Does Extended Sawyer–Eliassen Equation Effectively Capture the Secondary Circulation of a Simulated Tropical Cyclone?

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ABSTRACT: The validity of the gradient wind balance in a tropical cyclone (TC) remains controversial, especially for the boundary layer and the upper outflow layer, even though this balance is assumed in the derivation of the Sawyer–Eliassen (SE) equation. This study derives an extended SE equation with the relaxation of the gradient wind and hydrostatic balance in cylindrical $r$-$z$ coordinates, and then we diagnose the secondary circulation using this unbalanced SE equation and the azimuthally averaged tangential wind and thermodynamical fields from a three-dimensional numerical simulation of an intensifying TC. The gradient wind and hydrostatic imbalance produce two additional time-dependent forcing terms on the right-hand side (rhs) of SE equation, which are proved to be negligible for diagnosing the secondary circulation, even as the storm evolves rapidly. The use of the unbalanced basic state deforms the fields of coefficients that appear in the SE equation, and thus the forced secondary flows. The results indicate that the unbalanced solution captures the boundary layer inflow better than the balanced solution described by Bui et al. and the pseudobalanced solution described by Heng et al. The unbalanced solution is closer to the simulation because more unbalanced components are included. Many previous studies always employ the thermal wind balance relation to simplify the SE equation, which is invalid in an unbalanced vortex and result in an overestimation of the boundary layer inflow. These unbalanced dynamics could provide a reliable diagnosis of the secondary flow near the boundary layer.

KEYWORDS: Dynamics; Hurricanes/typhoons; Secondary circulation; Tropical cyclones

1. Introduction

The primary circulation of a tropical cyclone (TC) can generally be considered a warm-core, nearly circular vortex in a gradient wind and hydrostatic balance (LaSeur and Hawkins 1963; Hawkins and Rubsam 1968; Jorgensen 1984; Willoughby 1990, 1998). The gradient wind and hydrostatic balance imposes a strong constraint on the primary circulation. The heat and momentum forcings would drive the flow away from a thermal wind balance, and a transverse or secondary circulation is then required to oppose the effects of the forcings to keep the vortex in balance. This toroidal circulation can be determined by solving a diagnostic equation for the streamfunction of the overturning circulation. This diagnostic equation is commonly referred to as the Sawyer–Eliassen (SE) equation or often alternatively as the balance dynamics (Eliassen 1951).

The SE equation is a linear partial differential equation, which implies that solutions to different forcings are additive. The SE equation can thus be used to diagnose the contributions of different forcings to the toroidal flows and intensification of the vortex. Many earlier theoretical studies have focused on the idealized balanced vortex (Willoughby 1973; Shapiro and Willoughby 1982; Pendergrass and Willoughby 2009; Vigh and Schubert 2009). Despite the gradient wind balance that has been widely used in theoretical models of TCs, the validity of the gradient wind balance in TCs remains controversial in both observational and simulation studies, especially for the boundary layer (Gray and Shea 1973; Gray 1991; Kepert 2006, 2001; Kepert and Wang 2001; Zhang et al. 2001). In addition to the theoretical studies, the balance dynamics have been applied to diagnosing the secondary circulation and spin up of a simulated TC, in which the balanced assumption is not exactly valid and the diagnostic effectiveness remains controversial near the boundary layer (e.g., Fudeyasu and Wang 2011; Möller and Shapiro 2002; Abarca and Montgomery 2014, 2015; Sun et al. 2013; Ohno and Satoh 2015; Wu et al. 2016). The question then arises as to how to diagnose the secondary circulation reasonably and exactly in a simulated unbalanced vortex, especially near the boundary layer.

Bui et al. (2009) investigated the extent to which the secondary circulation of a TC simulated using a full-physics model can be captured by balance dynamics. They employed a model-derived azimuthally averaged tangential wind field and computed its corresponding balanced thermodynamical fields following Smith (2006) in solving the traditional SE equation. They found that the SE equation captures a major fraction of the secondary circulation except in the boundary layer, where the inflow is considerably underestimated and the gradient wind imbalance is important. Heng et al. (2017) instead adopted model-derived azimuthally averaged temperature fields directly and reexamined the SE balance solution.
of a TC in their idealized numerical simulation. They found that the SE equation reproduced the secondary circulation during the intensification period, even in the boundary layer. Their sensitivity calculations demonstrated that part of the underestimation of the boundary layer inflow by Bui et al. (2009) resulted from their global regularization for inertial instability in the upper troposphere and the one-sided finite-differencing scheme used at the surface with relatively coarse vertical resolution (500 m). Montgomery and Smith (2018) criticized that Heng et al. (2017) used model-derived azimuthally averaged tangential wind and temperature fields to calculate static stability, inertial stability, and baroclinicity in the SE equation. Consequently, the basic-state vortex in the boundary layer is not in gradient and thermal wind balance, although these are assumptions of the SE equation. Heng et al. (2018) argued that with the relaxation of the thermal wind balance in the boundary layer, Heng et al. (2017) better reproduced the simulated boundary layer inflow than did Bui et al. (2009). Bui et al. (2009) calculated the azimuthally averaged temperature field in the boundary layer based on a thermal wind balance with a tangential wind, which gives rise to an illusory large cold-core structure in the boundary layer. Montgomery and Persing (2021) confirmed that the “pseudobalanced SE formulation” used by Heng et al. (2017) improves the solution of inflow while sacrificing the strict thermal wind balance. Of course, the TC solution is closer to the simulation if more unbalanced components are included.

