlier, the feedback of \( XY_1 \) to the \( dZ/dr \) for the \( M_3 \) mode [Eq. (12)] can be mathematically illustrated using a pair of Jacobian functions, \( J(M_1, M_3) \) and \( J(M_1, M_3) \), depicting the nonlinear processes of downscale transfer and subsequent upscale transfer that extend the feedback loop. From a macroscopic view discussed in appendix A, \( XY \) is responsible for the transfer of the domain-averaged kinetic energy and potential energy; \( XY_1 \) is responsible for the transfer of the domain-averaged potential energy at different scales, which provides a path for dissipation via the \( 4bZ_1 \) term in Eq. (14).

To quantitatively measure the degree of chaotic responses in the LMs with the goal of understanding the system’s predictability, we calculate the eLE using the TS and GSR numerical methods that were discussed in section 2e. Figure 7a shows the eLEs of the 3DLM and 5DLM as a function of \( r \) with \( 20 \leq r \leq 120 \) and an increment of one (\( \Delta r = 1 \)), while Fig. 7b shows the eLEs of the 5DLM with \( 35 \leq r \leq 50 \) and \( \Delta r = 0.1 \). For the 3DLM, the eLEs using the TS scheme, as shown in a pink curve, suggest the appearance of chaos as \( r > r_c \), and \( r_c \) is approximately 23.7. This \( r_c \) is slightly smaller than the
(linear) theoretical value of 24.74 proposed by Lorenz (1963a) using the stability analysis of the linearized 3DLM. Note that the accuracy of the $r_c$ depends on many factors, including the values of the system’s parameters (e.g., $\sigma$, $b$, and/or $d_c$), different initial conditions, numerical schemes, and so on. As our goal is to illustrate the (negative) nonlinear feedback associated with the new modes in the generalized LM, we made no attempt at searching for a precise $r_c$. We use $\Delta r = 0.1$ to identify the $r_c$, which is defined as the lowest value of $r$ when the eLE becomes positive from negative. In addition to the transition from stable regions (eLEs < 0) to chaotic regions (eLEs > 0), two of the so-called window regions where the LEs are nearly zero can be identified in the vicinity of $r = 93$ and $r = 100$ in the pink curve. The results of the 3DLM, which display a relatively smaller $r_c$ and indicate the appearance of windows, are in good agreement with previous studies [e.g., Fig. 1 of Froyland and Alfsen (1984)]. To understand the sensitivity of the eLE calculations to a specific scheme, a comparison of the eLEs using the TS procedure and GSR scheme (e.g., the orange circle in Fig. 7a) was made, showing insignificant differences except near the window regions (e.g., the green curve in Fig. 7a).

As compared to the 3DLM, the eLEs of the 5DLM (the black curve in Figs. 7a and 7b) indicate the following: (i) that a larger $r$ ($r_c \sim 42.9$) is required for the onset of the chaos; (ii) that one window exists but appears at a slightly larger $r$ (i.e., $r = 107$); and (iii) that eLEs are comparable to the corresponding ones of 3DLM for $44 < r < 80$ and display large differences when $r < 44$ and $r > 80$ (e.g., near window regions).

Fig. 6. Forcing terms of $dZ/dt$, which are from Eq. (17) of the 3DLM and Eq. (12) of the 5DLM. Results from the 3DLM with (a) $r = 20$ and (b) $r = 25$. Results from the 5DLM with (c) $r = 25$ and (d) $r = 35$. The black and orange lines represent $XY$ and $bZ$, respectively, while the blue line represents $XY_1$. In the 3DLM, $XY$ and $bZ$ are balanced to reach a steady state. In the 5DLM, the additional term $XY_1$ is required to reach a steady state.
d. Results of the revised 3DLM

