A THEORY OF ENTRAINMENT IN CONVECTIVE CURRENTS

By Henry G. Houghton and Harrison E. Cramer

Massachusetts Institute of Technology[1,2]

(Manuscript received 12 January 1951)

ABSTRACT

It is held that entrainment is a necessary dynamic consequence of the vertical stretching of an accelerated convective column. On this basis, equations are developed for the rate of entrainment, the vertical divergence and the lapse rate, for both unsaturated air and cloud air. It is assumed that a steady state exists, the cross section of the rising column is invariant with height, the entrained air is uniformly mixed with the rising air and the environment is at rest. The equations are integrated numerically over height in a number of selected cases. In unsaturated air, entrainment results in a lapse rate which is always greater than the dry adiabatic; if the environmental lapse-rate is superadiabatic, the lapse rate of the rising air is intermediate between the lapse rate of the environment and the dry adiabatic lapse-rate. In a cloud, entrainment results in a lapse rate intermediate between the environmental lapse-rate and the moist adiabatic lapse-rate. The lapse rate of the rising air increases as the relative humidity of the environment decreases. As a result of entrainment, the cloud liquid-water content increases with height at a significantly slower rate than would result from a simple lifting process. A decrease in the humidity of the environment reduces the rate of increase of liquid water with height, but it does not appear possible to "dry out" the cloud even in a dry environment. The horizontal velocity-convergence is found to be of the order of 10^{-3} sec^{-1}, and the computed vertical velocities in the cloud are in general agreement with those observed by the Thunderstorm Project. It is pointed out that the entrained air may be added through an ordered inflow, by turbulent exchange or by a combination of the two. It is assumed here that an ordered inflow occurs.

1. Introduction

Evidence has been presented that a substantial amount of environmental air is entrained by convective updrafts. Stommel (1947) made temperature measurements inside and outside trade-wind cumulus clouds and concluded that the clouds entrained rather large amounts of air from their environment. Byers and Hull (1949) have reported measurements of the horizontal velocity-divergence, which show entrainment throughout the major vertical extent of cumulonimbus clouds. There is also evidence that the liquid-water content in cumulus clouds increases less rapidly with height than would result from a simple lifting process. This again suggests the entrainment of substantial amounts of the drier environment.

Two physical explanations for the entrainment process have been presented. Schmidt (1947) and Stommel (1947) feel that the entrainment is due to turbulent lateral mixing and that jet-stream theory may be applied to the problem. Austin (1948) expressed the idea that entrainment is required to satisfy continuity. It is generally agreed that vertical accelerations exist in active convection. This implies a vertical stretching of the air column, or vertical divergence. To maintain continuity, a compensating horizontal convergence is required. Similarly, a negative acceleration would require horizontal divergence. From this point of view, entrainment is a necessary dynamic consequence of vertical acceleration.

The two entrainment mechanisms are not entirely independent. The mechanism based on continuity, which will hereafter be called dynamic entrainment, does not specify the type of flow but merely the mass which must be entrained to compensate for the vertical stretching. This mass may be added by an ordered inflow, by turbulent exchange or by a combination of the two. This suggests that the total mass-entrainment may be determined by considering dynamic entrainment only. This is not entirely true, as will be apparent from the later discussion.

It is the purpose of this paper to derive the equations for dynamic entrainment in vertically accelerated air-columns. For simplicity, it is assumed that a steady state exists, the cross section of the column is invariant with height, the entrained air is uniformly mixed with the rising air and the environment is at rest. The vertical acceleration is assumed to be due to the buoyancy force resulting from the difference in density of the rising air and the environment. None of these assumptions is requisite for a solution, and they may be modified at the expense of increased complexity. Solutions are obtained for both the dry and the saturated case. Numerical integrations of the equations are carried out for several cases of meteorological interest.

[1] Research sponsored by the Office of Naval Research under Contract N5ori-07804.

[2] Based in part on a doctoral thesis presented by the junior author to the Massachusetts Institute of Technology.

2. The dry-air case

The dynamic entrainment process is illustrated in fig. 1. This figure represents a volume element of the rising air-column of cross section $d\sigma$ and height dz. Air of temperature T' and density ρ' enters the bottom of the element with a vertical velocity w'. The environmental air, surrounding the rising air, is assumed to be at rest and to have a temperature T and density ρ. If $T' > T$, the air entering the base of the volume element is subjected to a positive vertical acceleration, due to the buoyancy force. The resultant vertical stretching of the column requires the entrainment of air from the environment, and it is assumed that the entrained air is completely mixed with the rising air. As a result of the addition of air from the environment, the density and temperature of the rising air will change. The velocity of the rising air will increase due to the buoyancy acceleration and will decrease due to the addition of new mass with no vertical momentum. The temperature and density of the air leaving the top of the elementary volume are $T' + (\delta T'/\delta z)\, dz$ and $\rho' + (\delta\rho'/\delta z)\, dz$, where δ represents the change due to the air added from the environment. The changes in temperature and density due to adiabatic expansion are independent of those due to mixing and need not be included now. The velocity at the top of the element is $w' + (Dw'/Dz)\, dz$, where D stands for the combined change due to buoyancy acceleration and mixing. The equation of continuity is then