The present study investigated the validity of the SE equation when the gradient and thermal wind balance is relaxed in the simulated vortex. Following the appendix in Bui et al. (2009) and Montgomery and Smith (2017a), we extended the SE equation with the relaxation of the gradient wind and hydrostatic balance in cylindrical r–z coordinates. The remainder of this paper is organized as follows. Section 2 briefly describes the Weather Research and Forecasting (WRF) experiment, the simulation, and the structure of gradient wind imbalance. The extended SE formulations, the description of diagnostic methods, and the procedure of regularization are presented in section 3. The diagnostic results, the sensitivity to regularization, and the error analysis are presented in section 4. Finally, the conclusions and discussion are presented in section 5.

2. Simulation

2a. WRF setup

This study used version 3.9.1 of the WRF Model (Skamarock et al. 2005). The model was set on an f plane at 20°N, containing four domains (i.e., D01, D02, D03, and D04) with horizontal resolutions of 18, 6, 2, and 0.67 km and dimensions of 240 × 240, 240 × 240, 360 × 360, and 540 × 540, respectively. All domains had 64 vertical levels with the lowest half-σ model level at 21 m, approximately. In this study, the Kain–Fritsch cumulus scheme (Kain 2004) was implemented only for D01, while D02–04 adopted explicitly resolved convection. The other physics schemes used in all domains were the WRF single-moment 6-class scheme (Hong and Lim 2006), Dudhia shortwave radiation scheme (Dudhia 1989), Rapid Radiative Transfer Model longwave radiation scheme (Mlawer et al. 1997), Yonsei University scheme (Hong et al. 2006), and revised MM5 Monin–Obukhov surface layer scheme (Zhang and Anthes 1982; Jiménez et al. 2012).

The horizontally uniform environmental fields are quiescent, and the initial sounding of temperature and humidity profiles are based on Jordan (1958). The sea surface temperature was set uniformly to be 29°C. The model was initialized with an idealized axisymmetric cyclonic vortex, which had a surface radial structure of the vortex’s azimuthal velocity that followed Eq. (37) in Rotunno and Emanuel (1987). The initial cyclonic vortex had a maximum tangential wind speed at the surface of 25 m s⁻¹ at a radius of 82.5 km. The results for the D03 mesh with 2-min intervals were used in the diagnostic and analyses discussed below.

2b. WRF simulations

The time series of the maximum 10-m azimuthally averaged tangential wind is shown in Fig. 1. After an initial adjustment period of approximately 12 h, the TC vortex intensified rapidly from 12 h with a maximum 10-m tangential wind of 17 m s⁻¹ to 123 h with a maximum 10-m tangential wind of nearly 70 m s⁻¹. Subsequently, the vortex continued to evolve with fluctuations.

To examine the validity of the SE equation, we chose a strong stage of the storm averaged from 120 to 121 h in the simulation when the storm had a maximum azimuthal mean tangential wind of 70 m s⁻¹ and the gradient wind imbalance is also significant. Additionally, a moderate stage of the storm averaged from 62 to 63 h in the simulation and a relatively
weak stage of the storm averaged from 38 to 39 h are also investigated in the appendix.

c. Gradient wind imbalance

Figure 2 shows the radius–height cross sections of the azimuthal-mean tangential wind and agradient wind of the simulated TC at 120 h. The simulated storm at 120 h has a maximum azimuthal-mean tangential wind speed of approximately 85 m s$^{-1}$ at a height of 1 km and the corresponding radius of maximum wind is located approximately 30 km from the storm center. The agradient winds are weak compared with the tangential winds in the troposphere, except in the inflow boundary layer and upper troposphere. In the boundary layer, there is a subgradient wind below 1 km, which is attributable mainly to the reduction in the tangential flow due to surface friction, whereas there is a supergradient acceleration in the vicinity of the maximum wind speed, which is consistent with the results of previous studies (e.g., Zhang et al. 2001; Kepert and Wang 2001; Smith and Montgomery 2008). The extreme violation of the gradient wind balance, which occurs in the upper troposphere, is also reported in previous studies (Möller and Shapiro 2002; Fudeyasu and Wang 2011).

To facilitate the understanding of gradient wind imbalance, we performed a budget analysis for the azimuthal-mean radial wind tendency. The azimuthal-mean radial wind tendency equation in cylindrical coordinates can be written as

$$\frac{\bar{v}^2}{r} + f\bar{v} \frac{1}{\bar{p}} \frac{\partial \bar{p}}{\partial r} = \frac{d}{dt} \bar{r} - \frac{\partial^2 \bar{u}}{r} + \frac{\bar{u}}{\bar{p}^2} \frac{\partial \bar{p}}{\partial r},$$

(1)

where a prime denotes the deviation (eddy) from the azimuthal mean, the overbar denote azimuthal mean, $\bar{r}$ and $\bar{v}$ are the radial and tangential velocity components, $r$ and $z$ are the radial and vertical coordinates, $\bar{p}$ is the pressure, $\bar{p}$ is the density, $f$ is the Coriolis parameter, and $f_0$ is the radial friction. Figures 3a and 3b show that the pressure gradient force and the sum of centrifugal and Coriolis force are nearly counteracted, except in the lowest 3 km. We define $\bar{U} = \bar{v}^2/r + f\bar{v} - (1/\bar{p}) \partial \bar{p}/\partial r$ to quantify the degree of gradient wind imbalance. Figures 3c and 3d show that structure of $\bar{U}$ is well reproduced by the right-hand side (rhs) of Eq. (1). Figures 3e and 3f show that the supergradient in the vicinity of the maximum wind speed is primarily attribute to the advection and the subgradient below 1 km is attribute to the surface friction, which is consistent with Zhang et al. (2001). Note that our calculation shows that the perturbed terms [the third term on the rhs of Eq. (1)] are negligible compared to the first two items on the rhs of Eq. (1).

