Entropic Balance Theory and Variational Field Lagrangian Formalism: Tornadogenesis

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(Manuscript received 11 July 2013, in final form 30 January 2014)

ABSTRACT

The entropic balance theory has been applied with outstanding results to explain many important aspects of tornadic phenomena. The theory was originally developed in variational (probabilistic) field Lagrangian formalism, or in short, variational formalism, with Lagrangian density and action appropriate for supercell-storm and tornadic phenomena. The variational formalism is broadly used in in modern physics, not only in classical mechanics, with Lagrangian density and action designed for each physical problem properly. The Clebsch transformation (equation) was derived in the classical variational formalism but has not been used because of the unobservable and nonmeteorological Lagrange multiplier. The entropic balance condition is thus developed from the Clebsch transformation, changing the unobservable nonmeteorological Lagrange multiplier to observable meteorological rotational flow velocity with entropy and making it applicable to tornadic phenomena. Theoretical details of the entropic balance are presented such as the entropic right-hand rule, entropic dipole, source and sink, overshooting mechanism of hydrometeors against westerlies and the existence of single and multiple vortices and their relation to tornadogenesis. These results are in reasonable agreement with the many observations and data analysis publications. The Clebsch transformation and entropic balance are the new balance conditions, different from the known other balance conditions such as hydrostatic, (quasi-)geostrophic, cyclostrophic, Boussinesq, and anelastic balance.

The variations in calculus of variations and in the classical variational formalism are hypothetical. However, this article suggests that the hypothetical variations could be physical, relating to quantum variations and their interaction with the classical systems.

1. Introduction

There have been two approaches in flow dynamics: one is the deterministic Newtonian approach and the other is probabilistic (variational), stated in this article as *variational field Lagrangian formalism,* or in short, *variational formalism*. The governing equations, conventionally used in meteorology and oceanography, are deterministic, which can be derived from the variational formalism (sections 2, 3, and 4 and the appendix).

The fundamental methodology of the classical variational formalism, the Euler–Lagrange (E–L) procedure with Hamilton's least-action principle coincides with the methodology that has been broadly used, not only in classical mechanics (Lamb 1932; Bateman 1932; Oden and Reddy 1976; Sasaki 1955), but also in modern advanced physics, with Lagrangian density designed in a physically appropriate form for each problem (Feynman and Hibbs 1965; Kaku 1993; Weinberg 1995; Nair 2005; Mandl and Shaw 2010). The research cited here represents a great number of available references with respect to field Lagrangian formalism used to describe essential mechanisms in both classical and modern physics.

Feynman and Hibbs (1965) and section 5 of this manuscript clearly show the classical limit of the quantum theory to describe the classical systems. The order of magnitudes of the dimensions, mass, times, and so on of the classical system is so large that the action L (of the notation used in this article) is enormous, which is estimated 0.5×10^{-2} mksa or $\rho = 10^{-3}\,\mathrm{g}^{-1}$ and $|\mathbf{v}| = 10\,\mathrm{m\,s}^{-1}$. Since the Plank's constant $\hbar = h/(2\pi) = 1.054 \times 10^{-34}$ mksa rationalized system (J, s), the magnitude of $L/\hbar$ is on the order of 10^{32}, so that the quantum field formalism has not been considered normally in the classical flow formalism. We will reconsider it, showing

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DOI: 10.1175/JAS-D-13-0211.1

a possible need for both systems, especially their interactions between classical and quantum systems.

The entropic balance theory was originally developed on the same basis as the variational field Lagrangian formalism to derive the same equation of Clebsch transformation (Lamb 1932). The Clebsch transformation equation was developed in the variational formalism but has not been used directly for actual phenomena because of the unobservable, nonmeteorological, but mathematical Lagrange multiplier. The Lagrange multipliers for the constraints of mass and entropy that appeared in the Clebsch transformation are analogous to the mathematical vector potential (and gauge field) of the theoretical process found known as the "Aharonov–Bohm effect" (or "A-B effect" or "A effect") (Feynman et al. 1964; Feynman and Hibbs 1965; Landsman 1998; section 5 herein). Its experimental verification has been difficult and has not been made until two decades later (Tonomura 1997).

The entropic balance theory, changing the unobservable nonmeteorological Lagrange multiplier to observable meteorological rotational flow velocity with entropy, is now applicable to tornadic phenomena and is made applicable to solve mysteries of tornadogenesis. Furthermore, the balance is a newly found one, because it is different from the other known balance conditions, such as hydrostatic, (quasi-) geostrophic, cyclostrophic, Boussinesq, and anelastic balance.

The variational formalism is similar to the one observed by Euler in 1736 and referred to as the Euler equation (Oden and Reddy 1976; Lanczos 1970) and later as the Gateaux derivative (Gateaux 1913) in linear approximation with respect to the hypothetical variation that has two parameters, one infinitesimal and the other arbitrary, and the Frechet differential in normed space that deals with a variety of broader problems. A similar mathematical manipulation leads to a full set of dynamical and thermodynamical nonlinear equations of the ideal flow (Lamb 1932; Bateman 1932; Sasaki 1955). The E–L equations are all prognostic except for one that is diagnostic, so-called by Lamb as the Clebsch's transformation (Clebsch 1857) of flow velocity, although the terminology was not commonly used by authors who derived the same or a similar equation.

In the action Lagrangian of the variational formalism by Bateman (1932) and Lamb (1932), the thermodynamic property is assumed to be barotropic, that is, the flow pressure is a function of density only, in order to derive the equation of motion. Baroclinicity is included by Sasaki (1955, 1999, 2009, 2010) and Dutton (1976). Salmon (1988, 1998)'s Hamiltonian formalism leads to the Clebsch transformation equation. Note that Hamiltonian formalism with a Poisson bracket is a powerful tool for quantization. Dutton compared two types of variations, Lagrangian (material) variation and Eulerian variation, and obtained similar results to those of variational formalism. Sasaki further modified the Clebsch transformation and developed the entropic balance equation that is made applicable to real meteorological phenomena (section 4). The entropic balance theory was developed by introducing the two hypotheses in the Lagrangian density: hypothesis 1 of discontinuous sudden cloud physical phase changes, compared with longer time scales of a tornado, a supercell, and daily synoptic weather, and hypothesis 2, concerned with the ensemble property of such weather systems. Of course, it also includes the continuous change of entropy flux through the boundary of the domain.

