Characteristics of the Heat Flux in the Unstable Atmospheric Surface Layer

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(Manuscript received 22 September 2015, in final form 11 June 2016)

ABSTRACT

The behavior of the heat flux $H$ with respect to the stability parameter $\zeta$ ($=z/L$, where $z$ is the height above the ground, and $L$ is the Obukhov length) in the unstable atmospheric surface layer is analyzed within the framework of Monin–Obukhov similarity (MOS) theory. Using MOS equations, $H$ is expressed as a function of $\zeta$ and vertical surface-layer potential temperature gradient $\partial T/\partial z$. A mathematical analysis is carried out to analyze the theoretical nature of heat flux with the stability parameter by considering the vertical potential temperature gradient as (i) a constant and (ii) a power-law function of heat flux. For a given value of $H$, two values of $\zeta$ associated with different stability regimes are found to occur in both the conditions, suggesting the nonuniqueness of MOS equations.

Turbulent data over three different sites—(i) Ranchi, India; (ii) the Met Office’s Cardington, United Kingdom, monitoring facility; and (iii) 1999 Cooperative Atmosphere–Surface Exchange Study (CASES-99; United States)—are analyzed to compare the observed nature of $H$ with that predicted by MOS. The analysis of observational data over these three sites reveals that the observed variation of $H$ with $\zeta$ is consistent with that obtained theoretically from MOS equations when considering the vertical temperature gradient as a power-law function of heat flux having the exponent larger than $2/3$. The existence of two different values of the stability parameter for a given value of heat flux suggests that the application of heat flux as a boundary condition involves some intricacies, and it should be applied with caution in convective conditions.

1. Introduction

Monin and Obukhov similarity (MOS) theory (Monin and Obukhov 1954) is widely used to estimate the stability parameter $\zeta$ ($=z/L$, where $z$ is the height above the ground, and $L$ is the Obukhov length) and surface fluxes in atmospheric models for weather forecasting as well as for air quality and climate modeling (Arya 1988; Beljaars and Holtslag 1991; Garratt 1994; Oleson et al. 2008; Skamarock et al. 2008; Jimenez et al. 2012; Giorgi et al. 2012; Pielke 2013). These surface fluxes are of crucial importance not only because they influence the steady state of the atmosphere, but also because they determine the mean profiles in the atmospheric boundary layer (Holtslag and Nieuwstadt 1986; Beljaars and Holtslag 1991). The input parameters, such as exchange coefficients, boundary layer height, required in short-range forecasts and air pollution modeling are dependent on the surface fluxes. Further, in single-column and large-eddy simulation models for the study of the atmospheric boundary layer, surface layer sensible heat flux is frequently used as a boundary condition (Basu et al. 2008; Gibbs et al. 2015).

A number of studies (Taylor 1971; De Bruin 1994; Malhi 1995; Derbyshire 1999; van de Wiel et al. 2007; Basu et al. 2008; Wang and Bras 2010; van de Wiel et al. 2011; Gibbs et al. 2015) suggest that the MOS equations have multiple solutions in the stable atmospheric condition, and thus it is not legitimate to use sensible heat flux as a boundary condition in numerical models in stable conditions. Basu et al. (2008) have pointed out the shortcomings of using sensible heat flux as a surface boundary condition in numerical models and suggested that, in order to capture the very stable regime in numerical models, one should prescribe surface temperature as a boundary condition in place of sensible heat flux.

Wang and Bras (2010) have pointed out that the nonuniqueness of MOS equations is a numerical artifact.
and have postulated an extremum hypothesis of turbulent transport in the atmospheric surface layer to derive a unique solution for the MOS equations. The hypothesis of Wang and Bras (2010, 2011) leads to possible simplification of MOS equations by replacing the universal similarity functions with some empirical constants in both stable and unstable atmospheric conditions. The interpretations of Wang and Bras (2010, 2011) are contradictory to those reported in the literature by a number of researchers in the stable atmospheric conditions. Consequently, van de Wiel et al. (2011) have argued that the nonuniqueness of MOS equations is not a mathematical artifact and have provided a physical justification for the heat flux maxima and dual nature of heat flux resulting in nonexistence of any preferred stability state. However, the study of van de Wiel et al. (2011) is limited to the stable conditions, and, to the authors’ knowledge, the detailed theoretical and observational analyses of the behavior of sensible heat flux in daytime convective conditions have not yet been carried out to assess the possible nonuniqueness of MOS equations and its implications in numerical models. Hence, an attempt is made here to analyze the nature of sensible heat flux in an unstable atmospheric surface layer within the framework of MOS theory.

