FREQUENCY DISTRIBUTIONS OF VELOCITIES IN TURBULENT FLOW

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ABSTRACT

A conventional anemometer used in the atmosphere or in a wind tunnel measures velocities projected on a plane independently of their directions in the plane. The relation between the frequency distribution of those velocities and the frequency distribution of the turbulent components parallel to the direction of the mean velocity is discussed. At small intensities of the longitudinal turbulence the two distributions are approximately the same. However, at intensities of turbulence of the order of those measured in the atmosphere, the difference between the two frequency distributions becomes appreciable.

Some experimental frequency distributions of wind velocities are presented, and the intensities of atmospheric turbulence are determined from the experiments.

1. Introduction

The importance of atmospheric turbulence in meteorology has been realized for many years. The interest in this field increases now in connection with weather control, various micrometeorological problems, and particularly in relation to microwave propagation and to air pollution [1]. The development of the uranium industry and the construction of atomic piles emphasize still more the air-pollution problems. Methods of protection from radioactive contamination due to the burst of an atomic bomb must also be investigated [2].

The meteorologist will have to evaluate on which site a source of pollution will be the least dangerous. He will have to determine when the local meteorological conditions may become such as to produce a danger of contamination from the polluting source.

Let us consider, for instance, an area to be protected from pollutants emitted by a neighboring source. In this area we can place a net of instruments measuring the local mean velocities of the wind and the character of its turbulence. A number of possible trajectories of polluting clouds and their dispersion along these trajectories can then be determined. To each of these trajectories will correspond a distribution of pollutant concentration. If any of the computed concentrations reaches the dangerous proportion, the micrometeorological conditions will be considered as unfavorable. It will then be necessary to stop or to reduce the emission of polluting aerosols during the unfavorable time or to evacuate the dangerous area.

To be effective, the determination of the local meteorological conditions should be very rapid. The computation will have to be done, therefore, on high-speed computing machines. Two elements of information will have to be fed into the computing machines: the equations governing the dispersion in the atmosphere, and the numerical values describing the nature of the source, of the pollutants and of the atmosphere. Although a fully satisfactory theory is not yet completed, it is nevertheless possible to use many of the available results (see papers by Sutton, Calder and also [3]). In any case, it is necessary to determine as correctly as possible the mean velocities, in magnitude and in direction, and to measure the characteristics of the turbulent fluctuations. In the present paper, we shall discuss one question only: the measurement of turbulent velocities with a conventional meteorological anemometer.

2. Frequency distributions of velocities

When wind velocities are measured with a conventional pressure-tube or cup anemometer, the result is given as a function of the horizontal component of the velocity. Similarly, in a wind tunnel a hot-wire anemometer measures the component of the velocity perpendicular to the direction of the hot wire. At small intensities of turbulence, the difference between the horizontal component of the velocity,

$$V_H = [(\bar{u} + u')^2 + v'^2]^{\frac{1}{2}},$$

and the component parallel to the direction of the mean velocity, $u = \bar{u} + u'$, is small and, in general, can be neglected. When, however, the intensity of turbulence is large, as is most often the case in the natural wind and in some artificial flows (as for instance in jets and boundary layers), this difference can no longer be ignored.

We are going to determine the frequency distribution of velocities projected on a plane (V_H) under the assumption that the components of the turbulent velocity, u', v' and w', are distributed according to a normal law. This problem has already been treated

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by Hesselberg and Björkdal [4], Ertel and Jaw [5], Kampé de Fériet [6] and Koo [7]. However, the equations which they have derived were only applied to the most simple case of isotropic turbulence and lead to great difficulties in numerical applications to the case of non-isotropic turbulence. In another paper [8], the writer has shown how such equations can be used in atmospheric turbulence, assuming isotropy of the turbulence. In the present paper, the relations for a particular case of non-isotropic turbulence are derived and represented in a form which can be easily used for numerical applications. Some numerical results are given, and several experimental results are presented for comparison.

Assuming that the components of the turbulent velocity are distributed according to a normal law, one finds for the probability distributions of u' and v'

$$\mathrm{Prob}\left[\xi < \frac{u'}{\bar{u}} < \xi + d\xi\right] = \frac{1}{T_x\sqrt{2\pi}} \exp\left[-\frac{\xi^2}{2T_x^2}\right] d\xi, \quad (1)$$

$$\mathrm{Prob}\left[\eta < \frac{v'}{\bar{u}} < \eta + d\eta\right] = \frac{1}{T_y\sqrt{2\pi}} \exp\left[-\frac{\eta^2}{2T_y^2}\right] d\eta, \quad (2)$$

where $T_x = \sqrt{\overline{u'^2}}/\bar{u}$ and $T_y = \sqrt{\overline{v'^2}}/\bar{u}$ are, respectively, the x- and the y-components of the intensity of turbulence.