In sections 2, 3, and 4 and also in the appendix, we discuss Lagrangian density and the action Lagrangian physically appropriate for the tornado mechanism and derive the Clebsch transformation and entropic balance equation.

2. Action Lagrangian of tornadic storm

The entropic balance theory developed for tornadogenesis (Sasaki 1999, 2009, 2010; Sasaki et al. 2013) is extended for the atmospheric flow of all scales. The atmospheric flow is approximated as an ideal gas flow because a high Reynolds number R_e with the molecular viscosity of the air, and a moderately high Rossby number R_o of Earth's rotation, which allows us to neglect Coriolis force:

$$R_e = 10^8 \text{ to } 10^{12} \quad \text{and} \tag{2.1a}$$

$$R_o = 10^1 \text{ to } 10^2 . \tag{2.1b}$$

The entropic balance theory hypothesizes that changes in entropy are a quasi-adiabatic process; that is, the microphysical phase change of a small ensemble of hydrometeor molecules is instantaneous, creating a new entropy level, with adiabatic conditions before and after the phase change. It is hypothesized that this phase-change time scale is significantly shorter than the time scales of convective storms and tornadoes (hypothesis 1):

$$\Delta t_{\text{phase change}} \ll \Delta t_{\text{supercell,tornado}} . \tag{2.2}$$

Variations of the initial entropy levels are small enough and allow us to approximate them by their ensemble means (hypothesis 2).

The Lagrangian density $\mathcal{L}$ is thus formulated as

$$\mathcal{L} := \rho\{1/2\{\mathbf{v}^2 - U(\rho, S) - \Phi\} - \alpha\{\partial_t \rho + \nabla \cdot (\rho \mathbf{v})\} - \beta\{\partial_t(\rho S) + \nabla(\rho \mathbf{v} S)\}\} , \tag{2.3a}$$

where ρ, U, Φ, S, and $\mathbf{v}$ are the density of the air, internal energy, gravitational potential energy, entropy, and flow velocity, respectively; and α and β are the Lagrange multipliers to satisfy the constraints of conservation of mass and entropy, respectively.

Then, the action Lagrangian is defined as

$$L := \int_\Omega \mathcal{L}\, d\Omega, \tag{2.3b}$$

where Ω represents the temporal and spatial integration domain, and the four-dimensional time–space element $d\Omega = d^4x_j$ ($=dx_0$, dx_1, dx_2, dx_3) represents time t ($=x_0$) and three orthogonal spatial coordinates x_1, x_2, and x_3, respectively. An ensemble of air molecules is represented by the spatial integration. The Hamilton principle of the least action imposes

$$\delta L = 0, \tag{2.4}$$

where δ represents the first variation.

From (2.3a) and (2.3b), we see that the variables to be taken the first variation are $\mathbf{v}$, ρ, α, β, and S and (2.4) may be denoted

$$\delta_{\mathbf{v},\rho,\alpha,\beta,S} L = 0, \tag{2.5}$$

The first variation follows the calculus of variations in the variational formalism (section 3).

Note that the notation S instead of L is commonly used in the variational formalism, but S is used for entropy and L is used for the action Lagrangian in this study.

3. Variational formalism

The variational formalism is applied for the Lagrangian density [(2.3a)]. We write the action Lagrangian [(2.3b)] as $L(\phi_i)$, where ϕ_i stands for $\mathbf{v}$, ρ, and S, respectively for $i = 1$, 2, and 3 as

$$\delta L = \delta \int_\Omega \mathcal{L}(\phi_i, \partial_j \phi_i)\, d^4x_j, \quad j = 0,1,2,3, \tag{3.1}$$

In the domain Ω of four-dimensional integration, we consider the hypothetical variation $\delta\phi_i$ of ϕ_i,

$$\phi_i(\mathbf{x}) \rightarrow \phi_i(\mathbf{x}) + \delta\phi_i(\mathbf{x}), \tag{3.2}$$

and assume no hypothetical variation on the four-dimensional boundary $\partial\Omega$,

$$\delta\phi_i(\mathbf{x}) = 0 \quad \text{on} \quad \partial\Omega, \tag{3.3}$$

where $\mathbf{x}$ is the four-dimensional (time and space) vector.

Under the constraints (3.2) and (3.3), Hamilton's principle requires the action Lagrangian [(3.1)] should be stationary:

$$\partial L = 0. \tag{3.4}$$

Note that the Hamiltonian and associated Poisson bracket derived from the Lagrangian is essential for quantization, but the Lagrangian is sufficient for problems of classical field mechanics (Landsman 1998).

Substitution of (3.1) into (3.4) becomes

$$\begin{aligned}\delta L &= \int_\Omega \{\partial\mathcal{L}/\partial\phi_i \cdot \delta\phi_i + \partial\mathcal{L}/\partial(\partial_j\phi_i) \cdot \delta(\partial_j\phi_i)\}\, d^4x_j \\ &= \int_\Omega \{\partial\mathcal{L}/\partial\phi_i \cdot \delta\phi_i + \partial\mathcal{L}/\partial(\partial_j\phi_i) \cdot \partial_j(\delta\phi_i)\}\, d^4x_j.\end{aligned} \tag{3.5}$$

Applying the integral by part to the second term of the integrand of (3.5), we get

$$\delta L = \int_\Omega \{\partial\mathcal{L}/\partial\phi_i - \partial_j[\partial\mathcal{L}/\partial(\partial_j\phi_i)]\}(\delta\phi_i)\, d^4x_j, \tag{3.6}$$

where the vanishing boundary variation [(3.3)] was used.