The previous discussions indicated that the $XY_1$ plays a role in stabilizing the solutions in the 5DLM with $25 \leq r \leq 40$, and the $XY_1$ is the only difference between the first three equations of the 5DLM [Eqs. (10)–(12)] and the 3DLM [Eqs. (15)–(17)]. In section 2c, we proposed to emulate the $XY_1$ using $qX^2$ with a tunable parameter $q$ in the revised 3DLM, as shown in Eqs. (15), (16), and (18). The range of $q$ within $0$–$0.5$ can be roughly estimated by the following relation $Y_{c}^{3d} \leq Y_{c}^{5d} \leq Y_{c}$, which represents the analytical solutions of the critical point $Y$ [e.g., Eqs. (21b), (22b), and (19c)] in the 3DLM, 3DLMP, and 5DLM, respectively. To pin down the range of $q$ that can
effectively provide similar negative feedback, we conduct a limited number of runs using selected values of $q$. The eLEs of four runs with $q = 0.15, 0.17, 0.19$, and 0.36 are discussed below. For the case with $q = 0.36$, eLEs over the range of $r = 20–120$ are negative and thus suggest stable solutions (not shown). However, its critical point deviates from the corresponding one of the 3DLM by approximately 25% as a result of the relation $X_{c}^{3d}/X_{c}^{3d} = \pm \sqrt{1/(1 - q)} = \pm 1.25$ [see Eqs. (21b) and (22b)]. Unless stated otherwise, we mainly discuss the revised 3DLM with $q = 0.15, 0.17$, or 0.19 in the following. Figures 8a and 8b show the eLEs of the three cases for $20 \leq r \leq 120$ and $35 \leq r \leq 50$, respectively. Each of these cases displays a transition region between $38 < r < 46$, where the eLE turns from negative to positive (Fig. 8a). As compared to the original 3DLM, the transition regions for the three cases with the revised 3DLMP appear at a larger $r$. Within these transition regions, critical numbers for the onset of chaos can be determined as 38.5, 41.8, and 45.6 for cases with $q = 0.15, 0.17$, and 0.19, respectively (as shown Fig. 8b). For $50 \leq r \leq 80$, the eLEs of the revised 3DLMPs are comparable to those of the 3DLM (Fig. 8a).
Yt initially oscillate and later approach a steady state after solutions for the case using constant value of in the original 3DLM [e.g., Eq. (21)]. The black line indicates the parameterized feedback term (, and ). Among these three runs, the case for is the most comparable results to those of the 5DLM. This case is further analyzed below.

In addition to the long-term-averaged behavior of the solutions represented by the eLEs, we examine the time evolution of the solutions (, , ) from the revised 3DLM, normalized by the critical points of the original 3DLM [e.g., Eq. (21)]. Figure 9a displays the normalized solutions for the case using and that initially oscillate and later approach a steady state after . The steady-state solutions of the nondimensional and are approximately , consistent with the calculation using the relation (1 − ). Figure 9b shows the r–time diagram of the normalized solution (−) when with a maximum deviation of 10% from the critical point of the 3DLM.

The above experiments suggested that although the 3DLM becomes chaotic at , an additional nonlinear dissipative term that emulates the negative feedback, explicitly resolved in a higher-dimensional (5D) LM, can effectively and realistically stabilize the solutions of the revised 3DLM, leading to a (stable) steady-state solution. Using a given set of system parameters, the critical points (steady-state solutions) in a revised 3DLM are not exactly the same as those in the original 3DLM. However, the differences between the former and the latter can be remained within 10% if a value of is properly selected (i.e., ).

e. Stability analysis in the (, ) space

The previous sections discussed the stability problem by varying . Here we examine the dependence of solution stability on and address the question of whether the 5DLM still requires a larger for the onset of chaos when different values of are used. Although a task-level parallelism was implemented in the schemes for the eLE calculation, it is still computationally intensive for obtaining eLEs over a wide range of values for both and (i.e., , ). Therefore, to achieve our goal efficiently, we begin with the stability analysis of the linearized LMs at a nontrivial critical point and conduct the eLE analysis using selected values of . The former is to examine the local predictability, while the latter is to give a measure of the total predictability of the system.