2. Methodology

According to MOS theory, nondimensional gradients of the mean wind speed $U$ and virtual potential temperature $\theta$ in a homogeneous surface layer are universal functions of the stability parameter $\zeta$:

\[
\begin{align*}
\frac{kz}{u_0} \frac{\partial U}{\partial z} &= \varphi_m(\zeta), \\
\frac{kz}{\theta_0} \frac{\partial \theta}{\partial z} &= \varphi_h(\zeta),
\end{align*}
\]

where $k$ is the von Kármán constant, $z$ is the height above the ground, $u_0$ and $\theta_0$ are the friction velocity and temperature scales, and $\varphi_m$ and $\varphi_h$ are, respectively, the similarity functions describing dimensionless wind and temperature profiles in terms of $\zeta$. Here, $\zeta$ is defined as

\[
\zeta = \frac{kzg}{\bar{\theta}} \left( -\frac{H}{\rho C_p u_0^2} \right),
\]

in which $\bar{\theta}$ is the mean potential temperature in the layer, $g$ is the acceleration due to gravity, $\rho$ is density of the dry air, $C_p$ is specific heat capacity at constant pressure, and $H$ is the heat flux, defined as

\[
H = -\rho C_p u_0 \theta_0.
\]

Substituting the value of $u_0$ from Eq. (4a) in Eq. (3), one gets

\[
\left( \frac{H}{\rho C_p} \right)^2 = \frac{kzg}{\bar{\theta}} \frac{\theta_0^3}{\zeta}.
\]

We assume that $\zeta$ is finite ($0 < -\zeta < \infty$) and nonzero ($\zeta \neq 0$). In the unstable atmospheric conditions, there is a positive upward heat flux and negative temperature gradient $\partial \theta / \partial z$. Hence, the temperature scale $\theta_0$ and stability parameter $\zeta$ are negative in unstable conditions. As a result of this, the expression on the right-hand side of Eq. (4b) is positive.

Substituting the value of $\theta_0$ from Eq. (2) in Eq. (4b), one gets

\[
\left( \frac{H}{\rho C_p} \right)^2 = (kz)^3 \frac{g}{\bar{\theta} \varphi_h(\zeta)} \frac{(\partial \theta / \partial z)^3}{\zeta}.
\]

A number of expressions for the universal similarity functions $\varphi_m$ and $\varphi_h$ have been proposed in the literature by various researchers (Kazansky and Monin 1956; Ellison 1957; Yamamoto 1959; Panofsky 1961; Sellers 1962; Businger et al. 1971; Dyer 1974; Högström 1996; De Bruin 1999; Wilson 2001). However, the Businger–Dyer similarity functions are commonly used in atmospheric models (WRF; Skamarock et al. 2008). Thus, in the subsequent analysis, an expression for the function $\varphi_h$ in unstable conditions is taken as

\[
\varphi_h(\zeta) = \text{Pr}_r (1 - \gamma_h \zeta)^{-1/2},
\]

in which $\text{Pr}_r = (\varphi_h / \varphi_m)_{\zeta=0}$ is the turbulent Prandtl number for neutral stability and $\gamma_h$ is a constant. The functions $\varphi_m$ and $\varphi_h$ differ on the values of $\text{Pr}_r$, $\gamma_m$, and $\gamma_h$. The constant $\gamma_m$ appears in the expression of $\varphi_m$. The values of $\text{Pr}_r$, $\gamma_m$ and $\gamma_h$ are $\text{Pr}_r = 0.74$, $\gamma_m = 16$, and $\gamma_h = 9$ (Businger et al. 1971); $\text{Pr}_r = 1$ and $\gamma_m = \gamma_h = 16$ (Dyer 1974); and $\text{Pr}_r = 0.95$, $\gamma_m = 19$, and $\gamma_h = 11.6$ (Högström 1996). Substituting the expression of $\varphi_h$ from Eq. (6) into Eq. (5), we obtain

\[
\left( \frac{H}{\rho C_p} \right)^2 = (kz)^3 \frac{g}{\text{Pr}_r^3 (1 - \gamma_h \zeta)^{3/2}} \frac{(\partial \theta / \partial z)^3}{\zeta}.
\]

As $\partial \theta / \partial z < 0$ and $\zeta < 0$ in unstable conditions, without loss of generality, we replace $\partial \theta / \partial z$ with $-\partial \theta / \partial z$ and $\zeta$ with $-\eta$. Accordingly, Eq. (7) can be rewritten as

\[
\left( \frac{H}{\rho C_p} \right)^2 = (kz)^3 \frac{g}{\text{Pr}_r^3 (1 + \gamma_h \eta)^{-3/2}} \frac{(\partial \theta / \partial z)^3}{\eta}.
\]
Taking the square root of both the sides and retaining the positive sign, as $H$ is positive in unstable conditions, one obtains

$$H^* = (kz)^2 \sqrt{\frac{g}{\vartheta}} \left( \frac{\partial \theta_n}{\partial z} \right)^{3/2} \frac{(1 + \gamma_h \eta)^{-3/4}}{(1 + \gamma_h \eta)^{3/4}}. \quad (9)$$

in which $H^* = H/\rho C_p$.

The behavior of $H^*$ with $\eta$ depends on the value of the vertical potential temperature gradient ($\partial \theta_n/\partial z$) in surface layer or the lapse rate. In the subsequent analysis, the nature of $H^*$ with $\eta$ is analyzed for two assumed forms of vertical potential temperature gradient: (i) as a constant and (ii) as a power-law function of heat flux.