$$\rho' w'\, d\sigma + \Delta m\, dz\, d\sigma = \left[\left(\rho' + \frac{\delta\rho'}{\delta z} dz\right)\left(w' + \frac{Dw'}{Dz} dz\right)\right] d\sigma, \quad (1)$$

where Δm is the mass of environmental air added per unit volume of the rising column, per unit time. Expansion of (1) and neglect of second-order terms gives

$$\Delta m = w'\, \delta\rho'/\delta z + \rho'\, Dw'/Dz. \quad (2)$$

In similar fashion, the heat-balance equation for the elementary volume may be stated as

$$c_p T' \rho' w'\, d\sigma + c_p T\, \Delta m\, dz\, d\sigma = c_p[\rho' w'\, d\sigma + \Delta m\, dz\, d\sigma][T' + (\delta T'/\delta z)\, dz], \quad (3)$$

where $\delta T'/\delta z$ refers only to the change in temperature of the rising air produced by the mixing with the entrained air. Expansion of (3) and neglect of second-order terms gives

$$\rho' w'\, \delta T'/\delta z + (T' - T)\, \Delta m = 0. \quad (4)$$

The entrained air has no initial vertical velocity. When it is mixed with the rising air, the vertical momentum of the rising air is decreased by the addition of the entrained mass, in accordance with the principle of the conservation of momentum. Symbolically,

$$\rho' w'[w' + (\partial w'/\partial z)\, dz]\, d\sigma = [\rho' w' + \Delta m\, dz][w' + (Dw'/Dz)\, dz]\, d\sigma, \quad (5)$$

where $\partial w'/\partial z$ is the rate of change in velocity with height due to the buoyancy force. As before, Dw'/Dz is the total change in velocity with height, including both the effect of buoyancy and the effect of the entrained mass of air. Clearly, $\partial w'/\partial z > Dw'/Dz$ for a positive vertical acceleration. Expansion of (5) and neglect of higher-order terms gives

$$\Delta m = \rho'(\partial w'/\partial z - Dw'/Dz). \quad (6)$$

The buoyancy principle is usually applied to isolated air parcels rather than to a continuous column. If it is remembered that the buoyancy force is simply the difference in weight between an elementary column of the warm air and a similar adjacent column of the environment, it is readily seen that the buoyancy force on each element of the continuous column is the same as if the element were an isolated parcel. The buoyancy acceleration is

$$dw'/dt = g(\rho - \rho')/\rho' = g(T' - T)/T. \quad (7)$$

The vertical component of the equation of motion for the warm air is

$$dp' = -\rho' g\, dz - \rho'(dw'/dt)\, dz.$$

Insertion of (7) leads to $dp' = -\rho g\, dz = dp$. This shows further that the buoyancy acceleration balances the weight differences, and that the pressure in the rising column is the same as that of the environment.

Now,

$$dw'/dt = \partial w'/\partial t + u'\, \partial w'/\partial x + v'\, \partial w'/\partial y + w'\, \partial w'/\partial z. \quad (8)$$

In view of the assumption of a steady state, $\partial w'/\partial t$ is zero. The assumption of complete mixing of the entrained air means that there can be no horizontal variations of w' within the rising column. Thus,

$$\partial w'/\partial z = g(T' - T)/w'T. \quad (9)$$

Combination of (2) and (6), to eliminate Dw'/Dz, yields

$$\Delta m = \frac{\rho'}{2}\left(\frac{\partial w'}{\partial z} + \frac{w'}{\rho'}\frac{\delta\rho'}{\delta z}\right). \quad (10)$$

Logarithmic differentiation of the equation of state gives

$$\rho'^{-1}\, \delta\rho'/\delta z = p^{-1}\, \delta p/\delta z - T'^{-1}\, \delta T'/\delta z.$$

$\delta p/\delta z$ is zero, since the pressure is determined by the environment and is not affected by the mixing process. Thus,

$$\rho'^{-1}\, \delta\rho'/\delta z = -T'^{-1}\, \delta T'/\delta z. \quad (11)$$

Substituting (11), (9) and (4) into (10), and solving for $\delta T'/\delta z$, the mixing lapse-rate of temperature, we find

$$\delta T'/\delta z = -gT'(T' - T)^2/w'^2T(T' + T). \quad (12)$$

The total lapse-rate is obtained by adding the dry adiabatic lapse-rate to (12). This is permissible, since the expansional cooling represented by the dry adiabatic lapse-rate is independent of the cooling due to the admixture of environmental air.