Numerical procedures for the local (or linear) stability analysis in the (, ) space are discussed in appendix C and briefly summarized as follows. To perform a stability analysis of the 3DLM, 3DLM, or 5DLM, we linearize each of these LMs with respect to one of its nontrivial critical points [e.g., Eqs. (C2)–(C6)], obtain its characteristic or eigensystem [i.e., Eq. (C8)], and solve for their eigenvalues. The analytical solutions of critical points for the 5DLM [Eq. (19)], 3DLM [Eq. (21)], and revised 3DLM [Eq. (22)] are used for the analysis. An eigenvalue can be a real or complex number, and its real part is denoted . The appearance of a positive suggests an unstable solution near the critical point. In the following, we examine the solution stability by checking whether the largest is positive or negative.
Figure 10 shows the contour lines of the Re($\lambda$) in the ($s$, $r$) space, each of which describes the critical value $r_c$ as a function of $s$, where the superscript $l$ indicates the local (or linear) analysis. The pink, red, and black lines show the contour lines of Re($\lambda$) for the 3DLM, 3DLMP, and 5DLM, respectively. Solid circles with the same color scheme indicate the $r_c$ determined using the eLE analysis, as discussed in the next paragraph. The contour line of Re($\lambda$) = 0 for the 3DLM is identical to the curve describing the relation $r = \sigma(a + b + 3)/(\sigma - b - 1)$, which was solved analytically to meet $\lambda = 0$ by Lorenz (1963a) (as shown with green multiplication signs in Fig. 10a). The linear stability analysis produces comparable but slightly larger critical values. In general, given a fixed $\sigma$ in each of these LMs, the larger the value of $r$ is, the larger Re($\lambda$) is (e.g., Fig. 10b). Thus, unstable solutions [Re($\lambda$) > 0] appear as $r_c < r$. When $\sigma = 10$, the $r_c$ values for the 3DLM, 3DLMP, and 5DLM are 24.74, 43.54, and 45.94, respectively (Fig. 10b and Table 3). In realizing the stability dependence on $\sigma$ from the linear analysis, we performed additional eLE calculations using our LMs with $\sigma = 13, 16, 19, 22$, and 25 and plot the $r_c$ values as solid circles. In each of the selected runs, the eLE analysis produces a smaller critical value with respect to the linear stability analysis, that is, $r_c < r_c^l$, as shown in Fig. 10a. As $\sigma$ increases from 10, the tendency of the increasing $r_c$ in the 3DLM could be seen in both of the linear and eLE analyses (as shown with pink lines and pink solid circles). By comparison, the linear analysis on the revised 3DLMP and 5DLM shows information that $r_c$ first decreases and then increases, and the eLE analysis produces a similar tendency. It is clearly shown in Fig. 10a that when a $\sigma$ is given (e.g., over the range $5\rightarrow25$), the 5DLM (and 3DLMP) requires a larger $r$ for the onset of chaos than the 3DLM, suggesting improved stability over a wide range of $\sigma$.

4. Concluding remarks

In this study, we derived the generalized 5D Lorenz model (LM) to investigate the impact of two higher-wavenumber modes on the numerical predictability. The domain-averaged total energy of the 5DLM is conserved in the dissipationless limit. Distinct from other studies with generalized LMs, we provided physical justification for the choices of additional modes that can improve solution stability and focused on the interpretation of the nonlinear-scale interactions (i.e., increased degree of nonlinearity) enabled by these additional modes. We first illustrated the nonlinear feedback loop in the 3DLM and emphasized the importance of properly selecting new modes to extend the feedback loop and thus improving the degree of nonlinearity in the 5DLM. By comparing
with other generalized LMs, we found that the 5DLM might serve as the lowest-order generalized LM with increased system stability. The inclusion of new modes introduces both nonlinear terms and dissipative terms that have collective impact on the increase of solution stability. The additional nonlinear terms are mainly associated with the improved vertical advection of temperature. The mathematical simplicity of the 5DLM with respect to existing generalized LMs makes it easier to obtain the analytical solutions of its critical points, identify the major feedback process and its role in the solutions’ stability of the generalized LMs (e.g., 5DLM and 6DLM), and perform (linear) local stability analysis near the critical points over a wide range of parameters (, ). The analyses of both local stability and ensemble-averaged Lyapunov exponents (eLEs) show that the 5DLM requires a larger normalized Rayleigh number for the appearance of chaotic solutions than the 3DLM. While Lorenz demonstrated the association of the nonlinearity with the existence of the nontrivial critical points and strange attractors in the 3DLM, we emphasized the importance of the nonlinearity in both producing new modes and enabling subsequent negative feedback to improve solution stability. More details are given below.