In the present paper, it is assumed that the cross correlation coefficient

$$\overline{u'v'}/\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}} \quad (3)$$

is small compared to unity. The case when the cross correlation is not neglected will be considered at another time.

For the two-dimensional probability distribution, we find

$$\mathrm{Prob}\left[\xi < \frac{u'}{\bar{u}} < \xi + d\xi,\ \eta < \frac{v'}{\bar{u}} < \eta + d\eta\right] = \frac{1}{2\pi}\frac{1}{\kappa T_x^2} \exp\left[-\frac{1}{2T_x^2}\left(\xi^2 + \frac{\eta^2}{\kappa^2}\right)\right] d\xi\, d\eta, \quad (4)$$

where

$$\kappa = T_y/T_x = \sqrt{\overline{v'^2}}/\sqrt{\overline{u'^2}}.$$

Expression (4) represents the joint probability of having, at the same time, for the x-component of the turbulent velocity a value between $\bar{u}\xi$ and $\bar{u}(\xi + d\xi)$ and for the y-component of this velocity a value between $\bar{u}\eta$ and $\bar{u}(\eta + d\eta)$. If we trace the vector $(u'^2 + v'^2)^{\frac{1}{2}}$ (see fig. 1), (4) represents the probability distribution for $(u'^2 + v'^2)^{\frac{1}{2}}$ to have its tip inside the square element $d\xi$, $d\eta$.

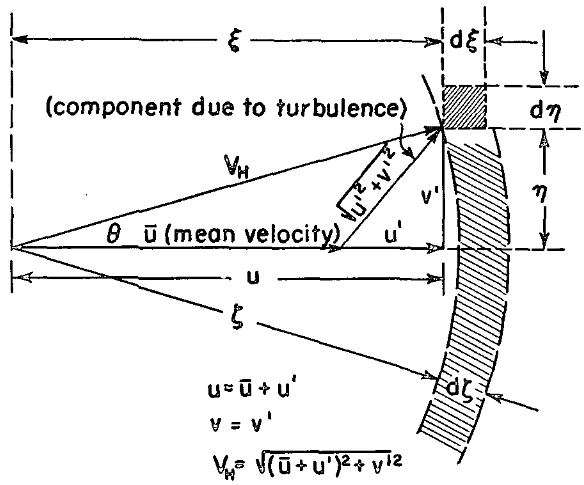

FIG. 1. Coordinate system.

Since our anemometer measures the velocity V_H, we shall determine the probability distribution

$$\mathrm{Prob}\left[\zeta < V_H/\bar{u} < \zeta + d\zeta\right] = D(\zeta)\, d\zeta, \quad (5)$$

which measures the probability of having the tip of the vector V_H within the annular area limited by the concentric circles of radii ζ and $\zeta + d\zeta$.

To find the frequency distribution function $D(\zeta)$, we shall determine the probability distribution of V_H by replacing ξ and η in (4) by ζ. Noting that $u' = V_H \cos\theta - \bar{u}$ and $v' = V_H \sin\theta$, we have $\xi = \zeta\cos\theta - 1$ and $\eta = \zeta\sin\theta$, and after a simple transformation we find

$$D(\zeta) = \frac{1}{2\pi}\frac{\zeta}{\kappa T_x^2} \exp\left[-\frac{1}{2T_x^2}\left(\frac{\kappa^2 + \zeta^2}{\kappa^2}\right)\right] \times \int_0^{2\pi} \exp\left\{\frac{\zeta^2}{2T_x^2}\left[\left(\frac{1-\kappa^2}{\kappa^2}\right)\cos^2\theta + \frac{2}{\zeta}\cos\theta\right]\right\} d\theta. \quad (6)$$

To simplify this equation, we must integrate the expression

$$J = \int_0^{2\pi} \exp\left(a\cos\theta + b\cos^2\theta\right) d\theta. \quad (7)$$