3. Methods

a. Extended SE formulation

The axisymmetric primitive equations in cylindrical $r$–$z$ coordinates on the $f$ plane may be expressed as follows (Bui et al. 2009):

$$\frac{\bar{v}^2}{r} + f\bar{v} - \frac{1}{\bar{p}} \frac{\partial \bar{p}}{\partial r} = \bar{U},$$

(2)

$$\frac{\partial \bar{v}}{\partial t} + \bar{v} (\bar{v} + f) + \bar{w} \frac{\partial \bar{f}}{\partial z} = F_\lambda + \bar{V},$$

(3)

$$\frac{1}{\bar{p}} \frac{\partial \bar{p}}{\partial z} + g = -W,$$

(4)

$$\frac{\partial \bar{X}}{\partial t} + \bar{v} \frac{\partial \bar{X}}{\partial r} + \bar{w} \frac{\partial \bar{X}}{\partial z} = -\chi^4 (\bar{Q} + \bar{\Theta}),$$

(5)

$$\frac{1}{r} \frac{\partial r \bar{v}}{\partial r} + \frac{\partial \bar{w}}{\partial z} = 0,$$

(6)
where \( r \), \( \lambda \), and \( z \) are the radial, azimuthal, and vertical coordinates, respectively; \( \pi \), \( \bar{v} \), and \( \bar{w} \) are the velocity components, \( g \) is the gravitational acceleration; \( \zeta = \partial \bar{v} / \partial r + \partial \bar{u} / \partial \lambda \) is the vertical component of the relative vorticity; \( \chi = 1/\bar{\theta} \), and \( \bar{\theta} \) is the potential temperature. The dotted terms \( \dot{U} \), \( \dot{V} \), \( \dot{W} \), and \( \dot{Q} \) represent the residual of gradient wind equation, tangential wind tendency equation, hydrostatic equation, and potential temperature tendency equation, and are calculated directly using

\[
\dot{U} = \frac{\bar{v}^2}{r} + f - \left( \frac{1}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial r} \right),
\]

\[
\dot{V} = \frac{\partial \bar{u} \partial \bar{v}}{r} + f \bar{\omega} + \frac{\partial \bar{u} \partial \bar{w}}{r} - F_z,
\]

\[
\dot{W} = -\left( \frac{1}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial z} \right) - g, \quad \text{and} \quad \dot{\Theta} = \frac{\partial \bar{u} \partial \bar{\omega}}{r} + \frac{\partial \bar{u} \partial \bar{w}}{r} + \frac{\partial \bar{w}}{\partial z} - \dot{Q},
\]

respectively. The terms \( Q \) and \( F_L \) represent the heat source and momentum source, respectively, are defined as

\[
Q = \tilde{\theta} - \frac{u' \partial \theta}{\partial r} - \frac{v' \partial \theta}{r \partial \lambda} - \frac{w' \partial \theta}{\partial z},
\]

\[
F_L = -\frac{u' \tilde{\bar{Q}}}{r} - \frac{w' \tilde{\bar{Q}}}{\partial z} + PBL,
\]

\( \tilde{\theta} \) is the tendency of the overall potential temperature (i.e., the sum of the microphysics latent heating \( \dot{\theta}_{\text{mic}} \), radiative heating \( \dot{\theta}_{\text{rad}} \), and heating in the planetary boundary layer \( \dot{\theta}_{\text{PBL}} \)), the final three terms on the rhs of Eq. (7) represent the contributions of eddies to heating, the first two terms on the rhs of Eq. (8) are the contributions from the momentum forcing associated with eddies, and PBL is the forcing of tangential momentum in the planetary boundary layer.

Combining Eqs. (2) and (4) and eliminating the pressure we obtain

\[
\frac{\partial}{\partial r} \left[ \bar{\rho} (g + \dot{W}) \right] + \frac{\partial}{\partial z} \left[ \bar{\rho} (\dot{\bar{U}} - \dot{U}) \right] = 0,
\]

where \( \bar{C} = \frac{\bar{v}^2}{r} + f \bar{v} \) is the sum of the centrifugal and Coriolis terms. With \( \ln \bar{\psi} = (1 - \kappa) \ln \bar{\rho} - \ln \bar{\theta} \) and \( \bar{\theta} = 1/\bar{\chi} \), Eqs. (2), (4), and (9) can be recast as

\[
\frac{g \partial \bar{\chi}}{\partial r} + \frac{\partial}{\partial z} (\bar{\chi} \bar{C}) = \frac{\partial}{\partial r} (\bar{\chi} \dot{\bar{U}}) - \frac{\partial}{\partial r} (\bar{\chi} \dot{\bar{W}}),
\]

which is the extended thermal wind equation in the unbalanced SE formulation, where \( \bar{U} \) and \( \bar{W} \) quantifies the degree of the gradient wind and hydrostatic imbalance, respectively.
and vertical wind components are obtained as
(i.e., \(\frac{\partial}{\partial t}\) substitution of the time derivatives from Eqs. (3) and (5)
where
\[\frac{\partial}{\partial r}(\zeta^2 + f) + \frac{\partial}{\partial z}(\zeta^2 v) + \frac{\partial}{\partial r}(\zeta^2 U) + \frac{\partial}{\partial \theta}(\zeta^2 W)\].
(12)
where \(\vec{\psi}\) is the transverse streamfunction from which the radial and vertical wind components are obtained as
\(\vec{\psi} = (-1/\rho)\partial\psi/\partial z\) and
\(\vec{\psi} = (1/\rho)\partial\psi/\partial \theta\), respectively.