Since $\delta\phi_i$ is an arbitrary hypothetical variation in (3.6), the steady-state condition [(3.4)] of (3.6) leads the following E–L equation:

$$\partial\mathcal{L}/\partial\phi_i - \partial_j[\partial\mathcal{L}/\partial(\partial_j\phi_i)] = 0. \tag{3.7}$$

In the application of (3.4) and (3.7) to the Lagrangian [(2.3a)] and the action Lagrangian [(2.3b)], we find that there exists a sole diagnostic E–L equation, that is, the Clebsch transformation equation and the entropic balance equation, among all other prognostic E–L equations. The Clebsch transformation has not been applied to real meteorological data in the past. The entropic balance equation is developed from the Clebsch transformation and is found to explain several mysteries of the tornadogenesis mechanism. We will discuss both the Clebsch transformation and the entropic balance equation and how the entropic balance equation plays roles to explain the tornadogenesis mechanism in the next section.

4. Diagnostic E–L equation: Clebsch transformation, entropic balance equation, and tornadogenesis

Now, we may apply the variational formalism to the first variation $\delta_{\mathbf{v}}$ of the action Lagrangian [(2.5)] with respect to $\mathbf{v}$;

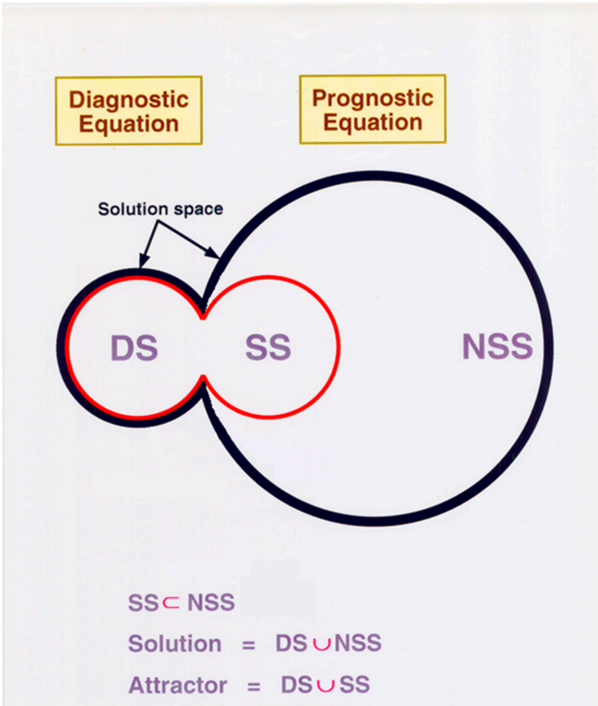

FIG. 1. Solution space. The domain of full solution of the E–L equations is schematically shown in the solution space by the thick solid line. It includes nonstationary state (NSS), stationary state (SS), and the solution of the diagnostic E–L equation (DS). The solution in the domain covered by DS and SS has long-lasting property mathematically similar to the attractor.

$$\delta_{\mathbf{v}} L = 0\,, \tag{4.1}$$

then, we get

$$\mathbf{v} = -\nabla\alpha - S\nabla\beta\,, \tag{4.2}$$

and the vorticity ($\boldsymbol{\omega}$) equation applying $\nabla\mathbf{x}$ to (4.2),

$$\boldsymbol{\omega} = -\nabla S \times \nabla\beta\,. \tag{4.3}$$

Equations (4.2) and (4.3) are diagnostic equations, called *Clebsch transformation* by Lamb (1932), which correspond to the solution's diagnostic state (DS) and stationary state (SS) in Fig. 1.

Because of the variational principle used in the entropic balance theory, the diagnostic equation [(4.2)] should be satisfied always with all other prognostic E–L equations. In the schematic diagram of the solution space (Fig. 1), it is shown that the solution subspace DS is expressed as a part of the other solution subspaces, nonstationary state (NSS) and SS. Since the helicity becomes nearly its maximum at the time of mesocyclone development and tornadogenesis; that is, the local change of vorticity vanishes and the long-lasting subspaces DS and SS are essential. The solution in the subdomain covered by DS and SS has a long-lasting property that is similar mathematically to the attractor in nonlinear dynamics. They appear in Fig. 1 as the subdomains of the solution spaces DS and SS. Note that

$$\mathrm{DS} \subset \mathrm{SS} \subset \mathrm{NSS}\,. \tag{4.4}$$

The relationships expressed by (4.4) emphasize the importance of the diagnostic E–L equation [(4.2)]; that is, the transition to a SS or DS from a NSS must satisfy (4.2). In other words, we can find the necessary conditions for the tornadogenesis and transition among different stages from the entropic balance theory.

The diagnostic, balance, Clebsch transformation [(4.2)], and derived vorticity [(4.3)] may provide insight to a long-lived tornado, presumably by DS and SS steady states, as expressed by (4.4). However, the vector relations [(4.2) and (4.3)] have not been applied to real atmospheric phenomena because of the nonmeteorological Lagrange multipliers. In order to express the nonmeteorological Lagrange multipliers in meteorological parameters, the entropic balance theory is developed and applied to gain clear insight of the development mechanisms of supercells and tornadogenesis (Sasaki 1999, 2009, 2010; Sasaki et al. 2013). The entropic balance condition is thus developed from the Clebsch transformation, changing the unobservable nonmeteorological Lagrange multiplier to observable meteorological rotational flow velocity with entropy. The Clebsch transformation [(4.2)] and associated vorticity equation [(4.3)] are rewritten respectively, making the Lagrangian β to couple with scalar S and rotational flow velocity $\mathbf{v}_R$, as

$$\mathbf{v} = -\nabla\alpha + \mathbf{v}_R\,, \tag{4.5}$$

and

$$\boldsymbol{\omega} = (1/S)\nabla S \times \mathbf{v}_R\,, \tag{4.6}$$

where

$$\mathbf{v}_R = -S\nabla\beta\,. \tag{4.7}$$

Equations (4.5)–(4.7) are the basic equations of the entropic balance theory, which reveal various important properties of supercells and tornadoes and furthermore the wraparound mechanism of tornadogenesis. In practice, S is estimated from radar reflectivity analysis, and $\mathbf{v}_R$ is obtained from the variational data assimilation of Doppler velocity radar observation, as planned to be reported separately, but estimated approximately from Doppler velocity observation. The approximate estimate