### a. Case I: Constant vertical temperature gradient

For a given constant $\delta > 0$ such that $\partial \theta_n/\partial z = \delta$, Eq. (9) can be rewritten as

$$H^* = (kz)^2 \sqrt{\frac{g}{\vartheta}} \frac{\delta^{3/2}}{\Pr_f^{3/2} (1 + \gamma_h \eta)^{-3/4}}. \quad (10)$$

For determining maximum and minimum values, the vanishing of the first-order derivative of the function leads to the critical points, and then the positive (negative) value of the second-order derivative at the critical point ensures the point is minimum (maximum). For this purpose, differentiating Eq. (10) with respect to $\eta$, one gets

$$dH^*/d\eta = \left( (kz)^2 \sqrt{\frac{g}{\vartheta}} \left( \frac{\delta}{\Pr_f} \right)^{3/2} \eta^{-3/2} (1 + \gamma_h \eta)^{-1/4} \right) \left[ -1 + \gamma_h \eta + \frac{3}{2} \eta \gamma_h \right].$$

(11)

For the critical points of $H^*$, $dH^*/d\eta = 0$, which leads to a critical point $\eta = \eta_c = 2/\gamma_h$. The value of the second-order derivative with respect to $\eta$ at the critical point $\eta_c = 2/\gamma_h$ turns out to be

$$\left. \frac{d^2H^*}{d\eta^2} \right|_{\eta=\eta_c} = \left( (kz)^2 \sqrt{\frac{g}{\vartheta}} \left( \frac{\delta}{\Pr_f} \right)^{3/2} \left( \frac{2}{\gamma_h} \right)^{-3/2} \left( \frac{1}{4} \right)^{-1/4} \right)^2 \left( \frac{2}{\gamma_h} \right)^{-3/2} \left( \frac{3}{4} \right)^{-1/4} \gamma_h.$$

which is positive, as $\gamma_h > 0$ and $\delta > 0$. Thus, $H^*$ attains its minimum value at $\eta_c$. This suggests that, for a given value of $\partial \theta_n/\partial z$, $H^*$ decreases with $\eta$ until it attains its minimum value at $\eta_c$ and then starts increasing with increasing instability. The corresponding minimum value of $H_{min}$ is obtained from Eq. (10) as

$$H_{min} = \frac{27 \gamma_h^4}{4} (kz)^2 \sqrt{\frac{g}{\vartheta}} \left( \frac{\delta}{\Pr_f} \right)^{3/2}. \quad (12)$$

In nondimensional form, Eq. (10) is written as

$$\frac{H}{H_{min}} = \left( \frac{4}{27 \gamma_h} \right)^{1/4} \frac{1}{\left( 1 + \gamma_h \eta \right)^{3/4}}. \quad (13)$$

Thus, for a given temperature gradient, there exist two values of $\zeta (-\eta)$ corresponding to the same magnitude of heat flux (Fig. 1a).

### b. Case II: Varied vertical temperature gradient

The vertical potential temperature gradient is considered to vary as a power-law function of heat flux: that is,

$$\left( \frac{\partial \theta_n}{\partial z} \right) = \beta (H^*)^\alpha,$$

in which $\alpha$ and $\beta$ are constants.

Using Eq. (14), Eq. (9) can be written as

$$H^* = \left( (kz)^2 \sqrt{\frac{g}{\vartheta}} \right) \frac{\beta^{3/2}}{\Pr_f^{3/2} (1 + \gamma_h \eta)^{-3/4}}. \quad (15)$$

To obtain the critical points, differentiating Eq. (15) with respect to $\eta$, one gets

$$\left( 1 - \frac{3\alpha}{2} \right) H^* dH^*/d\eta = \left( (kz)^2 \sqrt{\frac{g}{\vartheta}} \right) \frac{\beta^{3/2}}{\Pr_f^{3/2}} \left( \eta^{-3/2} (1 + \gamma_h \eta)^{-1/4} \right) \times \left[ -1 + \gamma_h \eta + \frac{3}{2} \eta \gamma_h \right].$$

(16a)

Thus, in this case $H^*$ has a critical point at $\eta = \eta_c = 2/\gamma_h$ provided $\alpha \neq \gamma_h$. The value of second-order derivative at the critical point $\eta_c = 2/\gamma_h$ turns out to be

$$\left. \frac{d^2H^*}{d\eta^2} \right|_{\eta=\eta_c} = \left( \frac{1}{\left( 1 - 3\alpha/2 \right)} \right) \frac{1}{H^*} \\left( \frac{\beta^{3/2}}{\Pr_f^{3/2}} \right)^{3/2} \left( \frac{2}{\gamma_h} \right)^{-3/2} \left( \frac{3}{4} \right)^{-1/4} \gamma_h.$$

(16b)