Through the mathematical analysis of the 3DLM, we discussed the feedback loop that includes the nonlinear terms  and . The inclusion of the mode in the 3DLM enables the appearance of the stable nontrivial critical points when , but leads to chaotic solutions when . In comparison, the inclusion of the mode in the 5DLM can improve the representation of the (X) by enabling a downscape transfer process and provide feedback to the mode via an upscale transfer process , which adds the term in [Eq. (12)]. Therefore, the nonlinear loop is extended through the Jacobian terms and [Eqs. (35) and (36)] and is further extended through and in the 5DLM, as shown in Fig. 1. Based on the eLE calculations, the critical value for the 5DLM with is approximately 42.9. The value of the 5DLM is comparable to the one determined by the local stability analysis of the linearized 5DLM that gives 45.94.

Both the eLE analysis and the local (or linear) stability analysis suggest that the 5DLM still produces stable steady-state solutions when ranges from 25 to 42, while the solution of the 3DLM becomes chaotic.

To understand the differences in the predictability between the 3DLM and 5DLM, the competing impact of the nonlinear term against other nonlinear and dissipation terms was illustrated with the use of Eq. (12) . While the first nonlinear term (XY) and the linear term (bZ) act as a forcing term and dissipative term, respectively, the second nonlinear term may work as an additional dissipative term. Therefore, chaotic responses that appear in the 3DLM can be suppressed further by the additional modes in the 5DLM, producing stable solutions such as . However, we would like to emphasize that the negative feedback by the term comes from the collective effects of the nonlinear and dissipative terms associated with the new modes and that it is not trivial to separate them. A macroscopic view suggests that enables the transfer of domain-averaged potential energy at different scales, which in turn enables the feedback associated with the dissipation of the mode [i.e., ] in Eq. (14). Although chaos may appear in the presence of nonlinearity as well as a heating term in the 3DLM, the increased degree of nonlinearity with additional dissipative terms (i.e., the extension of nonlinear feedback loop) in the 5DLM can reduce chaotic responses. Simply speaking, the appearance of small-scale processes that involve the nonlinear interactions with damping terms may help stabilize solutions. The role of the negative nonlinear feedback by was further demonstrated by parameterizing its effect into the revised 3DLM. Based on the analysis of the analytical solutions for the critical points of the 5DLM, the negative nonlinear feedback process through is emulated by a nonlinear eddy dissipation term . As the revised 3DLM produces stable solutions as , it is suggested that the predictability (or chaos) of the 3DLM can be improved (or suppressed) by the nonlinear dissipation term.

Since numerical solutions with the 5DLM display sensitive dependence on ICs after , the butterfly effect of the first kind exists. As the 5DLM...
(3DLM) contains only one horizontal and four (two) vertical wave modes, the predetermined nonlinear mode–mode interactions among the selected modes cannot generate any new modes and thus limit their spatial-scale interactions and upscale energy transfer. In addition, the inclusion of new modes could impact (i.e., increase) the stability of solutions in the 5DLM. Therefore, it is suggested that the appearance of the butterfly effect of the first kind cannot directly lead to the conclusion that small perturbations can alter large-scale structure, namely, the butterfly effect of the second kind, because 1) it requires further upscale transfer of energy by additional low-wavenumber modes and 2) the inclusion of new modes may have a significant impact on the solution stability (i.e., an extremely large $r$ for the onset of chaos).