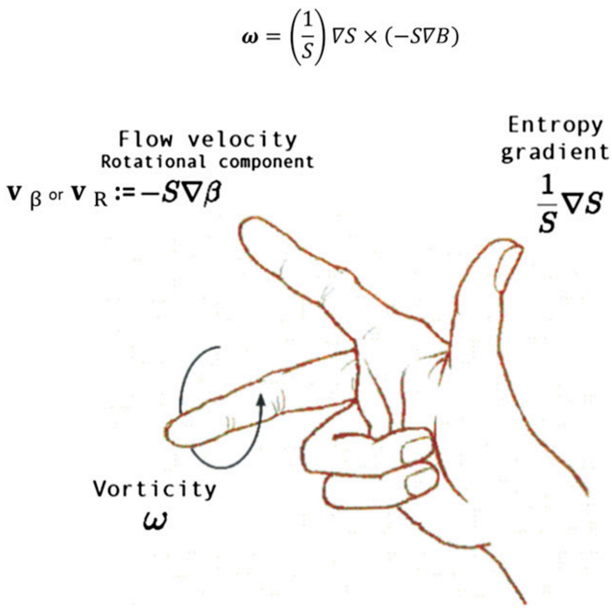

FIG. 2. Entropic right-hand rule. The rule shows the mutually orthogonal vector relation, similar to the so-called Fleming's right-hand rule of electromagnetic fields, which we call the entropic right-hand rule, among the orthogonal variables of three spatial dimensions: entropy gradient $(1/S)\nabla S$, rotational component of flow velocity $\mathbf{v}_R$ or $\mathbf{v}_\beta := -S\nabla\beta$, and vorticity $\boldsymbol{\omega}$.

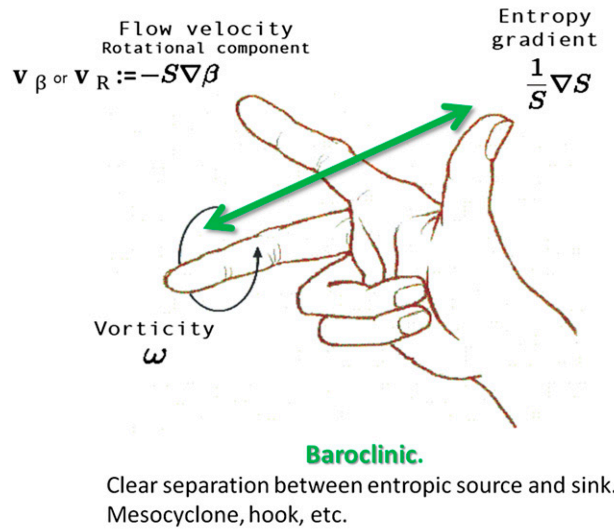

FIG. 3. Baroclinic case of the entropic balance expressed as the right-hand rule.

was determined to have worked well to estimate vorticity in testing. Note that the wind under entropic balance is much more general (there is less assumption) than the thermal wind balance that is under the geostrophic or quasigeostrophic balance.

Figure 2 illustrates schematically the entropic right-hand rule. The diagnostic velocity equation [(4.5)] is universal for the ideal flow. The vorticity equation [(4.6)] derived from (4.5) is conveniently demonstrated by the mutually orthogonal vector relation, similar to the so-called Fleming's right-hand law of electromagnetic fields (called the "entropic right-hand rule" by the author), among the orthogonal variables of the spatial three dimensions: the vorticity $\boldsymbol{\omega}$, the entropy gradient $(1/S)\nabla S$, the rotational flow velocity component $-S\nabla\beta$, denoted by $\mathbf{v}_\beta$ or $\mathbf{v}_R$, and the divergent component $-\nabla\alpha$, denoted $\mathbf{v}_\alpha$ or $\mathbf{v}_D$. These notations are used in the figure illustrations in this article. Based on the entropic balance theory, the wraparound mechanism is introduced by Sasaki (1999, 2009, 2010) to explain explicitly the nonlinear process of tornadogenesis. The results are consistent with tornadic storm observations and successful tornado simulations of phenomena in tornadic storms, such as overshooting hydrometeors against the upper-level westerlies. The development of a mesocyclone, a hook echo, and a wall cloud is easily explained by the entropic balance theory. The almost discontinuous transition from supercell to tornado suggested from visual observation and data analysis is also easily explained by the entropic balance theory as a transition from baroclinic to barotropic stages. Furthermore, an increase of the relative helicity to 1 (its maximum value) as a result of computer simulation is explained by the entropic balance theory. From the entropic balance theory, we can easily explain the observation of the tornado touching the ground in the perpendicular direction.

For simplicity, the vorticity in balance with the entropic gradient and rotational component of flow velocity as given by (4.6) derived from the entropic balance theory is schematically illustrated as the entropic right-hand rule (Fig. 2). Note that when the gradient of entropy is of larger magnitude, it is baroclinic (Fig. 3), while when it is of smaller magnitude, it is barotropic (Fig. 4). A schematic diagram of a supercell with a tornado is shown in Fig. 5, in which various known features of the supercell and tornado, together with a newly found mechanism, such as the overshooting of hydrometeors against the upper headwind westerlies, are explained by the entropic balance theory in the caption of the figure. Figure 6 shows a horizontal cross section of entropic source and sink that produces a baroclinic gradient in balance with the mesocyclone vortex at middle levels according to the entropic balance theory. The existence of a single vortex (Fig. 7) and that of multiple vortices (Fig. 8, for four vortices) are easily understood to exist by applying the entropic right-hand rule (Fig. 2) to each vortex.