Inserting (9), (11) and (12) into (10), and solving for Δm, we obtain

$$\Delta m = g\rho' T'(T' - T)/w'T(T' + T). \quad (13)$$

An expression for Dw'/Dz, the total change of w' with height, may be obtained from (6) with the aid of (9) and (13):

$$Dw'/Dz = g(T' - T)/w'(T' + T). \quad (14)$$

Since T'/T is nearly unity, it is clear from (13) and (14) that $\Delta m = \rho' Dw'/Dz$, very closely. Insertion of this value for Δm into (6) leads to the interesting result that $Dw'/Dz = \frac{1}{2}\,\partial w'/\partial z$. Equations (12), (13) and (14) describe the entrainment process for dry air in accordance with the postulates and assumptions made. In view of the complexity of these equations, no attempt has been made to integrate (14) over a finite height-interval. It has been considered preferable to perform numerical integrations, the results of which are presented later.

3. The cloudy-air case

Entrainment of dry environmental air into a growing cumuliform cloud is perhaps of more interest than the dry-air case considered above. The basic assumptions and the physical process envisaged are the same as in the dry-air case. The additional condition is introduced that the rising cloud-air remains saturated. The continuity condition expressed by (2) and the conservation of vertical momentum given by (6) apply unchanged to the cloudy-air case. Equations (7), (9) and (11) may also be used for the cloudy-air case, if the virtual temperatures are used instead of the actual temperatures. Insertion of the thusly modified equations (9) and (11) into (10) yields

$$\Delta m = \frac{\rho'}{2}\left(\frac{g}{w'}\frac{T_v' - T_v}{T_v} - \frac{w'}{T_v'}\frac{\delta T_v'}{\delta z}\right), \quad (15)$$

where the subscript v denotes virtual temperature.

When the entrained, unsaturated environmental air is mixed with the cloudy air, some of the liquid water in the cloudy air must be evaporated to maintain saturation. This introduces a latent-heat term into the dry-air heat-balance condition expressed by (4), which then becomes, for the cloudy air,

$$\rho' w'(\delta T'/\delta z) + (T' - T)\,\Delta m - \rho' w' L c_p^{-1}(\delta A'/\delta z)_e = 0, \quad (16)$$

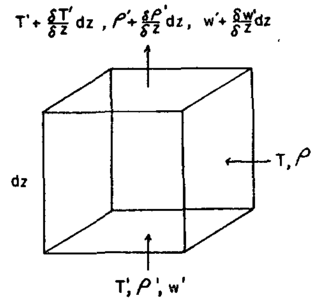

FIG. 1. Volume element of rising column.

where A' is the liquid-water content of the cloudy air in g/g, and $(\delta A'/\delta z)_e$ represents the change in liquid-water content due to evaporation. This is not the total change in liquid-water content due to the mixing, since it also decreases due to the dilution of the cloud air with environmental air which contains no liquid water.

A new condition of continuity for water vapor may be written. This condition states that the sum of the water vapor entering the base of the elementary volume of fig. 1, the water vapor in the entrained air and the water vapor gained by the evaporation of liquid water is equal to the water vapor leaving the top of the elementary volume. Symbolically,

$$\rho' w'[q' - (\delta A'/\delta z)\,dz]\,d\sigma + q\,\Delta m\,d\sigma\,dz = [\rho' w'\,d\sigma + \Delta m\,d\sigma\,dz][q' + (\delta q'/\delta z)\,dz],$$

where q' and q are the specific humidities of the cloudy air and of the environmental air, respectively. Expansion and neglect of higher-order terms reduces this to

$$\rho' w'(\delta q'/\delta z + \delta A'/\delta z) + \Delta m\,(q' - q) = 0. \quad (17)$$

The specific humidity q' of the rising air is the saturation specific-humidity, and the Clausius-Clapeyron equation may be used to obtain an expression for $\delta q'/\delta z$. The usual approximate expression, $q' = 0.623\,e_m'/p$ is differentiated with respect to z to give

$$\delta q'/\delta z = (0.623/p)(\delta e_m'/\delta T')(\delta T'/\delta z).$$

Substitution for $\delta e_m'/\delta T'$ from the Clausius-Clapeyron equation gives

$$\frac{\delta q'}{\delta z} = \frac{0.623LJ}{pR_w}\frac{e_m'}{T'^2}\frac{\delta T'}{\delta z}, \quad (18)$$

where J is the mechanical equivalent of heat, R_w is the gas constant of water vapor, and e_m' is the saturation vapor-pressure at T'.