As $\beta > 0$ and $\gamma_h > 0$, $d^2H^*/d\eta^2 > 0$, provided $\alpha < \gamma_h$ and $d^2H^*/d\eta^2 < 0$ for $\alpha > \gamma_h$ [Eq. (16b)]. This suggests that, for $\alpha > \gamma_h$, $H^*$ first increases with $\eta$ until it attains a maximum value at $\eta_c = 2/\gamma_h$ and then starts decreasing with increasing value of $\eta$. The corresponding maximum value of $H_{max}$ is obtained from Eq. (15) as
In nondimensional form, Eq. (15) is written as

$$H_{\text{max}} = \left[ \frac{27}{4\gamma_h^2} (kz)^{1/4} \left( \frac{g}{\beta} \right)^{3/2} \right]^{1/(1-3\alpha/2)} , \quad \alpha > 2/3 .$$

(17)

The same critical point (i.e., $\eta = 2/\gamma_h$) is a point of minimum for $\alpha < 2/3$ (Fig. 1c). Thus, there exist two values of stability parameter $\zeta (=-\eta)$, corresponding to the same magnitude of heat flux (Figs. 1b,c), provided $\alpha \neq 2/3$. For $\alpha = 2/3$, Eq. (15) leads to an identity

$$H_{\text{max}} = \left[ \frac{4}{27\gamma_h^2} \left( \frac{g}{\beta} \right)^{3/2} \frac{1}{(1 + \gamma_h \eta)^{-3/4}(\eta^{1/2})} \right]^{1/(1-3\alpha/2)} .$$

(18)

The parameter $\gamma_h$ in Eq. (17) is the heat conductivity, which is non-dimensionalized by its maximum value $H_{\text{max}}$. The same critical point (i.e., $\eta = 2/\gamma_h$) is a point of minimum for $\alpha < 2/3$ (Fig. 1c). Thus, there exist two values of stability parameter $\zeta (=-\eta)$, corresponding to the same magnitude of heat flux (Figs. 1b,c), provided $\alpha \neq 2/3$. For $\alpha = 2/3$, Eq. (15) leads to an identity

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(18)

The data used in the present study are obtained from a fast response sensor [Campbell Scientific 3D sonic anemometer CSAT3] installed at 10-m height to measure the three components of wind and temperature at a frequency of 10 Hz, mounted over a 32-m micrometeorological tower deployed in a remote and grassland area of the Birla Institute of Technology, Mesra, Ranchi (23.412°N, 85.440°E), India, with an average elevation 609 m above mean sea level in a tropical region (Dwivedi et al. 2015). This study analyzes turbulent measurements for the year 2009 taken at 10-m height with a sonic anemometer. The hourly fluxes are calculated using an eddy correlation technique. The whole dataset is divided based on the wind speed and the stability regimes (Srivastava and Sharan 2015). As the focus of the study is on unstable
conditions, data during the daytime conditions with \( \zeta < 0 \) are used. The data that correspond to a wind speed of less than 0.1 m s\(^{-1}\), the transition period between day and night, or before and after rainfall are excluded. A total of 2865 hourly data points with \( \zeta < 0 \) are obtained with 0.1 \(< U < 10 \) m s\(^{-1}\).

b. Met Office’s Cardington (United Kingdom) monitoring facility dataset

Another dataset used in this study is taken from the meteorological tower installed at Cardington in Bedfordshire (52.06\(^\circ\)N, 0.25\(^\circ\)W) in the United Kingdom with average height of 29 m above mean sea level. The site is a large grassy field and relatively flat (Luhar et al. 2009). The dataset contains the high-frequency wind and temperature observations recorded from the ultrasonic anemometers at 10, 25, and 50 m above the ground surface and slow-response temperature measurements at 1.2, 10, 25, and 50 m obtained from the platinum resistance thermometer. In the present study, 30-min-averaged data at the 10-m level for the months of June, July, and August of years 2007, 2008, and 2009 are used to calculate the hourly averaged turbulent quantities. After the analysis, data corresponding to daytime unstable conditions (i.e., \( \zeta < 0 \)) are chosen for the study. A total of 2382 hourly data points with \( \zeta < 0 \) and 0.1 \(< U < 10 \) m s\(^{-1}\) are selected for the study.

c. CASES-99 (United States) dataset

The CASES-99 experiment was performed near Leon (37.38\(^\circ\)N, 96.14\(^\circ\)W) in southeastern Kansas in the United States during the month of October 1999. The turbulence measurements were taken on a 60-m tower at 1.5 (0.5)-, 5-, 10-, 20-, 30-, 40-, 50-, and 55-m levels by eight sonic anemometers. The experiment was focused on the stable atmospheric boundary layer; however, it provides good observations for the study of surface layer characteristics during free convection (Poulos et al. 2002; Steeneveld et al. 2005). The terrain of the main experimental site was relatively homogeneously flat and lacked obstacles in the surroundings. The turbulence measurements taken from a sonic anemometer at the 10-m level on a 60-m tower (Poulos et al. 2002) are used in the present study. Detailed information on the experiment is given in Poulos et al. (2002). The turbulence at a sampling rate of 20 Hz was used in the analysis to calculate the fluxes. Hourly turbulent fluxes at the 10-m level are calculated from the observations by using an eddy correlation technique. After the analysis of turbulence data at the 10-m level, 249 hourly averaged data points corresponding to unstable conditions (i.e., \( \zeta < 0 \)) are selected for the study.