While chaotic solutions (associated with the butterfly effect of the first kind) occur in the low-dimensional LMs (e.g., 3DLM and 5DLM) that include very limited nonlinear-scale interactions (i.e., limited degree of nonlinearity), it was reported that stable solutions could be obtained in the “sufficiently high-resolution” model by Curry et al. (1984). Therefore, it is hypothesized that solution stability in high-dimensional LMs can be further increased through additional negative nonlinear feedback with additional modes in numerical modeling. However, the nonexistence of a nontrivial critical point in the 4DLM (as $r > d_o + 1$) may indicate the importance of proper mode truncation in improving the solution stability of the nonlinear system that has a finite degree of nonlinearity. Specifically, a comparison among the 3DLM, 4DLM, and 5DLM suggests that the inclusion of only the $M_5$ (e.g., $Y_1$) mode cannot effectively improve stability; while the inclusion of both the $M_5$ and $M_6$ modes can improve stability, the latter requires the former to help provide its feedback to the 3DLM through the new feedback loops, namely, $M_4 \rightarrow M_5 \rightarrow M_3$ and $M_5 \rightarrow M_6 \rightarrow M_5$. In addition, $M_6$ is required to conserve the domain-averaged total energy in the dissipationless limit. Therefore, we suggest that an incremental change in the degree of nonlinearity (e.g., with only $M_5$ mode) may not be a sufficient condition for improving stability particularly in the low-dimensional LMs. We will continue to examine this feature by incrementally increasing the number of modes in generalized LMs.

To achieve the above goals, we have derived a 6DLM with the inclusion of the $M_4$ mode [$M_4 = \sqrt{2} \sin(lx) \sin(3mx)$]. After finishing the derivations of the 6DLM in the fall of 2011, we became aware of the recent studies by Professor Z. E. Musielak and his colleagues who obtained the same 6DLM (Musielak et al. 2005). The 6DLM produces a slightly smaller $r_c$ ($= 41.1$) for chaotic solutions than the 5DLM. A comparison between the two LMs has been made to investigate the impact of an additional heating term associated with the $M_4$ mode on the solution’s stability, which is in preparation for publication (Shen 2014, manuscript submitted to J. Atmos. Sci.). To improve our understanding of the chaos dynamics and thus the short-term predictability (e.g., Legras and Ghil 1985; Nese and Dutton 1993; Nese et al. 1996), we will address if and how the changes of the critical points in the revised 3DLM and 5DLM, which have been solved analytically, can impact the transient evolution of chaotic solutions with respect to the original 3DLM. For example, the growth rate of the envelope of the numerical solutions (e.g., Fig. 4) from the nonlinear and linear systems in Eqs. (C2)–(C6) (with $FN = 1$ or 0) will be compared to the corresponding finite-time LE (e.g., Nese 1989; Zeng et al. 1991; Li and Ding 2011) and linear growth rate (e.g., Fig. 10). Fractal dimension in different LMs will be analyzed with different methods (e.g., Grassberger and Procaccia 1983; Nese et al. 1987; Zeng et al. 1992) to understand the solution’s stability. Our ultimate goal is to apply these analysis methods to examine the dependence of the solution’s stability on mesoscale resolutions (e.g., $\frac{1}{4^e}$ versus $\frac{1}{2^e}$) and on model physics (e.g., different moist processes) in global weather and climate simulations (Shen et al. 2006b, 2012).

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Appendix A

Energy Conservation in the 5DLM

The domain-averaged kinetic energy (KE) and potential energy (PE) are defined as follows (e.g., Treve and Manley 1982; Thiffeault and Horton 1996):

$$\text{KE} = \frac{1}{2} \int_0^{2H/a} \int_0^H (u^2 + w^2) dz \, dx,$$

$$\text{PE} = \frac{H}{2} \int_0^{2H/a} \int_0^H \left( \frac{\partial u}{\partial z} \right)^2 \, dz \, dx.$$
\[ PE = -\int_0^{2H/a} \int_0^H g(a \phi) \, dz \, dx. \]  
(A2)

With Eqs. (7) and (9), Eq. (A1) becomes
\[ KE = \frac{\pi^2}{2} \kappa^2 \left( \frac{1 + a^2}{a} \right)^3 X^2 = C_3 z^2, \]  
(A3)

where \( C_3 = \frac{\pi^2}{2} \kappa^2 \left( \frac{1 + a^2}{a} \right)^3 \). Since the integral of the \( M_2 \) and \( M_6 \) modes over the domain is equal to zero, \( \overline{PE} \) in Eq. (A2) is written as
\[ \overline{PE} = C_2 ga \int_0^{2H/a} \int_0^H z(ZM_3 + Z_1 M_6) \, dz \, dx, \]

and becomes
\[ \overline{PE} = -\pi^2 \nu k \left( \frac{1 + a^2}{a} \right)^3 \left( Z + Z_1 \right) = -C_3 \sigma \left( Z + Z_1 \right). \]  
(A4)