b. Balanced equations

With the assumptions of gradient wind and hydrostatic balance
(i.e., \(U = 0\) and \(W = 0\)), the extended thermal wind equation,
Eq. (10), in the unbalanced SE formulation can be simplified to
the balance thermal wind equation as in Bui et al. (2009) and
Smith (2006):

\[
\frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r}(\zeta^2 + f) + \frac{\partial}{\partial z}(\zeta^2 v) + \frac{\partial}{\partial r}(\zeta^2 U) + \frac{\partial}{\partial \theta}(\zeta^2 W) \right] = 0.
\]  
(13)

Additionally, by ignoring the residual of tangential wind tendency equation and potential temperature tendency equation
(i.e., \(V = 0\) and \(\Theta = 0\)), Eq. (12) can be simplified as

\[
\frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r}(\zeta^2 + f) + \frac{\partial}{\partial z}(\zeta^2 v) + \frac{\partial}{\partial r}(\zeta^2 U) + \frac{\partial}{\partial \theta}(\zeta^2 W) \right] = 0.
\]  
(14)

Conventionally, \(g\partial\vec{\psi}/\partial r\) in the second term on the left-hand side
(left-hand side (LHS) of Eq. (14) is replaced by \(-\partial(\zeta^2 + f)/\partial \theta\)
based on a balanced thermal wind equation, Eq. (13), (although we will
display that this replacement is unreasonable in the unbalanced
conditions), then we can obtain the balanced SE equation used in
Bui et al. (2009) and Heng et al. (2017):

\[
\frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r}(\zeta^2 + f) + \frac{\partial}{\partial z}(\zeta^2 v) + \frac{\partial}{\partial r}(\zeta^2 U) + \frac{\partial}{\partial \theta}(\zeta^2 W) \right] = 0.
\]  
(15)
c. Unbalanced equations

In the unbalanced SE formulation, assuming $\dot{V} = 0$ and $\dot{\Theta} = 0$, Eq. (12) can be rewritten as

$$
\frac{\partial}{\partial \tau} \left[ \frac{\partial}{\partial z} \left( \frac{\partial^2}{\partial r \partial z} \theta \right) + \frac{\partial}{\partial r} \frac{\partial}{\partial z} \left( \frac{\partial^2}{\partial r^2} \theta \right) \right] + \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial r} \left( \frac{\partial^2}{\partial z^2} \Gamma \right) + \frac{\partial}{\partial z} \left( \frac{\partial^2}{\partial z \partial F} \right) \right] = \frac{\partial}{\partial \tau} \left( \frac{\partial F}{\partial \tau} \right) + \frac{\partial}{\partial \tau} \left( \frac{\partial F}{\partial \tau} \right)
$$

in the balanced SE equation, and the two additional time-dependent forcing terms on the rhs of Eq. (12) (unit: K s$^{-3}$).

There are two differences between Eq. (16) and balanced SE equation, Eq. (15), i.e., the underlined term in Eq. (16), which is replaced using the balanced thermal wind equation, Eq. (13), in the balanced SE equation, and the two additional time-dependent forcing terms on the rhs of Eq. (16), which is explicitly in connection with the hydrostatic and gradient wind imbalance.

First, we examine the thermal wind equation in the unbalanced SE formulation. Figures 4a and 4b show that the balanced thermal wind relation in (13) is invalid in the vicinity of the maximum tangential wind and the lower boundary near the radius of maximum wind (RMW). This indicates that the replacement of underlined term in Eq. (16) by the balanced thermal wind relation should cause a serious error when the gradient wind imbalance is important. Figures 4c and 4d show that the extended thermal wind in Eq. (10) captures the thermal wind relation more reasonably and that the effects of the hydrostatic imbalance are thus negligible compared with those of the gradient wind.

**Table 1. Metrics of the secondary circulation.**

<table>
<thead>
<tr>
<th>Solution</th>
<th>Temperature fields</th>
<th>Diagnostic equation</th>
<th>Balance assumption</th>
<th>Min[$\alpha(z = 250,m)$]</th>
<th>$R[\text{Min}$(\alpha)$]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WRF</td>
<td></td>
<td>SE equation, Eq. (15)</td>
<td>Balance</td>
<td>-31.2</td>
<td>36</td>
</tr>
<tr>
<td>Smith</td>
<td>Thermal wind balance</td>
<td></td>
<td></td>
<td>-13.6</td>
<td>90</td>
</tr>
<tr>
<td>Heng</td>
<td>WRF output</td>
<td>SE equation, Eq. (15)</td>
<td>Pseudobalance</td>
<td>-42.1</td>
<td>44</td>
</tr>
<tr>
<td>Unbalanced</td>
<td>WRF output</td>
<td>Unbalanced SE equation, Eq. (14)</td>
<td>Unbalance</td>
<td>-29.7</td>
<td>42</td>
</tr>
</tbody>
</table>
imbalance (Zhang et al. 2000). This suggests that the substitution using the balance thermal wind in Eq. (13) should be forbidden in the unbalanced conditions, and the second term on the lhs of the unbalanced SE equation should be
\[
\left( -\frac{1}{r} \right) \left[ g \left( \frac{x}{r} \right) \left( \frac{1}{r^2} \right) \left( \frac{c}{z} \right) \right],
\]
instead of
\[
2 \left( -\frac{1}{r} \right) \left[ \left( \frac{1}{z} \right) \left( \frac{x}{c} \right) \left( \frac{1}{r^2} \right) \left( \frac{c}{z} \right) \right].
\]