Transition from the mesocyclone, in baroclinic state, shown in Figs. 5 and 6 to the single vortex and/or

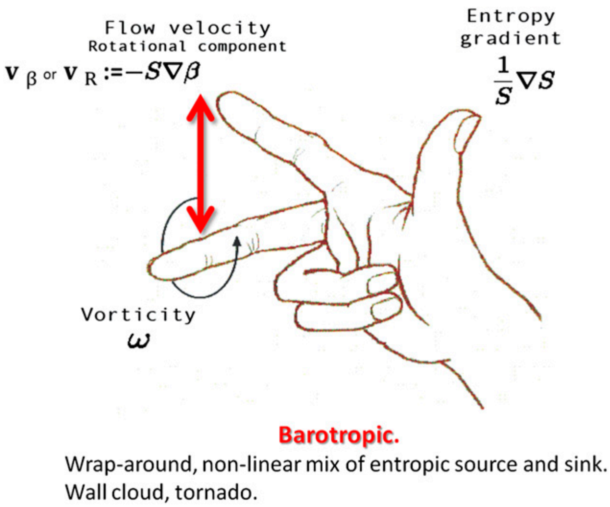

FIG. 4. Barotropic case of the entropic balance expressed as the right-hand rule.

multiple vortices, in barotropic state, as shown in Figs. 7 and 8, occurs through the wraparound process. The newly found wraparound mechanism for tornadogenesis is analogous to a nonlinear process, the so-called Baker's transformation, and the transition is discontinuous from baroclinic to barotropic stages by trapping the entropic sink core inside the vortex, like a nonlinear attractor (Figs. 1 and 9). Note also that the wraparound mechanism is two-dimensional, while Baker's transformation is one-dimensional.

The axisymmetric single vortex is theoretically well supported by the Noether's rotational invariant theorem (Kaku 1993) with the Lagrangian density [(2.3a)] and the application of the variational formalism. The existence of multiple vortices is also proven by the Noether's rotational invariant theorem extending it to application for sectional rotation.

Many visual observations of cases of single-vortex and multiple-vortex tornadoes are recorded. For instance, an excellent example is shown by the case of the tornado in central Oklahoma area on 3 May 1999. Multiple-vortex funnels were observed near Chickasha, Oklahoma, and a single EF5 tornado was observed near Moore, Oklahoma, likely from the same supercell during a few hours.

5. Conservation constraint of the classical system: Encompassing classical flow and quantum field mechanics

The Lagrangian density [(2.3a)] was originally used in the Euler and Lagrange procedure (section 3) to derive the Clebsch transformation (Clebsch 1857) and the entropic balance theory (Sasaki 1999). It is denoted as $\mathcal{L}_c$ because it does not include any quantum field procedure:

$$\mathcal{L}_c := \rho\{(1/2)\{\mathbf{v}^2 - U(\rho, S) - \Phi\} \\ - \alpha\{\partial_t\rho + \nabla\cdot(\rho\mathbf{v})\} - \beta\{\partial_t(\rho S) + \nabla(\rho\mathbf{v}S)\}\}. \tag{5.1}$$

As Feynman and Hibbs (1965) and Weinberg (1995) suggested, the classical flow field has direct physical connection to the quantum field variations, despite the huge order-of-magnitude differences in temporal and/or spatial scales, of these fields. An example of physical connection was theoretically predicted as the Aharonov–Bohm effect and experimentally proved by Tonomura (1997). The mksa rational unit system (Feynman and Hibbs 1965; Yoshida 2000; Kashiwa 2006) is used in this study: distance is in meters, mass is in kilograms, time is in seconds, force is in newtons ($\mathrm{N} = \mathrm{kg\,m\,s^{-1}}$), energy and work are in joules ($\mathrm{J} = \mathrm{N\,m}$) and watts ($\mathrm{W} = \mathrm{J\,s^{-1}}$), and electric current in is amperes (A), electric charge is in coulombs ($\mathrm{C} = \mathrm{A\ s}$), the charge of electron $e = 1.602\,177\,3 \times 10^{-19}$ C, mass of electron $m = 9.109\,389\,7 \times 10^{-31}$ kg, electric voltage is in volts (V), the intensity of electric filed is in volts per meter, magnetic bundle is in webers ($\mathrm{Wb} = \mathrm{A\ s}$), intensity of magnetic field is in amperes per meter, inductance is in henries ($\mathrm{H} = \mathrm{Wb\,A^{-1}}$), and static electricity is in faradays ($\mathrm{F} = \mathrm{C\,V^{-1}}$). The dielectric constant or permittivity ϵ_{oo} of the vacuum with the speed of light c and the magnetic permeability μ_o is $\epsilon_{oo} = 1/(\mu_o c^2)$.

The constraints in (2.3a) and (5.1) are concerned with the trajectory of flow particles. In the quantum field formalism, following Feynman and Hibbs (1965), the probability $P(b, a)$ of the trajectories is

$$P(b,a) = |K(b,a)|^2, \tag{5.2}$$

where $K(b, a)$ is the sum of the amplitude of each trajectory to go to b from a,

$$K(b,a) = \sum\phi[x(t)], \tag{5.3}$$

and sum is taken over all paths from a to b. The amplitude $\phi[x(t)]$, solution of the corresponding Schrödinger equation, appears as the phase proportional to the action L:

$$\phi[x(t)] \sim \exp\{(i/\hbar)L[x(t)]\} \\ \sim \exp(i\theta). \tag{5.4}$$

Note that the action is denoted usually by S in classical and modern physics; however, in the entropic balance