From Eqs. (A3) and (A4), the time derivative of the total energy is written as follows:
\[ \frac{d\overline{KE}}{d\tau} + \frac{d\overline{PE}}{d\tau} = C_3 \left[ X \frac{dx}{d\tau} - \sigma \left( \frac{dZ}{d\tau} + \frac{dZ_1}{d\tau} \right) \right]. \]  
(A5)

To examine Eq. (A5) in the dissipationless limit (\( \nu = \kappa = 0 \)), we derive the following equations by multiplying Eqs. (10), (12), and (14) of the 5DLM by \( X, -\sigma \), and \(-\sigma/2\), respectively,
\[ X \frac{dx}{d\tau} = -\sigma X^2 + \sigma XY, \]  
(A6)
\[ -\sigma \frac{dZ}{d\tau} = -\sigma XY + \sigma XY_1 + \sigma b Z, \]  
(A7)
\[ -\sigma \frac{dZ_1}{d\tau} = -\sigma XY_1 + \sigma b Z_1. \]  
(A8)

Here the crossed-out symbol indicates a dissipative term that is associated with either \( \nu \nabla^4 \phi \) or \( \kappa \nabla^2 \theta \) in Eqs. (1) and (2). The dissipative terms are neglected in the dissipationless limit. The term \( \sigma XY \) in Eq. (A6) is originally from the linear term \( g(a \partial \phi \partial x) \) in Eq. (1), while the other nonlinear terms in Eqs. (A7) and (A8) are from the advection term \( J(\phi, \theta) \) in Eq. (2). Equation (A6) represents the time derivative of the \( \overline{KE} \) and Eq. (A7) [Eq. (A8)] represents the time derivative of the \( \overline{PE} \). The nonlinear term \( XY \), which appears in Eqs. (A6) and (A7), is responsible for the conversion of \( \overline{KE} \) and \( \overline{PE} \), while the nonlinear term \( XY_1 \) is responsible for the conversion of \( \overline{PE} \) at different scales. As the summation of Eqs. (A6)–(A8) is zero when the crossed-out terms are excluded, we have
\[ d\overline{KE}/d\tau + d\overline{PE}/d\tau = 0. \]  
Therefore, the total energy is conserved. Note that in the 3DLM where both \( Y_1 \) and \( Z_1 \) are missing in Eqs. (A7) and (A8), Eq. (A5) is still equal to zero. However, \( Z \) has to be included to conserve the total energy in the 3DLM. In comparison, when \( Z_1 \) is not in Eq. (A8) but only \( Y_1 \) is included in Eq. (A7), Eq. (A5) is not equal to zero, except for the trivial solution \( X = 0 \). Therefore, it is important to include both \( Y_1 \) and \( Z_1 \) (i.e., both \( M_4 \) and \( M_6 \) modes) to conserve the total energy of the system.

When dissipation terms are included in Eqs. (A6)–(A8), the time derivative of the total energy becomes
\[ \frac{d\overline{KE}}{d\tau} + \frac{d\overline{PE}}{d\tau} = C_3 \sigma (X^2 + bZ + 2bZ_1). \]  
(A9)

In the above equation, the nonlinear terms (\( XY \) and \( XY_1 \)) are implicit while they are internally responsible for the energy conversion. When a steady state is reached, Eq. (A9) leads to \( X_i = \pm \sqrt{b/(Z_c + 2Z_{1i})} \), which is the same as Eq. (A9c). In addition, the \( M_1 \) mode is associated with \( d\overline{KE}/d\tau < 0 \). When the mode \( M_2 \) has a positive (negative) amplitude, it is associated with negative (positive) potential energy [Eq. (A4)], but the corresponding tendency \( d\overline{PE}/d\tau \) in Eq. (A9) adds positive (negative) potential energy to the system. The \( M_6 \) plays a role similar to the \( M_3 \) mode.