We calculated the forcing terms on the rhs of Eq. (16), which is shown in Fig. 5. Figures 5e and 5f manifest that the two time-dependent forcing terms which is related to the hydrostatic and gradient wind imbalance are negligible in comparison with the traditional forcing terms (appendix shows that even when the storm evolves rapidly, these time-dependent forcing terms should also be omitted safely for diagnosing the secondary circulation), which is consistent with the results of recent studies (Bui et al. 2009; Montgomery and Smith 2017a; Montgomery and Persing 2021).

Therefore, by omitting the additional time-dependent forcing term on the rhs of Eq. (16) and discarding the substitution using the balanced thermal wind equation, Eq. (13), in the second term on the lhs of Eq. (16), we could obtain the unbalanced SE equation, Eq. (14), in which the thermal wind balance is relaxed.

d. Diagnostic methods

As discussed above, a thermal wind balance is assumed in the derivation of the traditional SE equation, Eq. (15). The azimuthal-mean tangential wind and temperature fields obtained in the simulation do not satisfy this balanced condition. Montgomery and Persing (2021) presented two types of diagnostic method for obtaining the vortex used in the SE equation, Eq. (15): the balanced method and the pseudobalanced method. The balanced method, such as the Smith solution, gains the mean tangential wind field from the simulation directly and calculates the potential temperature, density, and pressure following Smith (2006) to make sure the balanced condition. The pseudobalanced method, as used in Heng et al. (2017) and Persing and Montgomery (2003), ignores the requirement of the thermal wind balance and solves the balanced

![Fig. 6. Radius–height cross sections of (a) A, (b) B, (c) B', and (d) C in Eqs. (17) and (22) (unit: m² K⁻¹ s⁻² kg⁻¹). (e) 4B₁² and (f) (B₁ + B₂)² below 3-km height (unit: 10⁻¹⁰ m² K⁻¹ s⁻² kg⁻¹). Yellow lines are the zero-degree contours of Dₛ in the Heng solution and green lines are the zero-degree contour lines of Dₘ in the unbalanced solution.](image-url)
SE equation, Eq. (15), with unbalanced coefficients. This method has been criticized in that the vortex is not in close gradient-wind balance. The method is referred to as the Heng solution in this study for convenience. To remove the inconsistency between the unbalanced vortex and the balanced SE equation, we modify the pseudobalanced method in Heng et al. (2017) and diagnose the secondary circulation using model-derived azimuthally averaged temperature fields and the extended SE equation, Eq. (14). In contrast with the Heng solution, this unbalanced method is referred to as the unbalanced solution in the following. Details of these three diagnostic methods are presented in Table 1.

e. Ellipticity and regularization

We rewrite the standard SE equation, Eq. (15), as

\[
A \frac{\partial^2 \bar{\psi}}{\partial r^2} + C \frac{\partial^2 \bar{\psi}}{\partial z^2} + 2B_1 \frac{\partial \bar{\psi}}{\partial z} \frac{\partial A}{\partial r} + \frac{\partial B_1}{\partial z} \frac{\partial \bar{\psi}}{\partial r} + \left( \frac{\partial B_1}{\partial r} + \frac{\partial C}{\partial z} \right) \frac{\partial \bar{\psi}}{\partial z} - F = 0,
\]

(17)

where

\[
A = -\frac{\partial \bar{\psi}}{\partial z} \frac{1}{pr^2}.
\]

(18)
The ellipticity condition for convergent numerical solutions of Eq. (17) requires the standard discriminant
\[ D_s = 4B_2^2 - 4AC, \]
0.

Similarly, the extended SE equation, Eq. (14), is rewritten as

\[ B_1 = -\frac{\partial}{\partial z}(\bar{X}C)\frac{1}{pr}, \tag{19} \]
\[ C = \left[ \frac{\partial}{\partial z}(\bar{X}f + f) + C\frac{\partial}{\partial r}\frac{1}{pr} \right], \tag{20} \]
\[ F = \frac{\partial}{\partial r}(\bar{X}^2Q) + \frac{\partial}{\partial z}(C\bar{X}^2Q) - \frac{\partial}{\partial z}(\bar{X}\bar{F}_l). \tag{21} \]

The ellipticity condition for convergent numerical solutions of Eq. (17) requires the standard discriminant \( D_s = 4B_2^2 - 4AC \). Typically, regions where the SE equation is hyperbolic correspond to regions where the flow is inertially unstable \( [\bar{X}(\bar{X} + f) + C\bar{X}\partial \bar{X}/\partial r < 0] \) or statically unstable \( [-\bar{X}\partial \bar{X}/\partial z < 0] \) or the baroclinicity measured as \( (\partial / \partial z)(\bar{X}C) \) is sufficiently large. The distributions of \( A, B_1, B_2, \) and \( C \) with model-derived azimuthally averaged temperature fields and the regions where the ellipticity condition is violated are shown in Fig. 6. There are mainly two regions in which the equation is hyperbolic: one near the lower boundary, where the vertical shear is large (Fig. 6b), and the other in the outflow layer, where the flow is inertially unstable (Fig. 6d). Figures 6e and 6f show that the values of \( (B_1 + B_2)^2 \) are much smaller than \( 4B_2^2 \) in the vicinity of the maximum tangential wind and

where

\[ B_2 = \bar{s}\frac{\partial \bar{X}}{\partial r}\frac{1}{pr}, \tag{23} \]

and the corresponding modified discriminant is \( D_{m} = (B_1 + B_2)^2 - 4AC \).
the lower boundary, which indicate that the Heng method introduces more hyperbolicity in the unbalanced vortex.