Before combination of (15), (16) and (17) to obtain an expression for the mixing lapse-rate, it must be noted that (15) contains the mixing lapse-rate of virtual temperature. Combination of the derivative of the expression for virtual temperature in saturated air, expressed in terms of the specific humidity, with

(18) leads immediately to

$$\frac{\delta T_v'/\delta z}{\delta T'/\delta z} = \beta = \frac{1 + 0.606 LJT_v'q'/R_w T'^2}{1 - 0.606q'}. \quad (19)$$

β is evidently greater than unity and increases with increasing q' or T_v'. The mixing lapse-rate may now be obtained by combining (16), (17), (19), (15) and (18). The result, after some simplifications, is

$$\frac{\delta T'}{\delta z} = -\frac{\dfrac{g(T_v'-T_v)}{2T_v w'^2}\left[(T'-T)+\dfrac{L}{c_p}(q'-q)\right]}{1+\dfrac{0.623L^2Je_m'}{c_pR_wpT'^2}-\dfrac{\beta}{2T_v'}\left[(T'-T)+\dfrac{L}{c_p}(q'-q)\right]}. \quad (20)$$

Insertion of numerical values shows that the last term in the denominator of (20) is two to three orders of magnitude smaller than the other two terms. This term may usually be neglected but, in any event, β may be taken as unity without introducing an error as large as those contained in previous assumptions.

As in the dry case, the total lapse-rate is obtained by adding the appropriate moist adiabatic lapse-rate to (20). Once $\delta T'/\delta z$ is found, Δm, $\partial w'/\partial z$ and Dw'/Dz may be computed from the appropriate equations given above or combinations of them.

Austin and Fleisher (1948) have also presented an equation for the lapse rate in a convective cloud under the influence of entrainment. They impose an arbitrary rate of entrainment and compute the resultant lapse-rate. In the present development, the rate of entrainment is determined by the continuity condition and is not one of the imposed boundary conditions.

4. Numerical integrations

In view of the complexity of the expressions and the ultimate need for numerical results, no attempt was made to integrate the equations analytically over height; rather, numerical integrations were performed to study the effects of dynamic entrainment in a number of selected situations. It will be recalled that the solutions given above are for a steady state. It is therefore necessary to select values for w' and for $T' - T$, or $T_v' - T_v$, at the height chosen for the start of the numerical integration. Observational data on these quantities, particularly $T' - T$, are very meager, so that the choice was largely arbitrary. Inspection of (12), (13), (14) and (20) shows that $(T' - T)/w'$ is the quantity which largely determines the numerical values of $\delta T'/\delta z$, Δm and Dw'/Dz. The choice of the initial value of this parameter directly affects all of the computed quantities. For this reason, the numerical results given must be considered only as examples of the effects of dynamic entrainment and

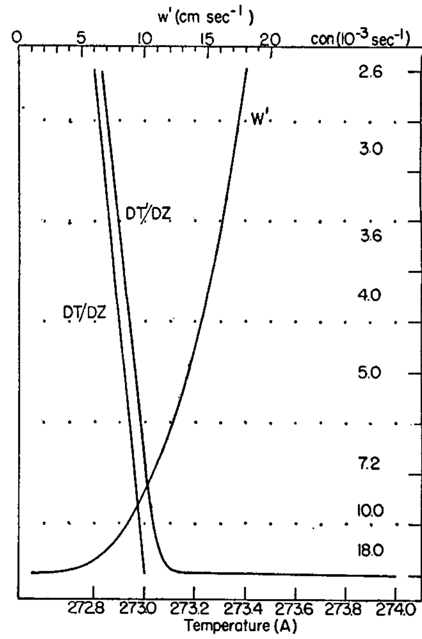

FIG. 2. Temperature and vertical velocity of rising dry air-column versus height in environment with dry adiabatic lapse-rate. Initial values of temperature difference and vertical velocity are 1C and 1 cm sec^{-1}, respectively.

not as definitive of what actually occurs in the atmosphere.

The first numerical integrations were carried out for dry air and an isothermal lapse-rate. These cases are of minor significance but served to establish the numerical procedures and aided in the choice of the initial conditions. The lapse rate of the rising air in an isothermal environment is very steep initially and

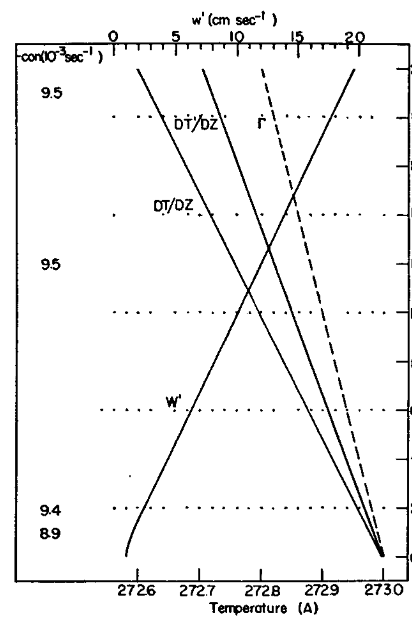

FIG. 3. Temperature and vertical velocity of rising dry air-column versus height in environment with superadiabatic lapse-rate (20C/km). Initial values of temperature difference and vertical velocity are 0.001C and 1 cm sec^{-1}, respectively. Dashed line follows dry adiabatic lapse-rate.