### 4. Results and discussion

#### a. Theoretical analysis

An analysis is carried out to analyze the behavior of heat flux with stability in unstable conditions within the framework of MOS theory. The sensible heat flux obtained from MOS equations in gradient form is expressed as a function of stability parameter \( \zeta \) and the vertical potential temperature gradient \( \partial \theta / \partial z \). The behavior of the heat flux is analyzed by considering the vertical surface layer potential temperature gradient as (i) a constant and (ii) a power-law function of heat flux. For a prescribed value of temperature gradient, heat flux attains its minimum value at \( \zeta = -2/\gamma_h \). The non-dimensional heat flux \( H/H_{\text{min}} \) is observed to decrease until it attains a minimum value at \( \zeta = -2/\gamma_h \) and then starts increasing with increasing instability (Fig. 1a). This suggests that, for a given value of heat flux, there exist two different values of stability parameter. Out of these two values, one corresponds to weakly unstable conditions, and the other one characterizes a relatively stronger instability.

By considering the power-law profile for vertical temperature as \( -\partial \theta / \partial z = \beta (H/\rho C_p)^{\alpha} \), the analysis shows that, if \( \alpha > \gamma_s \), \( H \) attains its maximum value at \( \zeta = -2/\gamma_h \). The magnitude of nondimensional heat flux \( H/H_{\text{max}} \) is small in near-neutral conditions. This increases with increasing instability until it attains the maximum value at \( \zeta = -2/\gamma_h \) and then starts decreasing with further increasing instability (Fig. 1b). The theoretical behavior of heat flux for \( -\partial \theta / \partial z = \beta (H/\rho C_p)^{\alpha} \) remains similar to that for \( -\partial \theta / \partial z = \delta \) if \( \alpha < \gamma_s \) (Fig. 1c).

It is interesting to note that the extremum point remains the same in all three conditions (i.e., \( \zeta = -2/\gamma_h \)). The extremum point implies the value of \( \zeta \) at which heat flux takes a maximum or minimum value.

The theoretical behavior of heat flux \( H \) with stability appears to be consistent with that observed if one takes \( -\partial \theta / \partial z = \beta (H/\rho C_p)^{\alpha} \) with \( \alpha > \gamma_s \). This behavior of the heat flux with stability may be justified as follows. In the weakly unstable conditions, the magnitude of heat flux is small because of a weak temperature gradient. In this condition, the mechanical production of turbulence is dominant. As the instability increases, the role of mechanical as well as thermal turbulence is significant, and heat flux attains a maximum value at a critical point \( \zeta = -2/\gamma_h \). However, beyond this point of maximum heat flux, the thermal turbulence is more significant than the mechanical turbulence, resulting in a large value of \( -\zeta \) but comparatively small value of \( H \). Here, we have analyzed the nature of \( H \) as a function of \( \zeta \) and \( \partial \theta / \partial z \) and found that existence of multiple values of \( \zeta \) for a given value of heat flux persists in the case of unstable
conditions in a manner similar to that in stable conditions. In the next section, we have analyzed three different datasets to support the theoretical behavior of $H$ in unstable conditions with the observed behavior with respect to $\zeta$.

b. Observational analysis

Figures 2a–e show the variation of heat flux $H$ with stability in unstable conditions for the Ranchi, Cardington, and CASES-99 datasets. The variation of $H$ with the stability parameter $\zeta$ is bounded by a curve. This curve first shows increasing behavior with $-\zeta$ until it attains a peak at $\zeta \approx -0.12$ and then decreases further with increasing instability. In the near-neutral condition, the value of the heat flux is small because of a weak temperature gradient. It starts increasing with instability and attains a maximum value around $\zeta \approx -0.12$. Beyond this value, it again starts decreasing with increasing instability. This shows that a given value of heat flux characterizes two different
maximum at the transition point of the DNS–DCS and creases with increasing instability until it reaches its average value of the DCS to the FCS sublayers suggesting that transition layers. However, the values decrease from in each sublayer. It is clear from Table 2 that the stability. Table 2 shows the observed average value of H is observed from its peak value with increasing instability. This observed nature of H with ζ appears to be similar to the stable conditions, where the same value of H corresponds to near-neutral and very stable atmospheric conditions (Taylor 1971; De Bruin 1994; Malhi 1995; Derbyshire 1999; van de Wiel et al. 2007; Basu et al. 2008; Wang and Bras 2010; van de Wiel et al. 2011; Gibbs et al. 2015). The observations show that the value of heat flux remains small in both near-neutral conditions and very unstable conditions. It attains a maximum value in between these two stability regimes at approximately ζ ≈ −0.12.