Developing the exponents in series, we find

$$J = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{n-2m} b^m}{\Gamma(m+1)\,\Gamma(n-2m+1)} \int_0^{2\pi} \cos^n\theta\, d\theta.$$

Since in the above expression the integrals with odd powers of cosines are zero, we find after integration

$$J = 2\sqrt{\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n-2m} b^m}{\Gamma(m+1)\,\Gamma(2n-2m+1)} \times \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}. \quad (8)$$

If we consider now the Bessel function of an imaginary argument of the zero order,

$$I_0(a) = \sum_{n=0}^{\infty} \frac{1}{\Gamma^2(n+1)\,2^{2n}}\,a^{2n},$$

we find for its $(2s)$-th derivative

$$I_0^{(2s)}(a) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)}{\Gamma(2n-2s+1)\,\Gamma^2(n+1)\,2^{2n}}\,a^{2n-2s}. \quad (9)$$

It can be easily shown that

$$J = 2\pi\Upsilon_0(a, b), \quad (10)$$

where

$$\Upsilon_0(a, b) = \sum_{s=0}^{\infty} \frac{b^s}{\Gamma(s+1)}\,I_0^{(2s)}(a). \quad (10a)$$

We can now solve the integral which appears in (6), and we find

$$D(\zeta) = \frac{1}{\kappa}\exp\left[-\frac{1}{2T_x^2}(1-\zeta)^2\right]a \times \exp\left[-(a+b)\right]\Upsilon_0(a, b), \quad (11)$$

with $a = \zeta/T_x^2$ and

$$b = \frac{\zeta^2}{2T_x^2}\left(\frac{1-\kappa^2}{\kappa^2}\right).$$

Fig. 2 represents the frequency-distribution function $D(\zeta)$ when the longitudinal intensity of turbulence $T_x = 0.5$ and for ratios $\kappa = T_y/T_x$ of 0.8, 1, 1.5, and 2. In addition, a normal distribution curve is given for comparison. The marked difference between the normal distribution, the case of isotropic turbulence ($\kappa = 1$) and the frequency distribution for non-isotropic turbulence should be noted. These differences will still be very appreciable at smaller intensities of turbulence.

While in atmospheric measurements [9] we most often meet the case of $\kappa \geqslant 1$, in wind-tunnel investigations (particularly for hot-wire measurements in the boundary layer) the case of $\kappa \leqslant 1$ will be of interest.

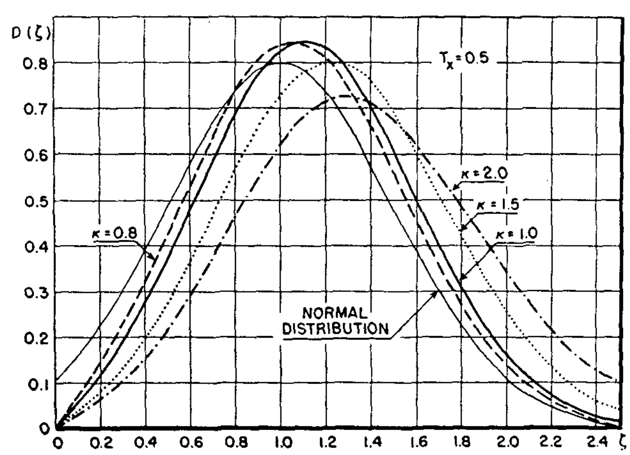

FIG. 2. Frequency distribution of horizontal velocities for longitudinal intensity of turbulence $T_x = 0.5$ and various ratios $\kappa = T_y/T_x$ (T_y: transverse intensity of turbulence). Normal distribution is given for comparison.

If we consider now the case of isotropic turbulence, for which $\kappa = 1$ and $T_x = T_y = T$, $b = 0$, $\Upsilon_0(a, b) = I_0(a)$ and

$$D(\zeta) = \frac{\zeta}{T^2}\exp\left[-\frac{1}{2T^2}(1+\zeta^2)\right] I_0\left(\frac{\zeta}{T^2}\right). \quad (12)$$

This equation, which was already given by other authors [5; 6], can be replaced by

$$D(\zeta) \approx \frac{1}{T}\sqrt{\frac{\zeta}{2\pi}}\exp\left[-\frac{1}{2T^2}(1-\zeta)^2\right], \quad (13)$$

when ζ/T^2 is large enough (say of the order of 12). If the intensity of turbulence is very small,

$$D(\zeta) \approx \frac{1}{T\sqrt{2\pi}}\exp\left[-\frac{1}{2T^2}(1-\zeta)^2\right]. \quad (14)$$

The last equation is equivalent to (1), representing the normal distribution function.