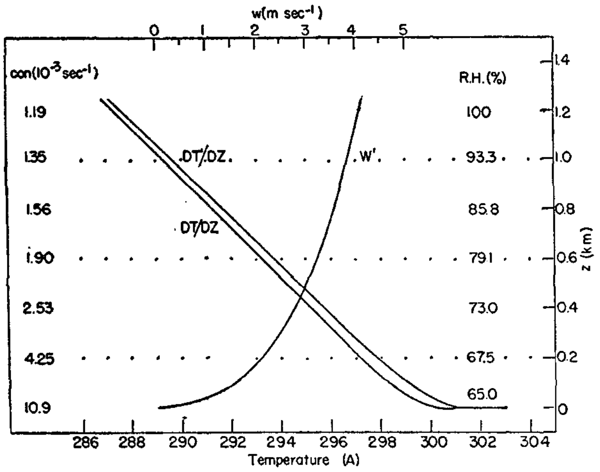

FIG. 4. Temperature and vertical velocity of rising unsaturated air-column versus height in environment with lapse rate as shown by curve DT/Dz. Initial vertical velocity was taken as 1 cm sec^{-1} at height of 1 cm. Initial temperature-difference is that which would result from adiabatic displacement of surface air to height of 1 cm.

approaches the dry adiabatic lapse-rate as the temperature of the rising air becomes equal to that of the environment.

Figs. 2, 3 and 4 show the results of integrations for dry air for three environmental lapse-rates: dry adiabatic, superadiabatic and a composite lapse-rate intended to represent typical conditions during summer convective activity. This composite lapse-rate was derived from data given by Geiger (1925) from the surface to 2 m, from data of Johnson and Heywood (1938) from 2 m to 100 m and by assuming a dry adiabatic lapse-rate above 100 m. As shown in fig. 2, the lapse rate of the rising air in a dry-adiabatic environment is initially very steep and approaches asymptotically a dry adiabatic lapse-rate; however, the rising air always remains warmer than the environment by a finite amount. The vertical velocity likewise increases very rapidly at first and then approaches a constant rate of increase with elevation. A similar computation for an initial temperature-difference of 0.01C and an initial vertical velocity of 1 cm sec^{-1} yielded similar results, except that the temperatures of the rising and environmental air were more nearly equal at all levels and the vertical velocities were smaller.

The behavior of the rising air in an environment with a superadiabatic lapse rate, as shown in fig. 3, is interesting in that the lapse rate of the rising air does not approach the dry adiabatic lapse-rate but, rather, one intermediate between the dry adiabatic and the environmental lapse-rates. As might be expected, the vertical velocities are rather large, and Dw'/Dz approaches a constant as z increases. The temperatures and vertical velocities of the rising air in the environment with the composite lapse-rate, as shown in fig. 4, are about as would be anticipated from the two previous cases. The computations for this case were continued to z = 1250 m, to yield appropriate initial conditions for the cloud-air cases. At 1250 m, the temperature difference is 0.282C and the vertical velocity is 4.15 m sec^{-1}. These values do not seem unreasonable for the base of an active cumulus cloud, and they were used as the initial conditions for all of the cloud-air integrations. The relative humidity of the ascending air was obtained by computing the variation in water-vapor content produced by complete mixing with the entrained air.

The results of the first integration for cloud-air are given in fig. 5. In this case, the environment was assumed to have a dry adiabatic lapse-rate and a constant relative humidity of 60 per cent. Note that the results are somewhat similar to those of fig. 3 in

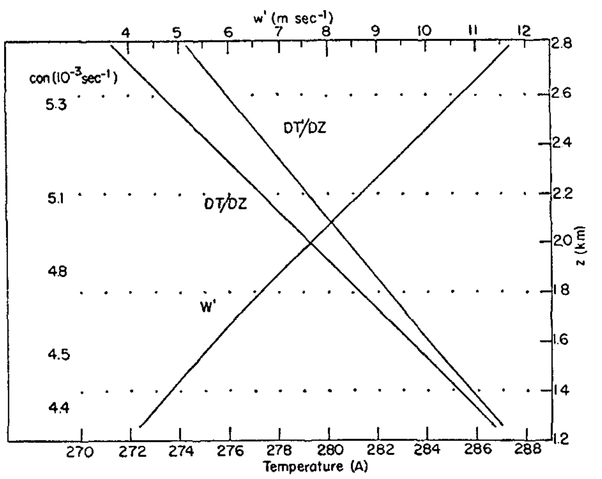

FIG. 5. Temperature and vertical velocity of rising column of cloudy air versus height in environment with dry adiabatic lapse-rate and relative humidity 60 per cent. Initial values of temperature difference and vertical velocity (at 1.25 km) are 0.282C and 4.15 m sec^{-1}, respectively.