The nature of H is further analyzed in the five unstable sublayers, similar to Srivastava and Sharan (2015), as suggested by Kader and Yaglom (1990) and Bernardes and Dias (2010). These sublayers are dynamic (DNS), dynamic–dynamic convective transition (DNS–DCS transition), dynamic convective (DCS), dynamic convective–free convective transition (DCS–FCS transition), and free convective (FCS) based on the strict ranges of the stability parameter ζ in each sublayer. The quantitative description of the data in each sublayer is given in Table 1. The data lying in the DNS and DNS–DCS transition layers characterize weakly to moderately unstable conditions, while data belonging to the DCS, DCS–FCS transition, and FCS layers represent moderately to strongly convective conditions. The heat flux H shows an increasing behavior with −ζ from DNS to the DNS–DCS transition layer until it attains a peak at ζ ≈ −0.12 (Figs. 2b,d,f). However, a decreasing trend of H is observed from its peak value with increasing instability. Table 2 shows the observed average value of H in each sublayer. It is clear from Table 2 that the average value of H increases from DNS to DNS–DCS transition layers. However, the values decrease from the DCS to the FCS sublayers suggesting that H increases with increasing instability until it reaches its maximum at the transition point of the DNS–DCS and DCS sublayers and then decreases with increasing instability.

The behavior appears to be similar for all three datasets chosen in the present study (Figs. 2a,e). However, the relationship is relatively more pronounced in Ranchi and Cardington data as compared to CASES-99 data. There is a variation in the point of extremum over the different datasets. In the Cardington dataset, the extremum point is observed to lie in the DCS sublayer. However, the heat flux attains its maximum value in the DNS–DCS transition sublayer over the Ranchi and CASES-99 datasets. There might be a number of reasons for the differences in the point of maximum heat flux in unstable conditions. These include frequency of occurrence of low-wind conditions, roughness lengths of heat and momentum of the sites, and the orographic features of the surrounding area. For example, significant differences are observed in the values of the stability parameter for the same wind and temperature values with different values of momentum roughness length (Sharan and Srivastava 2014). However, a precise physical reason for the differences in the point of maximum heat flux among the different datasets is not clear to us at the moment.

The theoretical value of ζ corresponding to maximum heat flux is found to be ~−0.22, −0.125, and −0.172 for the values of γh from Businger et al. (1971), Dyer (1974), and Högström (1996). The difference in the theoretical value arises as a result of the choice of the constant γh in the functional form of φh. Also, the observed

<table>
<thead>
<tr>
<th>Sublayers</th>
<th>Range of ζ</th>
<th>Number of hours</th>
<th>Ranchi, India</th>
<th>Cardington</th>
<th>CASES-99</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>U ≤ 2</td>
<td>U &gt; 2</td>
<td>U ≤ 2</td>
<td>U &gt; 2</td>
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<tr>
<td>DNS</td>
<td>−0.04 ≤ ζ ≤ 0</td>
<td>65</td>
<td>164</td>
<td>0</td>
<td>485</td>
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<tr>
<td>DNS–DCS transition</td>
<td>−0.12 ≤ ζ &lt; −0.04</td>
<td>152</td>
<td>685</td>
<td>3</td>
<td>775</td>
</tr>
<tr>
<td>DCS</td>
<td>−1.20 ≤ ζ &lt; −1.12</td>
<td>720</td>
<td>840</td>
<td>127</td>
<td>853</td>
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<tr>
<td>DCS–FCS transition</td>
<td>−2.00 ≤ ζ &lt; −1.20</td>
<td>58</td>
<td>26</td>
<td>28</td>
<td>31</td>
</tr>
<tr>
<td>FCS</td>
<td>ζ &lt; −2.0</td>
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<td>38</td>
<td>52</td>
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<table>
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<tr>
<th>Sublayers</th>
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<td>DNS–DCS transition</td>
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<td>DCS</td>
<td>111.54</td>
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<td>FCS</td>
<td>77.84</td>
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</table>

Table 2. Average value of heat flux in each of the sublayers.
behavior of $H$ with $\zeta$ from all the datasets is found to be consistent with that obtained theoretically from MOS equations when considering the vertical temperature gradient as a power-law function of heat flux having an exponent larger than $2/3$. This is also in line with the value taken by Rao and Narasimha (2006) in their study.

Wang and Bras (2010) have analyzed the behavior of $H^*$ and $\partial H/\partial z$ with respect to $u^*$ and observed that a given value of $\partial H/\partial z$ corresponds to two values of $u^*$ unless $\partial H/\partial z$ reaches a minimum value, which results in a unique solution of $u^*$. They argued that the non-uniqueness of the MOS equations in unstable conditions is a mathematical artifact, and it persists as long as the similarity functions are dependent on the stability parameter $\zeta$. To overcome this problem, they have proposed an extremum hypothesis to have a unique solution for MOS equations, which leads to a possible simplification of MOS equations by replacing the universal similarity functions with some empirical constants. However, the present study suggests that nonuniqueness of MOS equations in the case of convective conditions does not appear to be a mathematical artifact and is consistent with the observations. Analysis of the observational data over three different sites suggests that the value $\zeta = -2/\gamma_h$, corresponding to the extremum value of heat flux, which is interpreted by Wang and Bras (2010, 2011) as the only physically consistent value of $\zeta$, appears to be one out of many possible values in unstable conditions similar to the stable conditions, as pointed out by van de Wiel et al. (2011). The present study supports the three-sublayer model of Kader and Yaglom (1990) rather than the existence of any preferred stability state, as speculated by Wang and Bras (2010, 2011).