Following Heng et al. (2017), we first set \( \zeta = 0.01f \) at points where \( f = \bar{\omega} + f < 0.01f \) to suppress the presence of inertial instability. If the ellipticity condition was still violated at any grid point, the baroclinicity values \( B_1 \) and \( B_2 \) were reduced by a factor of \( b(0 \leq b \leq 1) \) at those points where \( D \approx 0 \) to ensure that the solution converged. The SE equation was solved numerically with the successive overrelaxation scheme following previous studies (Press et al. 2007). The computational domain for the SE equation was 0–18 km in the vertical with 250-m grid spacing and 0–300 km in the radial direction with 2.0-km grid spacing. The boundary conditions were \( c = 0 \) at \( r = 0, z = 0 \) and 18 km and \( \partial \psi / \partial r = 0 \) at \( r = 300 \) km.

4. Results
a. Secondary circulation

The azimuthally averaged radial and vertical winds of the simulated storm at 120–121 h are shown in Fig. 7a. There is strong inflow (in excess of 30 m s\(^{-1}\)) that is maximized at a radius of 36 km in the boundary layer, relatively weak inflow in the midtroposphere outside the eyewall, relatively deep outflow in the upper troposphere, and a secondary outflow jet immediately above the boundary layer. The vertical velocity field shows an intense updraft (>0.4 m s\(^{-1}\)) between 20 and 90 km from the storm center that extends to the upper troposphere at approximately 14 km above the sea level; the strongest updraft (i.e., >4.0 m s\(^{-1}\)) is evident at a height of approximately 6–9 km. At smaller radius than the main updraft region, there is weak subsidence in the simulated eye. The diagnostic secondary circulations with the three methods are shown in Figs. 7b–d, respectively. It is evident that the broad features of secondary circulation simulated in the WRF (e.g., the intense updraft, weak subsidence inside the eye, deep outflow in the upper troposphere, boundary layer inflow, and secondary outflow jet above the boundary layer inflow) are represented by all three solutions.

The inflow below 6-km height are shown in more detail (in Fig. 10) and the minimum values of the radial inflow, as well as the location of the extrema, are summarized in Table 1 to facilitate a further comparison of the inflow. The peak inflow of the Smith solution at a height of 250 m is 56% weaker than that in the WRF simulation, which is consistent with the results of recent studies (Bui et al. 2009; Montgomery and Persing 2021). The boundary layer inflow in the Smith balanced solution is maximized at a radius of 90 km, which is about 54 km outside of that obtained in the WRF simulation, implying that the balance dynamics is unable to reproduce the inward penetration of the low-level radial inflow into the eye of the storm and thus the contraction of the radius of maximum wind (Heng and Wang 2016; Smith and Vogl 2008; Smith and Montgomery 2016; Montgomery and Smith 2017b). Relative to the WRF
simulation, the peak inflow of the Heng solution at a height of 250 m is 35% stronger and displaced radially outward by 8 km. The overestimation of the boundary inflow was also found by Heng et al. (2017). Compared with the Smith balanced solution and Heng solution, the unbalanced solution captures the inflow layer better in terms of both the pattern and magnitude. The peak inflow at a height of 250 m occurs at a radius of 42 km and is 4.8% lower than simulation values. The low-level inflow in the Smith solution (at a height exceeding 3 km height) is noticeably deeper than the simulated inflow (confined below a height of 1.5 km). In contrast, the depths of the low-level inflow layer in the Heng solution and unbalanced solution are more consistent with those in the WRF simulation.

As described by Heng et al. (2018), in the Smith method, the calculated azimuthal–mean temperature field in the boundary layer based on the gradient wind balance equation, Eq. (13), should generate a large cold-core structure in the boundary layer (data not shown) and thus an appreciable underestimation of the boundary layer inflow. In the Heng solution, which uses model-derived azimuthally averaged temperature fields, the basic-state flow is not in gradient and thermal wind balance as should be assumed for a balanced calculation according to the SE equation, Eq. (15) (Montgomery and Smith 2018). This inconsistency introduces more illusory hyperbolicity and results in an overestimation of the boundary layer inflow. In the perspective of unbalanced dynamics, the Heng solution is not a balanced or quasi-balanced solution. Inversely, it exaggerates the imbalance upon the unbalanced solution by employing the balance relation $B_2 = B_1$ to simplify the SE equation in the unbalanced vortex. As a
consequence, the unbalanced solution captures the boundary layer inflow more accurately than the Heng and Smith solutions.