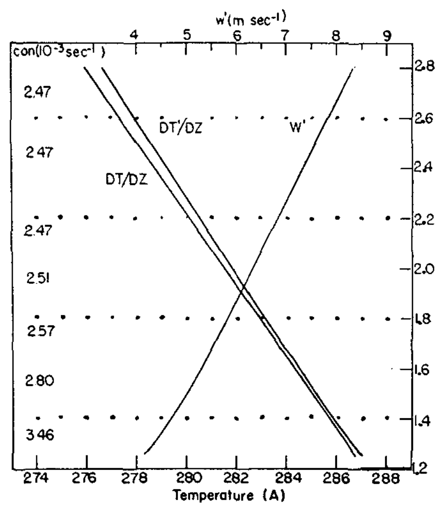

FIG. 6. Temperature and vertical velocity of rising column of cloudy air versus height in environment with lapse rate of 7C/km and relative humidity 60 per cent. Initial values of temperature difference and vertical velocity (at 1.25 km) are 0.282C and 4.15 m sec^{-1}, respectively.

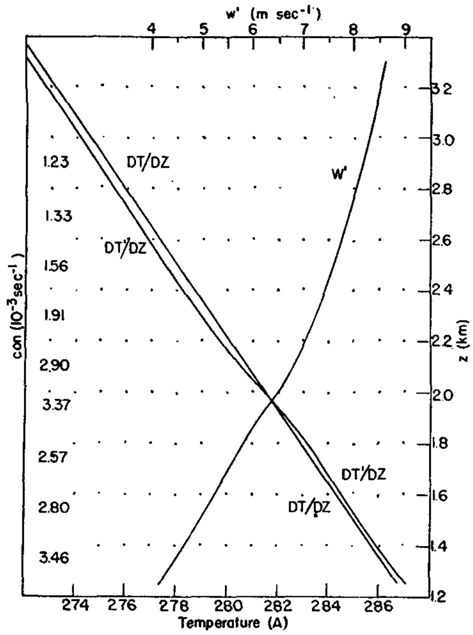

FIG. 7. Temperature and vertical velocity of rising column of cloudy air versus height in environment with lapse rate of 7C/km. Relative humidity is 60 per cent from 1.25 to 1.75 km, 50 per cent at 1.8 km, then decreases 10 per cent each 0.05 km to 10 per cent at 2 km and becomes zero at and above 2.1 km. Initial values of temperature difference and vertical velocity (at 1.25 km) are 0.282C and 4.15 m sec^{-1}, respectively.

that the lapse rate of the cloud-air approaches a value intermediate between the environmental lapse-rate and the moist adiabatic lapse-rate. The vertical velocity increases quite rapidly at a nearly constant rate and is almost 12 m sec^{-1} at 2800 m.

The effect of a somewhat more stable lapse-rate in the environment is illustrated in fig. 6. The environmental lapse-rate was taken as 7C/km with a relative humidity of 60 per cent throughout. The results are very similar to the previous case, except for the smaller values of the temperature differences and the vertical velocities to be expected with the more stable lapse-rate.

The case illustrated in fig. 7 was chosen in order to investigate the effect of an environment in which the relative humidity decreases sharply with height. The environmental lapse rate was taken as 7C/km. The relative humidity was assumed to be 60 per cent from 1250 to 1750 m, 50 per cent at 1800 m, then decreasing 10 per cent each 50 m to 10 per cent at 2000 m and becoming zero at and above 2100 m. An inspection of fig. 7 shows that the immediate effect of a decrease in the environmental humidity is to steepen the lapse rate of the rising air. The reduction of humidity is great enough to produce cloud-air temperatures which are several tenths of a degree colder than those of the environment; however, the virtual-temperature difference does not change sign. At 3100 m, the cloud lapse-rate is intermediate between the moist adiabatic lapse-rate and that of the environment. The temperature of the rising air finally exceeds that of the environment at approximately 4.5 km. This example tends to show that a cloud will not be "dried out" as a result of the entrainment of very dry environmental air. It should be noted, however, that the liquid-water content of a cloud in such a dry environment will be significantly lower than in a moist environment. Computations made for a saturated environment, the results of which are not reproduced here, suggest that, other things being equal, an increase in the humidity of the environment results in a higher liquid-water content, a cloud lapse-rate closer to the moist adiabatic, a larger temperature-difference between the cloud air and the environment and a larger vertical velocity.

Attention is called to the values of the horizontal velocity-convergence entered at intervals on figs. 2 to 7, inclusive. These are the values of Dw'/Dz which, in accordance with the assumptions made, is equal to the horizontal convergence. Although it cannot be anticipated that the numerical values arrived at in this study will be directly applicable to natural convection processes, it is of interest to compare them with the results of the Thunderstorm Project as given by Byers and Hull (1949). The horizontal convergence in the vicinity of a thunderstorm was measured by tracking three free balloons by radar. The values reported range up to about 10^{-3} sec^{-1} and average about 4×10^{-4} sec^{-1}. Most of the values computed in this study are nearly one order of magnitude larger than this. The vertical velocities found by the Thunderstorm Project, as reported by Byers and Braham (1949) are of the same order of magnitude as those computed here. It is interesting to note that, in those cases for which vertical velocities are given at two or more levels in the same updraft, Dw'/Dz is of the order of 10^{-3} sec^{-1}. In view of the errors of measurement, these results are probably not as good as those from the balloons. On the other hand, the latter may often give the integrated effects of several up- and downdrafts and thus underestimate the convergence in a single updraft.