The frequency distributions for isotropic turbulence are given in fig. 3. Curves corresponding to (12), (13), and (14) are represented for intensities of turbulence of 0.5, 0.1, 0.05, and 0.01. It will be noticed that, when $T = 0.01$, the three curves are superposed and the frequency distribution function can be considered as a normal distribution.

3. Mean velocity and turbulent intensity

To find the correct mean velocity $\bar{u}$ and the correct intensity of turbulence T from the measurements of V_H, it is necessary to take into account the difference between the assumed normal distributions of the turbulent components u', v', w' and the distribution function $D(\zeta)$.

In the present paper, we shall only give approximate relations for $\bar{u}$ and T in terms of V_H. Developing $V_H = [(\bar{u} + u')^2 + v'^2]^{\frac{1}{2}}$ in a series and neglecting the terms of higher order, we find

$$\frac{V_H}{\bar{u}} \approx 1 + \frac{u'}{\bar{u}} + \frac{v'^2}{2(\bar{u})^2}. \quad (15)$$

Upon averaging, this gives the relation [6]

$$\bar{V}_H/\bar{u} \approx 1 + \tfrac{1}{2}T_y^2. \quad (16)$$

Taking the square of (16) and again neglecting the terms of higher order, we find

$$(\bar{V}_H)^2/(\bar{u})^2 \approx 1 + T_y^2 \quad (17)$$

while, taking the square of (15) and averaging, we obtain

$$\overline{V_H^2}/(\bar{u})^2 \approx 1 + T_x^2 + T_y^2. \quad (18)$$

From the last two equations we find

$$(\bar{u})^2 \approx (\kappa^2 + 1)(\bar{V}_H)^2 - \kappa^2\overline{V_H^2}. \quad (19)$$