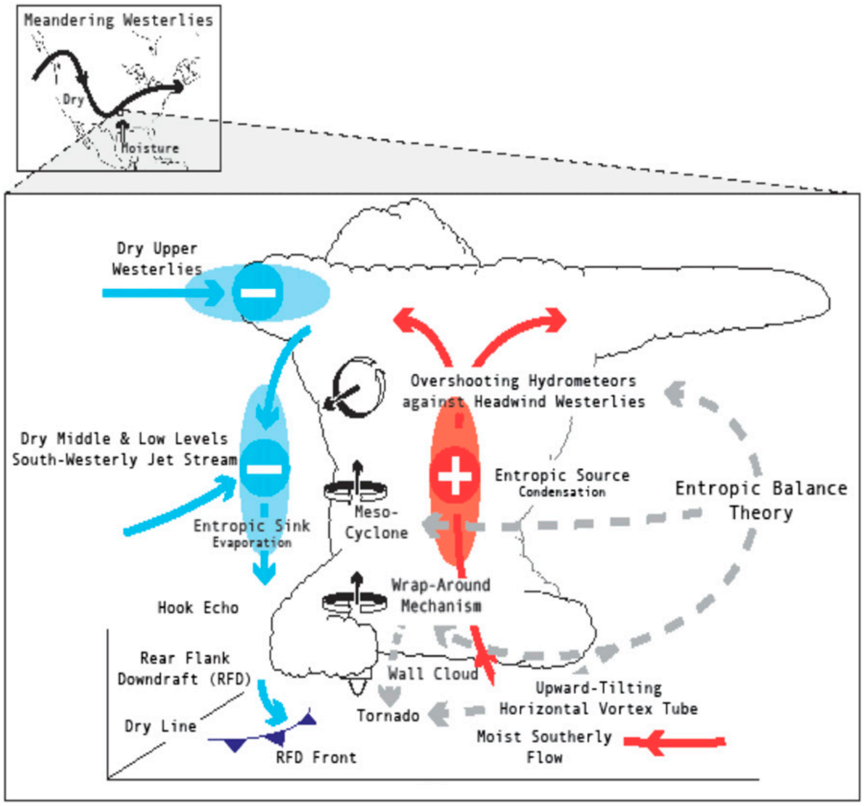

FIG. 5. Schematic diagram of tornadogenesis based on the entropic balance theory. Meandering westerlies transport water vapor evaporated from Gulf Stream deeper inland to the central U.S. Great Plains area and meet with the dry air to onset a supercell. The moisture of the southerly flow condenses and releases the latent heat to the surrounding air resulting in an entropic source that is shown by the circled plus sign, called the entropic source here. The hydrometeors, such as raindrops and ice particles, created by the condensation in the entropic source area are lifted by the updraft of the thermal convection of the storm reaching near the cloud top and are blown away toward the downstream (eastern) side of the storm. However, some of the lifted hydrometeors are overshot toward the upstream direction against the upper-air westerlies. This is due to the upper-air horizontal vortex, where the rotational flow direction is shown by the arrow with double solid lines. The overshot hydrometeors will fall down, evaporating because of the dry surrounding air, thus cooling the air. The descending hydrometeors with cooled air meet with a dry midlevel southwesterly jet and are cooled further to produce the rear-frank downdraft. Thus, an entropic sink forms nearly at the same level as the entropic source. The horizontal spatial gradient entropy is generated by the source–sink pair. A vortex (mesocyclone) is formed, and the wraparound mechanism is organized. The wraparound mechanism becomes activated by the mesocyclone that existed between the entropic source and sink. The wraparound mechanism is a nonlinear process (see Fig. 9), similar to the attractor, that generates a hook echo, a low-level mesocyclone, a wall cloud, and a tornado. These processes are explained by the diagnostic E–L equation, the entropic right-hand rule, and the wraparound mechanism that are derived by the entropic balance theory.

theory we have been using S as entropy and L as the action following convention [(2.3a), (2.3b), and (5.1)]. The magnitude of kinetic energy, the first term in (5.1), is estimated as 0.5×10^{-2} mksa for $\rho = 10^{-3}\,\mathrm{g}^{-1}$ and $|\mathbf{v}| = 10\,\mathrm{m\,s}^{-1}$. The magnitudes of U and Φ are assumed similar. Since the terms of the constraints vanish in classical mechanics, because

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \text{and} \tag{5.5a}$$

$$\partial_t(\rho S) + \nabla(\rho\mathbf{v}S) = 0, \tag{5.5b}$$

the E–L equations of (5.1) require that the temporal and spatial gradients do not vanish and lead the Euler–Lagrange formalism, in which conditions are usually fulfilled but not

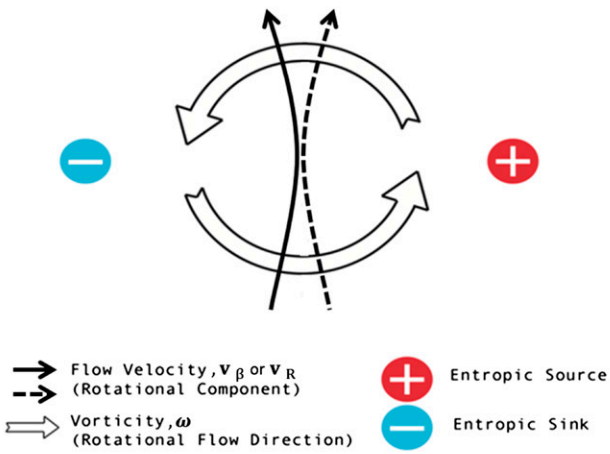

FIG. 6. An entropic vortex exists in balance with the gradient of entropy in the baroclinic field. The horizontal vortex is formed due to the slanted vertical entropy gradient at the upper levels above the entropic source and overshoots the hydrometeors upstream against the headwind westerlies. The vertical vortex is formed at the middle levels due to the horizontal gradient between the source and the sink of entropy. The entropic vortex formation is explained by the entropic right-hand rule derived from the entropic balance theory.

always satisfied. The quantum field formalism solves the difficulty.