In the upper levels of the vortex, the outflow structures of all three diagnostic solutions have a double outflow jet and show considerable discrepancies compared to simulation, principally in the region of regularization and regions adjacent to it. The inability of the diagnostic solutions to capture quantitatively the simulated upper-level structure is also found in many related studies (e.g., Wang and Smith 2019; Montgomery and Persing 2021) and most likely due to the regularization in the inertially unstable region of the outflow (Wang and Smith 2019). The diagnosed eyewall updraft of the unbalanced solution is marginally weaker than that of the simulation, which needs further research. In the Heng solution, the updraft and outflow jet at a height of 2–3 km are both evidently exaggerated, owing to the overestimation of the boundary layer inflow.

The above results demonstrate that, with the relaxation of the gradient wind balance, the unbalanced solution effectively captures the secondary circulation simulated in the WRF, even in the boundary layer. Therefore, the contributions of different forcing processes to the secondary circulation in the simulated storm can be calculated using the unbalanced SE equation. As shown in Fig. 8, the relative contributions of the different terms, including the diabatic heating [the first term in Eq. (7)], surface friction [the final term in Eq. (8)], and eddy forcing [the final three terms in Eq. (7) and the first two terms in Eq. (8)], were calculated. These calculations indicate that diabatic heating drives a deep inflow layer in the middle–lower troposphere and a broad outflow layer in the upper troposphere (Fig. 8a). Meanwhile, the upward motion forced by diabatic heating is dominant in the eyewall (Fig. 8b). Additionally, surface friction drives a shallow boundary layer radial inflow that accounts for approximately 40%–50% of the simulated inflow in the lower part of the boundary layer (Fig. 8c), as reported previously by Fudeyasu and Wang (2011) and Heng et al. (2017). Figures 8e and 8f show that the asymmetric eddy forcing contributes more trivially than other forcings because the simulated storm in our study is largely symmetric.

b. Regularization

As stated in section 3e, the inertial instability $\left[\frac{\partial z}{\partial t} + f - \frac{\partial z}{\partial r}\right]$ which often occurs in the outflow layer and the large baroclinicity measured as $\left(\frac{\partial z}{\partial C} + f \chi C\right)$ or $B_z$, which occurs near the lower boundary, could introduce hyperbolicity. For the solution of the diagnostic equation to be effectively posed, we set...
\[ \tau = 0.01f \] at points where \[ \tau = \frac{z}{f} < 0.01f, \] and if there are remaining points where \( D > 0 \), we then reduce the terms \( B_1 \) and \( B_2 \) by a factor of \( b \) (0 \( \leq b \leq 1 \); Heng et al. 2017). The adjustments of the inertial instability have little effect on the boundary layer inflow (Wang and Smith 2019), and therefore, in this section, we explore the sensitivity of the boundary layer inflow to the adjustments of the vertical shear of the tangential wind, or in other words, the artificial selection of parameter \( b \). The vertical profiles of the radial wind averaged within a radius of 100 km in the unbalanced solution and Heng solution when \( b = 0.45, 0.2, 0.1, 0.01 \) are shown in Fig. 9. It is worth noting that the Heng solution is nonconvergent when \( b = 0.45 \), the inconsistency between the unbalanced vortex and the balanced SE equation in Heng method cause more illusory nonellipticity and should use more stringent regularization to make sure the convergence of the solution. Compared with the case for the Heng solution, the boundary layer inflow of the unbalanced solution is much more insensitive to the artificial selection of parameter \( b \). The average radial wind velocities of the Heng solution are much more overestimated as \( b \) increases, which seems to be counterintuitive. In essence, the more stringent “regularization” (i.e., smaller \( b \)) reduces the coefficient of \( \frac{\partial^2 \Theta}{\partial r \partial z} \) and mitigates the exaggeration of the vertical shear of tangential wind in the Heng method, which reduces the overestimation of the boundary layer inflow.

**c. Other error source**

In practice, in addition to the gradient wind and hydrostatic imbalance, the residuals of tangential wind tendency equation and potential temperature tendency equation are often nonnegligible (i.e., \( \dot{V} \neq 0 \) and \( \dot{\Theta} \neq 0 \)), which might be introduced due to the definition of the circulation center, interpolating the values into the cylindrical coordinates, and the use of different finite-differencing schemes. In this study, each variable is interpolated from the model \((x, y, \sigma)\) to \((x, y, z)\), then transformed to cylindrical coordinates, finally averaged azimuthally and temporally (2-min intervals) over the 1-h period. Compared with the gradient wind and hydrostatic imbalance, which appear as two time-dependent forcing terms on the rhs of Eq. (12), \( \dot{V} \) and \( \dot{\Theta} \) behave as the same as the momentum source \( F_\alpha \) and heat source \( Q \), respectively. Figure 5 show that the effect of residual of tangential wind tendency equation \( \dot{V} \) on SE equation is nonnegligible, whereas the
forcing terms associated with the $\dot{\Theta}$, $\dot{U}$, and $W$ are negligible compared to that with heat source $Q$ and momentum source $F_l$. We solve the extended SE equation with the $\dot{V}$ included (which is referred to as the modified solution), and the modified inflow below 6-km height is shown in Fig. 10d. Compared to unbalanced solution, the modified solution captures the inflow better above 1-km height, which indicates that the error of unbalanced solution is primarily attributed to the residual of tangential wind tendency equation. In the boundary layer, the peak inflow of modified solution is slightly weaker than WRF output, which may attribute to the regularization in the boundary layer.