**APPENDIX B**

**Downscale and Upscale Transfer Processes in the Nonlinear Feedback Loop: A Simple Illustration**

In this section, we use trigonometric functions to discuss the downscale and upscale transfer associated with the nonlinear Jacobian \( J(\phi, \theta) \) term, both of which may form a nonlinear feedback loop. The Jacobian term can be written as \( \partial \phi / \partial \phi + \partial \phi / \partial x \). The first and second terms represent the nonlinear vertical and horizontal advection of temperature, respectively. The four Jacobian terms in Eq. (28) are briefly analyzed below. With no loss of generality, we can assume two modes as \( \sin (lx) \) (e.g., \( M_1 \) or \( M_2 \)) and \( \cos (lx) \) (e.g., \( M_2 \) or \( M_3 \)) or \( \sin (qz) \), e.g., \( M_3 \) or \( M_4 \)), respectively. Here \( p \) and \( q \) represent vertical wavenumbers: \( p = m \) or \( 3m \) and \( q = m, 2m, 3m, \) or \( 4m \). Therefore, the corresponding Jacobian becomes
\[ \text{lq} \left[ \cos^2 (lx) \sin (pz) \cos (qz) + \text{lp} \sin^2 (lx) \cos (pz) \sin (qz) \right] \]
and is proportional to the following:
\[ q \sin (pz) \cos (qz) + p \cos (pz) \sin (qz), \]  
(B1)
when $\cos^2(lx) = \frac{1 + \cos(2lx)}{2} - \frac{1}{2}$ and $\sin^2(lx) = \frac{1 - \cos(2lx)}{2} - \frac{1}{2}$ because of the truncation of the horizontal wave modes. Equation (B1) is dominated by the first part (i.e., $w \partial \theta / \partial z$) when $p < q$ or by the second part (i.e., $u \partial \theta / \partial x$) when $p > q$. Since we are mainly concerned with the representation of $J(M_1, M_3)$ and subsequent nonlinear processes, we simply discuss the Jacobian of the $M_1$ and one of the other modes, which is represented dominantly by $\sin(pz) \cos(qz)$ because $p < q$. Thus we have

$$\sin(pz) \times \cos(qz) = \frac{1}{2} \{\sin[(p \pm q)z] - \sin[(q \mp p)z]\} \quad \text{as} \quad p < q. \quad \text{(B2)}$$

The above equation indicates that the nonlinear interaction could lead to the generation of two new wave modes with wavenumbers $(p + q)$ and $(q - p)$ or to the modification of these two modes if they already exist. Therefore, downscale and upscale transfer processes may occur. The appearance of the new mode at a higher wavenumber $(p + q)$ enables its subsequent interaction with the $M_1$ mode that leads to

$$\sin(pz) \times \cos((p + q)z) = \frac{1}{2} \{\sin(2p + q)z - \sin(q - p)z\}. \quad \text{(B3)}$$

Therefore, Eqs. (B2) and (B3) collectively suggest that the “new” (or influenced) mode, $\sin[(p + q)z]$, generated (or modified) by the nonlinear downscale transfer process associated with the incipient wave mode $[\sin(qz)]$, can provide feedback to the incipient wave mode via a subsequent nonlinear upscale transfer process (as $q < p + q$). Thus, a feedback loop forms with Eqs. (B2) and (B3). Although these equations represent only the first part of the Jacobian function [e.g., Eq. (B1)], they are representative for $J(M_1, M_2)$, where $j = 2, 3, 5,$ and 6, that includes all of the nonlinear terms for the 5DLM as well as the major nonlinear terms for the 6DLM (B.-W. Shen 2013, unpublished manuscript). More specific discussions are given sections 3a and 3b with the calculation of the Jacobian.