Since $\hbar = h/(2\pi) = 1.054 \times 10^{-34}$ mksa-rationalized system (J s), the magnitude of $L/\hbar$ is on the order of 10^{32}, so that the quantum field formalism has not been

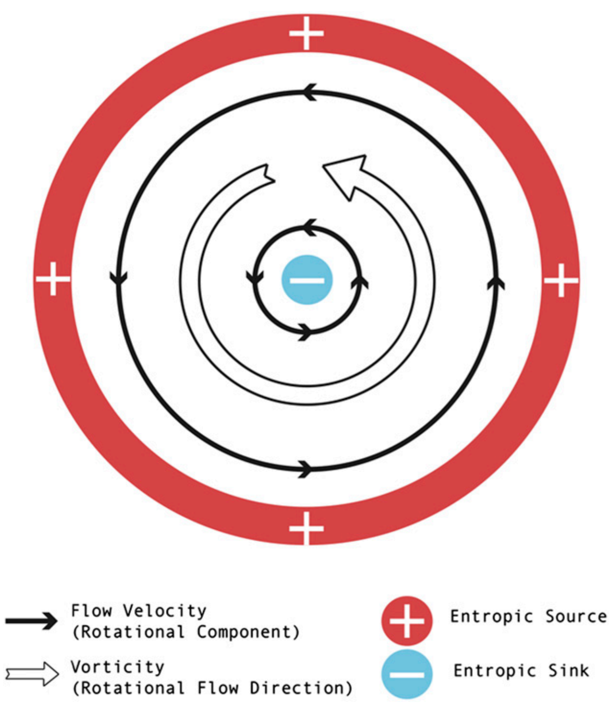

FIG. 7. Schematic illustration of single-vortex tornadogenesis explained by the wraparound mechanism. The wraparound mechanism is modeled by the entropic right-hand rule that is derived from the entropic balance theory.

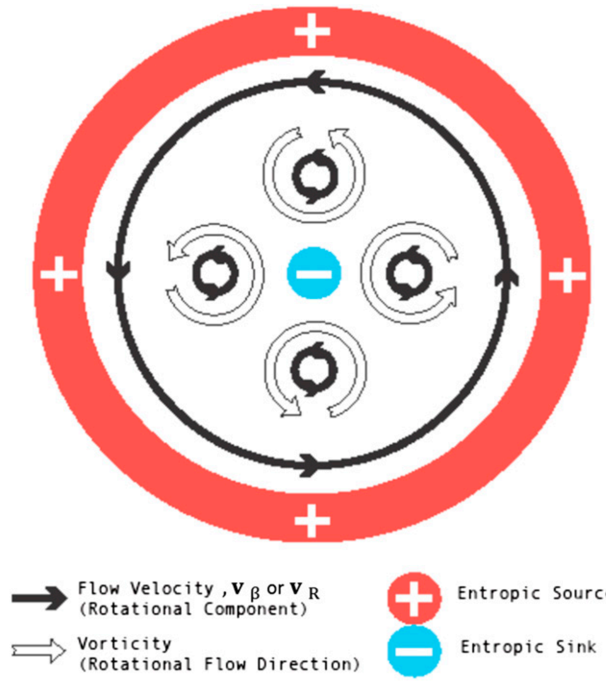

FIG. 8. Schematic illustration of multiple-vortex tornadogenesis, for four vortices as an example, explained by the wraparound mechanism.

considered in the classical flow formalism, which is on the order of magnitude of $1/10^{32}$. However, the constraints (5.5a) and (5.5b) are of the value of 0, and the value of $1/10^{32}$ is extremely small compared with the

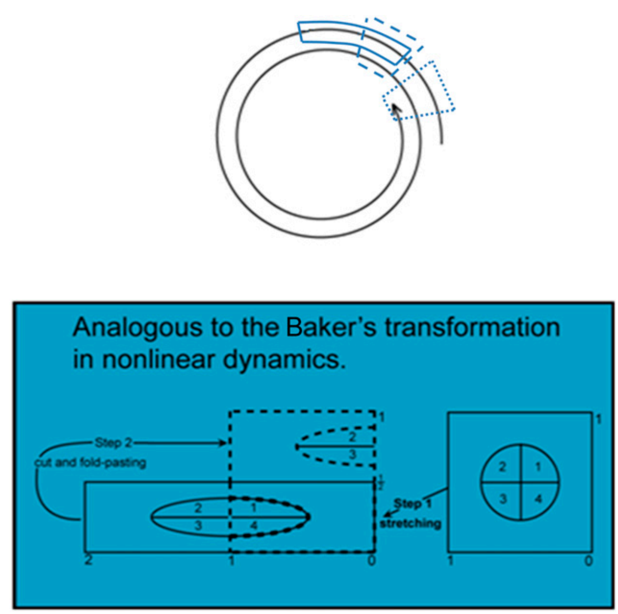

FIG. 9. Wraparound mechanism (two dimensional) analogous to the nonlinear Baker's transformation (one dimensional). The wraparound mechanism is modeled by the entropic right-hand rule that is derived from the entropic balance theory.

classical flow terms but of the same magnitude of quantum field formalism. However, the interaction between the classical and quantum systems, like the Aharanov–Bohm effect, could be the same order of magnitude as the classical flow system. Consequently, we may modify the Lagrangian density [(5.1)] to include a quantum field formalism instead of (5.5a) and (5.5b) as

$$\partial_t \rho_q + \nabla \cdot (\rho_q \mathbf{v}_q) - f_{q,\alpha} = 0 \quad \text{and} \tag{5.6a}$$

$$\partial_t (\rho_q S_q) + \nabla(\rho_q \mathbf{v} S) - f_{q,\beta} = 0, \tag{5.6b}$$

where the last terms of (5.6a) and (5.6b) do not vanish if there is a physical connection between the classical and quantum fields, such as shown by the Aharonov–Bohm effect.