The rate of entrainment has frequently been expressed in terms of the vertical pressure-interval within which the mass of the rising air is doubled. The fractional increase in mass per unit time due to the vertical stretching is equal to the vertical divergence, which is equal, in turn, to the horizontal convergence. Thus,

$$dm/m = (Dw'/Dz)\, dt = (\partial u'/\partial x + \partial v'/\partial y)\, dt. \quad (21)$$

With the introduction of the hydrostatic equation, (21) becomes

$$dp = -\frac{g\rho w'}{(\partial u'/\partial x + \partial v'/\partial y)} \frac{dm}{m}. \quad (22)$$

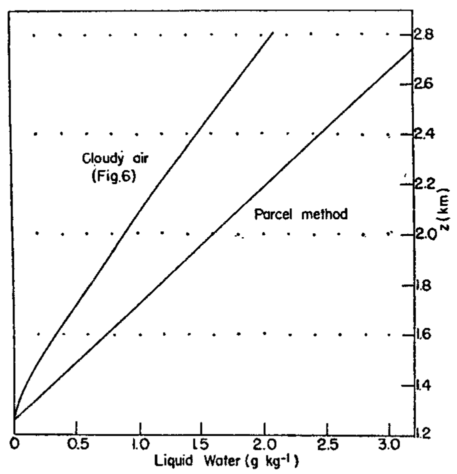

FIG. 8. Effect of entrainment on liquid-water content of rising cloudy air. Curve marked "Cloudy air" is for case presented in fig. 6 and includes effect of entrainment. Curve marked "Parcel method" shows liquid-water content which would result from simple lifting process without entrainment.

Thus, the pressure interval corresponding to a given fractional increase in mass is inversely proportional to the horizontal convergence but is also directly proportional to the vertical velocity. The numerical results of the present study show that the divergence varies slowly with height, whereas the vertical velocity increases rapidly with height; consequently, the pressure interval required to double the rising mass increases with height. For example, from fig. 6, this pressure interval increases from about 100 mb near the cloud base to about 330 mb at 2800 m. The data collected in Florida by the Thunderstorm Project [see Byers and Hull (1949)] show a marked increase in divergence with height, beginning usually at about 15,000 ft. This would tend to cause the pressure interval, required to double the rising mass, to increase more slowly or even to decrease with height in this region.

One of the consequences of the entrainment of environmental air is that the liquid-water content of the cloud is smaller than would be computed from the conventional "parcel method." To illustrate the magnitude of this effect, the liquid-water content for the case presented in fig. 6 is represented as a function of height in fig. 8, together with the values computed from the parcel method. The case considered in fig. 7 would show a greater departure from the parcel method values because of the drier environment.

5. Discussion

The validity of the results presented in this paper is dependent on the extent to which the assumptions are justified. One of the basic assumptions is that of a steady state; it is evident that natural convection is not steady, but it would be difficult to prescribe a realistic non-steady state. It is this assumption that leads to the problem of selecting initial values of the vertical velocity and the temperature difference. It might be somewhat better to start from rest, with a finite temperature-difference, but there is no assurance that such an initial condition would be, in fact, more realistic.

It has been assumed that the pressures of the rising air and of the environment are equal at all levels. It has been shown that the smaller weight of the warm air is just balanced by the vertical acceleration resulting from the temperature difference. However, this result applies only in the absence of friction. The upthrust of the warm air results in a form and body drag, the magnitude of which is difficult to estimate [see Schmidt (1947)]. There is also frictional resistance to the horizontal inflow, particularly in the lower levels. Finally, the cloud drops and hydrometeors exert a restraining force equal to their weight. This latter effect may be considerable, since one gram of liquid water in a cubic meter of air imposes a downward acceleration of about 1 cm sec^{-2}; this is the acceleration which would be produced by a temperature difference of about 0.3C. Qualitatively, the effect of these several frictional forces is to reduce the vertical acceleration, the vertical velocity and the rate of entrainment, and to increase the temperature difference because of reduced entrainment. It would also be expected that the pressure within the rising air would be less than that of the environment, and this is borne out by observations of surface pressure-falls during the cumulus stage of growth.

The liquid-water content in cumulus clouds often exceeds 1 g m^{-3}, and it might appear from the temperature differences indicated on figs. 5, 6 and 7 that this would be sufficient to halt the vertical motion. The downward force exerted by heavy precipitation is probably an important cause of the downdraft of a mature thunderstorm, but consideration must be given to the compensatory action of the entrainment mechanism. The vertical acceleration adjusts itself, so that the cooling of the rising air by the entrainment results in a balancing temperature difference. The addition of a restraining force, such as the weight of the liquid water, will result in a smaller amount of entrainment, a larger temperature-difference and a somewhat smaller vertical velocity. The limit is set by the temperature difference resulting from moist-adiabatic ascent with no entrainment and zero vertical acceleration.