5. Conclusions and discussion

To provide a reliable diagnosis of the secondary flow near the boundary layer, this study extended the SE equation in height coordinates with the relaxation of the gradient wind and hydrostatic balance. The unbalanced SE equation distinguishes the original SE equation, Eq. (15), from two aspects: one is the second term on the lhs of Eq. (15), in which the balance thermal wind equation, Eq. (13), is invalid in the unbalanced conditions; the other is the additional time-dependent forcing terms on the rhs of Eq. (15) from which the time derivative cannot be eliminated as before when the thermal wind balance is violated. The calculations show that these time-dependent forcing terms could be ignored reasonably, even when the storm evolves rapidly. As a result, for the simulated unbalanced mean vortex, which is not in a close gradient and thermal wind balance, we proposed an unbalanced method extending the Heng method (Heng et al. 2017), in which we diagnose the secondary circulation using the extended SE equation, Eq. (14), and the model-derived azimuthally averaged temperature fields.

The azimuthally averaged radial and vertical winds from the WRF simulation of a quiescent tropical cyclone were subsequently compared with those derived using the unbalanced method and two other available diagnostic methods: a balanced method and pseudobalanced method. The balanced method,
referred to as the Smith solution (Bui et al. 2009) in this study, uses a model-derived azimuthally averaged tangential wind field and calculates the corresponding balanced thermodynamical fields using the thermal wind equation, generating an illusory cold-core structure in the boundary layer. The pseudobalanced method, referred to as the Heng solution in this study, uses the model-derived azimuthally averaged temperature fields and original SE equation. The comparison showed many commonalities throughout the troposphere but also large differences in the boundary layer. All three solutions captured the overall structure of the secondary circulation of the simulation and exhibited differences from the WRF simulation in the region of outflow where $\left[\frac{\partial \eta}{\partial r} + \frac{\partial \sigma}{\partial \tau}\right]$ is negative. In the boundary layer, the Smith solution underestimated the peak inflow velocity by 56% and overpredicted the radial location of the peak inflow by 54 km, whereas the peak inflow of the Heng solution was stronger by 35% and displaced radially outward, away from the center by 8 km. In comparison, the unbalanced solution captured the boundary layer inflow well. The peak inflow was at a radius of 42 km and 4.8% weaker than that in the WRF simulation. The Heng method uses the traditional SE equation, where the gradient wind balance is strictly assumed and is inconsistent with the unbalanced vortex, which results in an overestimation of the boundary layer inflow. Compared to the unbalanced solution, Heng solution is not a balanced or quasi-balanced solution as they claim. On the contrary, it exaggerates the unbalanced effect. With the relaxation of the gradient wind balance, the gradient wind imbalance is included and thus the unbalanced solution is closer to the simulated. The unbalanced components deform the fields of coefficients that appear in the SE equation, and correct the overestimation in the Heng method.

A sensitivity analysis showed that the boundary layer inflow of the Heng solution is highly sensitive to the adjustments of the vertical shear of the tangential wind. The
artificial regularization in the boundary layer can mitigate
the overestimation of the boundary layer inflow by reducing
the exaggerated vertical shear of the tangential wind in the
Heng solution. Consequently, it is necessary to elucidate details
of the procedure of regularization in the boundary layer when
the Heng method is used to diagnose the secondary circulation.
In contrast, the boundary layer inflow of the unbalanced solution
is much more insensitive to the artificial regularization in
the boundary layer. In addition, the residuals of tangential wind
tendency are nonignorable, which may introduce errors
between the diagnostic and simulated inflow and indicates that
the budget analysis for the azimuthal-mean tangential wind
tendency deserves more careful handling when solving the SE
equation.

Much recent debate has centered on the validity of the SE
equation in diagnosing the simulated secondary circulation,
especially in the inner core and boundary layer (Bui et al.
2009; Heng et al. 2017, 2018; Montgomery and Smith 2018;
Montgomery and Persing 2021). This study shows that when
the more unbalanced components are included, the TC boundary
layer structures from the unbalanced SE solution would
be closer to the simulation. These results highlight the need
for unbalanced boundary layer dynamics to complete the
description of the spin up in the boundary layer, which is
consistent with the findings of recent studies (Bui et al. 2009;
Montgomery and Smith 2018; Montgomery and Persing 2021).

APPENDIX

Diagnosis in a Moderate Stage and Weak Stage of the
Simulated Storm

To further confirm the reliability of the unbalanced method for
the rapid intensification (RI) period, we computed the forcing
terms on the rhs of Eq. (12) at the simulation time of 38–39 and
62–63 h (see Fig. 1). Figures A1 and A2 show that the time-
dependent forcing terms caused by the hydrostatic and gradient
wind imbalance are insignificant, even as the storm evolves
rapidly. Likewise, only the residuals of tangential wind tendency
equation are nonnegligible compared to the heat source $Q$ and
momentum source $F_k$.

Figures A3 and A4 show the radial wind below 6-km height from the simulation and diagnosed solutions at the
simulation time of 38–39 and 62–63 h, respectively. The unbalanced solution also effectively captures the inflow below
6-km height compared to Heng solution, especially in the
boundary layer (also see Table A1). Figures A3d and A4d
indicate that the error of unbalanced solution could be
primarily attributed to the residual of tangential wind ten-
dency equation, especially above 1-km height.

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(DCC-OCC) and the approved Programme of the Ocean to
climate Seamless Forecasting system (OSF).

Data availability statement. The numerical model simula-
tions upon which this study is based are too large to archive
or to transfer. Instead, we provide all the information needed
to replicate the simulations and diagnoses; we used WRF Model version 3.9.1. The model code, initial condition files,
and namelist settings are available at http://data.fio.org.cn/
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