Accordingly, the inclusion of the quantum flow formalism based on Feynman and Hibbs (1965) may serve as a comprehensive way for the variational field Lagrangian approach, a new area of research. The variations in calculus of variations and the classical variational formalism are hypothetical. However, this study suggests that the variations are physical, relating to quantum variations and their interaction with classical systems. It would cast some light on the still mysterious lightning and microphysical processes of storm clouds (Petersen et al. 2008). It would justify the use of the variational formalism better than a deterministic Newtonian approach for the classical system.

Acknowledgments. The author expresses his appreciation for the encouragement and support given to this research by Dr. Kelvin Droegemeier, Vice President for Research, Dr. Robert Palmer, Director, Advanced Radar Research Center, University of Oklahoma, and Dr. Seon Ki Park of Ewha Women's University, and also to Weathernews, Inc., for generous support. Also, the author expresses his appreciation to faculty and scientists especially of NWC, University of Oklahoma, University of Tokyo, Kyoto University, Japan, for their useful discussion and to Vivek Mahale and Mark Laufersweiler of the University of Oklahoma for his assistance with computer graphics. I would like to acknowledge that Springer publisher gave permission to the author for the use of nine figures for this manuscript from the published book *Data Assimilation for Atmospheric, Oceanic, and Hydrostatic Applications* (Vol. 2), Chapter 18: Entropic Balance Theory and Radar Observation for Prospective Tornado Data Assimilation, Yoshi K. Sasaki, Mathew R. Kumjian, and Bradley M. Isom.

APPENDIX

Prognostic E–L Equations

Prognostic equations may be obtained by applying the variational formalism for the action Lagrangian L with respect to ρ, α, β, and S. The corresponding E–L equations become, respectively,

$$\delta_{\rho,\alpha,\beta,S} L = 0, \tag{A.1}$$

and then, the E–L equation from the first variation of (2.4) with respect to ρ,

$$\partial_t \alpha + S \partial_t \beta - \frac{1}{2}\mathbf{v}^2 - U(\rho, S) - \rho U_\rho(\rho, S) - \Phi = 0, \tag{A.2}$$

the mass continuity equation from (2.4) with respect to α,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{A.3}$$

the entropy equation from (2.4) with respect to β,

$$\partial_t S + \mathbf{v} \cdot \nabla S = 0, \tag{A.4}$$

and the E–L equation from (2.4) with respect to S,

$$\partial_t \beta + \mathbf{v} \cdot \nabla \beta = 0. \tag{A.5}$$

The equation of motion on any orthogonal coordinates on the rotating Earth is obtained from the above E–L equations as follows. Operating ∇ on (A.2) and using (A.3)–(A.5), we may prove

$$\nabla(\partial_t \alpha + S \partial_t \beta) = \partial_t(\nabla \alpha + S \nabla \beta) - \partial_t \mathbf{v}, \tag{A.6}$$

and

$$\partial_t \mathbf{v} + \nabla\left(\frac{1}{2}\mathbf{v}^2\right) + \nabla[U(\rho, S) - \rho U_\rho(\rho, S)] + \nabla \Phi = 0. \tag{A.7}$$

Because of the field Lagrangian formalism, the following vector equation is obtained:

$$\nabla\left(\frac{1}{2}\mathbf{v}^2\right) = (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{A.8}$$

which is appropriate to get the correct equation of motion on any orthogonal coordinates, polar, cylindrical, or spherical.

Thus, using (A.8), we easily obtain

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla[U(\rho,S) - \rho U_\rho(\rho,S)] + \nabla\Phi = 0. \tag{A.9}$$

Let us consider a moving coordinate system R, say the rotating Earth, relative to the fixed absolute coordinate system A. A position vector $\mathbf{x}$ that is defined on the absolute coordinate is expressed on the relative coordinates of the rotating system with the angular velocity $\boldsymbol{\Omega}$ measured on the absolute coordinates as follows:

$$(\partial_t \mathbf{r})_A = (\partial_t \mathbf{r})_R + \boldsymbol{\Omega}\times\mathbf{x}, \tag{A.10a}$$

or

$$(\mathbf{v}_A)_A = (\mathbf{v}_A)_R + \boldsymbol{\Omega}\times\mathbf{x}. \tag{A.10b}$$

The local change of (A.10b), recognizing that $\mathbf{x}$ is measured on the absolute coordinates, becomes

$$(\partial_t \mathbf{v}_A)_A = (\partial_t \mathbf{v}_A)_R + \boldsymbol{\Omega}\times\mathbf{v}_A. \tag{A.11}$$

Substituting $\mathbf{v}_A$ of (A.10b) into (A.11), and recognizing $\partial_t\mathbf{v}_A = \partial_t\mathbf{v}_R = \partial_t\mathbf{v}$, we get

$$(\partial_t \mathbf{v})_A = (\partial_t \mathbf{v})_R + 2\boldsymbol{\Omega}\times\mathbf{v}_R + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}). \tag{A.12}$$

From (3.7), we get the equation of motion in the vector form on the relative coordinates of the rotating Earth since the centrifugal force $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$ is included in the gravity field $\nabla\Phi$, denoted by g:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} + 1/\rho\nabla p + g = 0, \tag{A.13}$$

where

$$(1/\rho)\nabla p = \nabla[U(\rho,S) - \rho U_\rho(\rho,S)]. \tag{A.14}$$

Since the specific internal energy U has the relations with density ρ and entropy S as

$$\partial_S U = p/p^2 \quad \text{and} \quad \partial_\rho U = T, \tag{A.15}$$

and

$$\nabla U = \partial_\rho U \nabla\rho + \partial_S U \nabla S, \tag{A.16}$$

we get

$$\rho\nabla(U + p/\rho + \Phi) - \rho T\Delta S = \nabla p. \tag{A.17}$$

The equation of motion [(A.13)] is in vector form, and it is valid for all orthogonal coordinates systems, such as linear x, y, and z coordinates, cylindrical, and spherical coordinates.

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