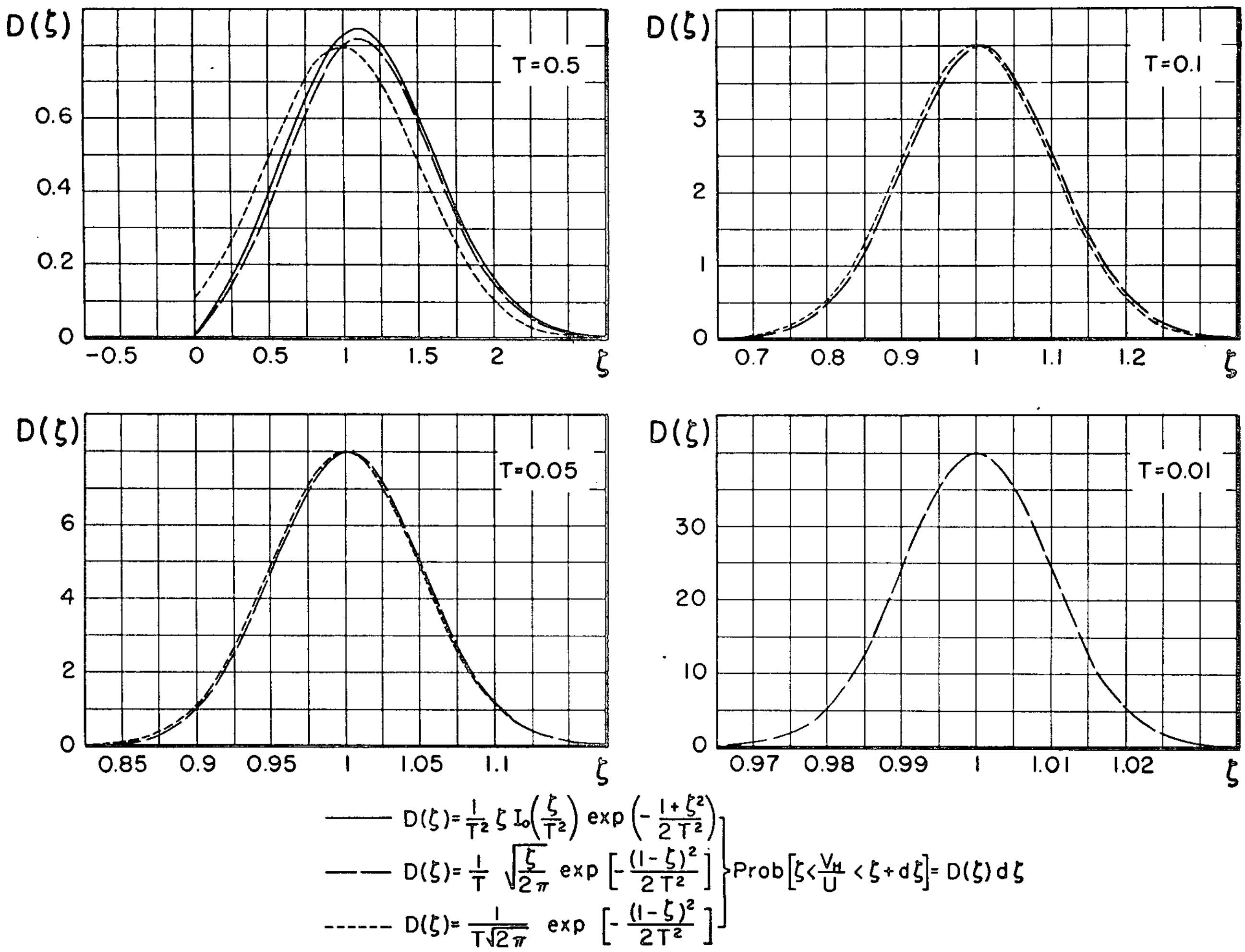

FIG. 3. Frequency distribution of horizontal velocities in isotropic turbulence for several values of intensity of turbulence.

When the turbulence is isotropic,

$$(\bar{u})^2 \approx 2(\bar{V}_H)^2 - \overline{V_H^2} \tag{20}$$

and

$$T^2 \approx \frac{\overline{V_H^2} - (\bar{V}_H)^2}{2(\bar{V}_H)^2 - \overline{V_H^2}}. \tag{21}$$

These approximate equations give the mean velocity $\bar{u}$ and the intensity of turbulence T in terms of V_H, measured with the anemometer.

Figs. 4 and 5 represent experimental frequency distributions measured, respectively, at the University of Texas in Austin and at the U. S. Weather Bureau in Washington.

The mean velocity and the intensity of turbulence indicated on the figures were obtained by applying the approximate equations (20) and (21). It will be noticed that the shape of the experimental distributions is

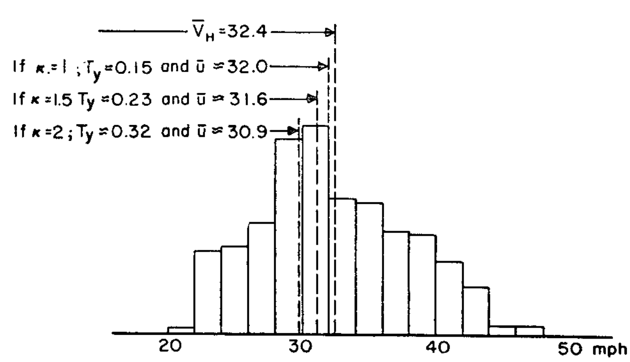

FIG. 4. Experimental frequency distribution measured with cup anemometer 24 ft above ground. Recorded at University of Texas, Austin, on 16 November 1950 between 2134 and 2140 GCT.

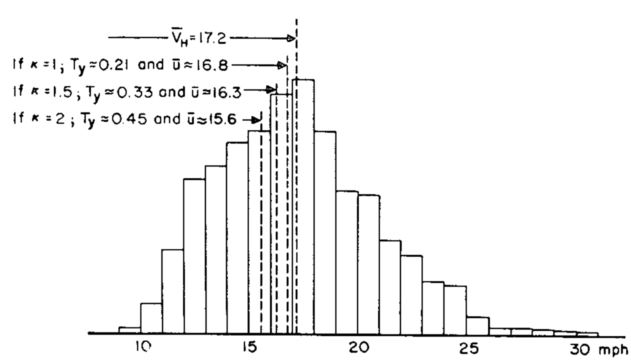

FIG. 5. Experimental frequency distribution measured with cup anemometer 100 ft above ground. Recorded at U. S. Weather Bureau, Washington, on 17 January 1951 between 1615 and 1649 GCT.

similar to the theoretical curves given in fig. 2. It seemed premature to use experiments for a more complete comparison with the theoretical results. It is believed that measurements made under well-known experimental conditions should be used for such a purpose.

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