5. Issues and limitations

In this study, the behavior of heat flux is analyzed with respect to the stability parameter in the unstable surface layer. Some underlying issues and the limitations associated with the analysis are discussed.

a. Functional form of similarity functions

The conclusions drawn from the present study rely on the nature of the similarity function $\phi_m(\zeta)$ and the assumptions on the form of the negative potential temperature gradient. According to Monin–Obukhov similarity theory, the nondimensional wind and temperature profiles are universal functions of stability parameter $\zeta$. Detailed forms of these functions are not given by the theory but must be determined using field experiments. However, based on the theory, certain asymptotic predictions can be made for the nature of these similarity functions (Högström 1996). In the neutral limit, $\phi_{m,h}(\zeta)$ must be a constant. For strong instability, when buoyancy dominates the local turbulence production, the dimensional analysis (Monin and Yaglom 1971) suggests that $\phi_{m,h}(\zeta)$ should vary as $-1/3$ power of the stability parameter. However, the various field experiments have not always verified these theoretical limits (Högström 1996). In view of these asymptotic limits, various expressions for the similarity functions are proposed in the literature by researchers (Kazansky and Monin 1956; Ellison 1957; Yamamoto 1959; Panofsky 1961; Sellers 1962; Businger et al. 1971; Dyer 1974; Högström 1996; De Bruin 1999; Wilson 2001). The historical Kazansky–Ellison–Yamamoto–Panofsky–Sellers (KEYPS) formula (Lumley and Panofsky 1964) for the similarity function for momentum $\phi_m$ (Kazansky and Monin 1956; Ellison 1957; Yamamoto 1959; Panofsky 1961; Sellers 1962) is given by

$$\phi_m^5(\zeta) - \gamma_6 \phi_m^{10}(\zeta) = 1,$$

in which $\gamma$ is a constant. This can be interpreted as an interpolation function that recovers the theoretical scaling for near-neutral conditions while capturing the $-1/3$ power law in free convection as predicted by MOS theory. However, theoretical justification of the KEYPS equation remained heuristic and assumed a constant heat-to-momentum eddy diffusivity, an assumption that was negated by the Kansas experiment, which prevented the widespread acceptance of the KEYPS equation (Katul et al. 2011). Based on the simple mixing-length model (Fleagle and Businger 1980), Businger derived the relationships $\phi_m(\xi) = (1 - \gamma_4)^{-1/4}$ and $\phi_h(\xi) = Pr(1 - \gamma_5)^{-1/2}$ (Businger et al. 1971; Businger 1988), which are commonly referred to in the literature as the Businger–Dyer (BD) relationships. The derivation of the KEYPS formula requires the ratio of heat to momentum eddy diffusivity ($K_h/K_m$) to be a constant, while the Businger–Dyer formula does not assume a constant ratio of heat to momentum eddy diffusivity (Businger 1988). The original KEYPS formula can be easily recovered from the BD profiles for a constant ratio of heat to momentum eddy diffusivity (Businger 1988). The BD functions have been successfully evaluated using the famous Kansas dataset. A major limitation of Businger–Dyer similarity functions is their failure to recover the well-known classical free-convection limit. However, Fleagle and Businger (1980) pointed out that this might be attributed to the fact that, in free convection, large eddies are associated with varying vertical shear, and, therefore, there is shear...
production of turbulent kinetic energy near the surface even though the shear averaged over many eddies vanishes (Fleagle and Businger 1980). Businger–Dyer similarity functions are still commonly used in all the climate, atmospheric, hydrological, and ecological applications or models of land surface processes when land surface fluxes are to be coupled to the state of the atmosphere (Parlange et al. 1995; Katul et al. 2011).

De Bruin (1999) revised the mixing-length model (Fleagle and Businger 1980) and proposed the expression for similarity functions as \( \phi_h = (1 - 3\zeta)^{2/3}(1 - 10\zeta)^{-1} \). The expression proposed by De Bruin (1999) follows the classical \(-1/3\) power law in free convection [i.e., \( \phi_h \propto (-\zeta)^{-1/3} \) for \(-\zeta \gg 1\)]. Based on the reanalysis of the Kansas dataset, Wilson (2001) proposed the expression \( \phi_h = \Pr \left[1 + \gamma(\zeta)^{2/3}\right]^{1/2} \) with \( \gamma \) as a positive constant, satisfying the classical free convection behavior.

We have carried out a similar analysis for these functional forms of similarity functions. In the case of the similarity function of De Bruin (1999), for a given temperature gradient, heat flux decreases with increasing instability until \( \zeta = -1/11 \) and then slightly increases and attains a constant value (see supplementary material file Hflux_1.pdf). By considering the power-law profile for vertical temperature as \(-\partial\theta/\partial z = \beta(H/\rho C_p)^\alpha\), the analysis shows that, if \( \alpha > 2/3 \), \( H \) attains its maximum value at \( \zeta = -1/11 \) and then slightly decreases and attains a constant value with further increasing instability. For the similarity functions of Wilson (2001), no critical points are found, which suggests the unique solution of MOS equations (see supplementary material file Hflux_1.pdf).