It has been assumed that the cross section of the rising air-column remains constant with height. It is clear that the continuity condition may be satisfied without entrainment, if the cross section of the rising air-column decreases upward at the appropriate rate. There is some evidence that the cross section of a cumulus cloud decreases somewhat with height, but this does not necessarily mean that the cross section of the rising column decreases. If the updraft acts as a jet stream, its cross section would tend to increase with elevation, thus requiring more entrainment. Until more observational evidence on this question is available, it seems best to retain the simple assumption of a constant cross section.

The assumption of complete mixing of the entrained air with the rising air is probably not entirely valid, in spite of the severe turbulence characteristic of cumulus clouds. The idealized diagram of the cumulus stage, given by Byers and Braham (1948), shows a considerable variation of vertical velocity and temperature across the rising column. As a second approximation, it would be possible to divide the rising air into annular rings, separated by a reasonable mixing-length. The entrained environmental air could then be assumed to mix with the outer annulus, the entrained air from the outer annulus with the next inner one, and so on. It is clear that this would lead to a variation of vertical velocity, acceleration and temperature across the rising column, with maximum values in the central core. The results obtained on the assumption of complete mixing perhaps may be considered as mean values for the rising column.

Although it has been assumed that there is no vertical motion in the environment, this could be taken into account simply by including an additional term in the momentum equation.

As pointed out earlier, there have been two explanations for entrainment. Regardless of the inflow mechanism, continuity must be satisfied. The inflow may take place either as an ordered motion or by turbulent exchange (or probably by a combination of the two). In the present theoretical development, an ordered inflow has been assumed. This means that environmental air is considered to flow into the rising column as required, and that none of the rising air passes to the environment as long as there is a positive acceleration. If the inflow were accomplished by turbulent exchange, there would be flow in both directions, with a net inflow as required to maintain continuity. It would seem that this mechanism would lead to an increase of the cross section of the rising air with height and probably to smaller vertical velocities. It appears reasonable to expect that both ordered and turbulent inflows occur, but it is not possible to assess their relative importance on the basis of present information. It is interesting to note the statement by Byers and Braham (1948) that the regions between cells were relatively free of turbulence. In any event, the continuity principle, as here utilized, permits the direct computation of the entrainment and gives results which are probably nearly correct, if it can be established that the cross section of the rising air is constant with height. (Slight modifications only are required if the cross section varies in any simple fashion with height.)

The above discussion suggests that the entrainment and vertical velocities computed here are larger than those which would be observed in nature, and that the actual temperature-differences would be larger than those computed here. If the assumption of a constant cross-section is valid, the computed results probably represent upper limiting values.

What has been presented here is a theory of the entrainment of air in accelerated vertical motion and not a theory of convection. This entrainment principle applies as well to accelerated downward motion as to the upward accelerated motions considered specifically. It also applies to accelerated horizontal motions, but a different set of equations would be required. Both upward and downward branches of convective motion are presumably accelerated, although not necessarily simultaneously. Entrainment into both ascending and descending branches plays an important role and, as pointed out by Byers and Braham (1949), entrainment is an important cause of the cold thunderstorm downdraft. However, entrainment does not prescribe the major convective circulation, and further studies of the dynamics of such circulations are required.

REFERENCES

Austin, J. M., 1948: A note on cumulus growth in a nonsaturated environment. *J. Meteor.*, **5**, 103–107.

—— and A. Fleisher, 1948: A thermodynamic analysis of cumulus convection. *J. Meteor.*, **5**, 240–243.

Byers, H. R., and R. R. Braham, 1949: *The thunderstorm.* Washington, U. S. Government Printing Office, 287 pp.

——, 1948: Thunderstorm structure and circulation. *J. Meteor.*, **5**, 71–86.

Byers, H. R., and E. C. Hull, 1949: Inflow patterns of thunderstorms as shown by winds aloft. *Bull. Amer. meteor. Soc.*, **30**, 90–96.

Geiger, R., 1925: As quoted in Haurwitz, B., and J. M. Austin, 1944: *Climatology.* New York, McGraw-Hill Book Co., p. 173.

Johnson, N. K., and G. S. P. Heywood, 1938: An investigation of the lapse rate of temperature in the lowest hundred meters of the atmosphere. *Geophys. Mem.*, No. 77, **9**, 15.

Schmidt, F. H., 1947: Some speculations on the resistance to the motion of cumuliform clouds. *K. ned. meteor. Inst. Meded. Verh.*, (B), **1**, No. 8.

Stommel, H., 1947: Entrainment of air into a cumulus cloud. *J. Meteor.*, **4**, 91–94.