**APPENDIX C**

**Numerical Method of the Stability Analysis near a Critical Point**

The solutions with initial conditions near a nontrivial critical point are analyzed as follows. We decompose the total field into the basic part and perturbation, which can be written as

$$A = A_c + A', \quad \text{(C1)}$$

where $A$ represents $(X, Y, Z, Y_1, Z_1)$, $A_c$ represents the basic state that is from the solution of the critical point, and $A'$ is a perturbation that measures the departure from the critical point. With Eq. (C1), the 5DLM [Eqs. (10)–(14)] becomes

$$\frac{dX'}{d\tau} = -\sigma X' + \sigma Y', \quad \text{(C2)}$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - Y' - X_c Z' - FN(X'Z'), \quad \text{(C3)}$$

$$\frac{dZ'}{d\tau} = (Y_c - Y_{1c})X' + X_c Y' - bZ' - X_c Y_1' + FN(X'Y' - X'Y_1'), \quad \text{(C4)}$$

$$\frac{dY_{1c}}{d\tau} = (Z_c - 2Z_{1c})X' + X_c Z' - dY_1' - 2X_c Z_1' + FN(X'Z' - 2X'Z_1'), \quad \text{(C5)}$$

$$\frac{dZ_{1c}}{d\tau} = 2Y_{1c}X' + 2X_c Y_1' - 4bZ_1' + 2FN(X'Y_1'). \quad \text{(C6)}$$

Here the flag FN indicates if the system is fully nonlinear (FN = 1) or not (FN = 0). The system with FN = 0 is linear with respect to the critical point. However, as the solutions of the basic state (critical point) are from the time-independent nonlinear 5DLM, the “linear system” with FN = 0 still poses the nonlinearity of the basic state. Numerical solutions with FN = 1 and FN = 0 will be compared to understand the evolution of solution’s growth rates that are impacted by the nonlinearity. Here, for local stability analysis, we only consider Eqs. (C2)–(C6) with FN = 0, which can be written as follows:

$$\frac{ds}{d\tau} = A s, \quad \text{(C7)}$$

where $s$ and $A$ are a column vector and matrix, respectively. The term $s$ is $(X, Y, Z, Y_1, Z_1)$, and the matrix $A$ for the 5DLM, denoted $A^{5d}$ is written as follows:

$$A^{5d} = \begin{pmatrix}
-\sigma & \sigma & 0 & 0 & 0 \\
r - Z_c & -1 & -X_c & 0 & 0 \\
Y_c - Y_{1c} & X_c & -b & -X_c & 0 \\
Z_c - 2Z_{1c} & 0 & X_c & -d & -2X_c \\
2Y_{1c} & 0 & 0 & 2X_c & -4b \\
\end{pmatrix}. \quad \text{(C8)}$$

Similarly, the matrix $A$ with the nontrivial critical point for the 3DLM and 3DLMP are denoted $A^{3d}$ and $A^{3d}$, defined as follows:
$$A^{3d} = \begin{pmatrix}
-\sigma & \sigma & 0 \\
r - Z^{3d}_c & -1 & -X^{3d}_c \\
Y^{3d}_c & X^{3d}_c & -b
\end{pmatrix},$$

$$A^{3d} = \begin{pmatrix}
-\sigma & \sigma & 0 \\
r - Z^{3d}_c & -1 & -X^{3d}_c \\
Y^{3d}_c & 2qY^{3d}_c & X^{3d}_c & -b
\end{pmatrix}.$$  

The critical points are analytically defined in Eq. (21) for the 3DLM, in Eq. (22) for the 3DLMP, and in Eq. (19) for the 5DLM. By assuming $s = s_o e^{lt}$, we obtain the following characteristic equation:

$$A s = \lambda s,$$  \hspace{1cm} (C8)

where $\lambda$ is the eigenvalue of the system and $I$ is the identity matrix. The number of eigenvalues is equal to the number of the dimensions in these LMs, and each of these eigenvalues can be a real or complex number. Let $\text{Re}(\lambda)$ represent the real part of $\lambda$, so the appearance of a positive $\text{Re}(\lambda)$ suggests an unstable solution near the critical point. Given any pair of $(\sigma, r)$, we calculate the eigenvalues by solving Eq. (C8) using EISPACK (e.g., Smith et al. 1976) (http://www.netlib.org/eispack/) and only analyze the maximum value of $\text{Re}(\lambda)$. Figure 10 shows the results of the $\text{Re}(\lambda)$ in the $(\sigma, r)$ space where $5 \leq \sigma \leq 25$ with $\Delta \sigma = 0.01$ and $20 \leq r \leq 50$ with $\Delta r = 0.01$. Discussions are made in section 3e.

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