**b. Assumption on vertical potential temperature gradient**

The assumption that heat flux varies as a power-law function of temperature gradient or vice versa is our hypothesis, which can be partially addressed on the basis of the classical flux gradient equation in which the nonlinearity is through the diffusivity parameter that is a nonlinear function of heat flux (Wang and Bras 2010). Also, this is justifiable from the observations (see supplementary material file Hflux_2.pdf). Hence, for the analysis, we have taken a simple hypothesis that the temperature gradient can be expressed as a power-law function of heat flux. However, the hypothesis that \((\partial\theta/\partial z) = \beta(H^\alpha)\eta\) leads to an identity for \( \alpha = 2/3 \) given as

\[
(kz)^2 \sqrt{\frac{g}{\beta}} \frac{\beta^{3/2}}{\Pr_r^{3/2}} \frac{1}{(1 + \gamma_h \eta)^{-3/4}(\eta)^{1/2}} = 1
\]

Since the heat flux no longer appears in the expression, it is the limitation of our hypothesis in this case. However, in view of studies available in the literature, the following interpretation may be applicable. When the ground is heated under low-wind conditions (i.e., wind shear is negligible), Priestley (1955) pointed out that the heat flux \( H \) follows the relation

\[
\frac{H \rho C_p}{\sqrt{g/\beta}} \frac{\partial \theta/\partial z}{(\eta)^{3/2}z^2} = H_N,
\]

where \( H_N \) is a constant. This constant \( H_N \) is termed the nondimensional heat flux by Priestley (1955) and is subsequently studied by Taylor (1956) and Dyer (1965, 1967). In the case of free convection, when the nondimensional heat flux is observed to be independent of the stability of the convection-controlled layer, the heat flux follows the \( 3/2 \) power law of the negative potential temperature gradient. In the case of forced convection, when the contribution of both the buoyancy and shear to produce turbulent kinetic energy is significant, \( H_N \) must be a function of the stability of the atmosphere: that is,

\[
H_N = f(\zeta).
\]

Thus, although our hypothesis for \( \alpha = 2/3 \) leads to an identity, it has an apparent significance in unstable conditions.

**6. Conclusions**

A systematic mathematical analysis of the MOS equations is carried out to analyze the theoretical behavior of heat flux \( H \) with respect to the stability parameter \( \zeta \) in unstable conditions. The heat flux \( H \) is expressed as a function of \( \zeta \) and temperature gradient \( \partial\theta/\partial z \), and two cases—(i) \(-\partial\theta/\partial z = \delta \), (ii) \(-\partial\theta/\partial z = \beta(H/\rho C_p)^\alpha\)—are considered for the analysis. For a given constant \( \partial\theta/\partial z \), the heat flux is observed to decrease until it attains a minimum value at \( \zeta = -2/\gamma_h \) and then starts increasing with increasing instability. However, for \(-\partial\theta/\partial z = \beta(H/\rho C_p)^\alpha\), with \( \alpha > 2/3 \), the heat flux increases with increasing instability until \( \zeta = -2/\gamma_h \) and then starts decreasing with further increasing instability. From the analysis, it is found that multiple values of \( \zeta \) exist for a given value of \( H \) in both the conditions with an identical point of extremum (i.e., \( \zeta = -2/\gamma_h \)). Turbulent data over three different sites—(i) Ranchi, India; (ii) the Met Office’s Cardington monitoring facility; and (iii) CASES-99—are analyzed to compare the observed nature of \( H \) with its theoretical nature as predicted by MOS theory. The analysis of observational data over these three different sites reveals that the behavior of \( H \) with \( \zeta \) is consistent with that obtained theoretically from MOS equations by considering the
vertical temperature gradient as a power-law function of heat flux having an exponent larger than 2/3. The data do not suggest any preferred stability state in unstable conditions, as speculated by Wang and Bras (2010, 2011). The value $\zeta = -2/3$, corresponding to the extremum value of heat flux, appears to be one out of many possible states in unstable conditions, similar to the stable conditions as argued by van de Wiel et al. (2011). The present study suggests that the existence of multiple values of $\zeta$ for a given value of $H$ persists in the case of convective conditions, similar to the case of stable atmospheric conditions that was not reported in the earlier studies.

Acknowledgments. The authors wish to thank Dr. Manoj Kumar for providing observational data for Ranchi, the Met Office for use of Cardington measurements, and the National Center for Atmospheric Research (NCAR) for CASES-99 observations. This work is partially supported by the Ministry of Earth Sciences, Government of India, under the CTCZ program (MoES/CTCZ/16/28/10); JC Bose Fellowship to MS from DST-SERB, Government of India (SB/SC/JCB-79/2014); and SRF to PS from University Grants Commission. The authors thank the reviewers for their valuable comments and suggestions.

REFERENCES


