What Are the Balanced and Unbalanced Dynamics of a Tropical Cyclone?

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(Manuscript received 26 July 2022, in final form 6 April 2023, accepted 7 April 2023)

ABSTRACT: Recently, Ji and Qiao took into account the unbalanced components and derived an extended Sawyer–Eliassen (SE) equation. This study developed a new derivation of this extended SE equation from the perspective of restoring forces, and gives a physical interpretation for the coefficients that appear in the SE equation. For an unbalanced vortex, we demonstrated that the thermodynamic fields are only determined by the distribution of gradient wind, and thus, the gradient wind and thermodynamic fields always remain in balance as the unbalanced vortex evolves. Consequently, we attributed the gradient wind imbalance to the agradient wind rather than to the thermodynamic fields. Subsequently, we explored the effect of the agradient wind on the secondary circulation, and showed that the agradient wind strengthens the secondary circulation in its vicinity, which can be explained as a consequence of the restoring forces and mass continuity. Furthermore, we speculated that the SE equation, together with the radial velocity equation, could reproduce the primary characteristic of the axisymmetric boundary layer dynamics by prescribing the parameterization of subgrid-scale turbulent mixing. Specifically, the noU\textsubscript{BL} and noVa\textsubscript{BL} experiments conducted by Fei et al. in an article published in 2021 were reinterpreted, and the oscillation wavelength of the agradient wind in the eyewall was approximated based on this framework. Additionally, a new numerical solution algorithm to overcome the hyperbolicity near the boundary layer was proposed. This study attempts to develop a complete dynamic theory for tropical cyclones in both qualitative and quantitative perspectives.

KEYWORDS: Secondary circulation; Tropical cyclones; Hurricanes/typhoons

1. Introduction

Tropical cyclones (TCs), especially typhoons or hurricanes, are intense storm systems that usually originate over warm tropical oceans. They represent one of the most destructive weather processes, often causing considerable loss of human life and substantial damage to infrastructure. Over the past several decades, the skill in forecasting the track of TCs has shown improvement statistically, whereas forecasting the intensity and size of TCs remains a substantial challenge (Gall et al. 2013; DeMaria et al. 2014). The lack of skill in forecasting TC intensity and size might be partly attributable to insufficient understanding of the inner-core dynamics.

Tropical cyclones can be viewed as warm-core, nearly axisymmetric vortices in an approximate gradient wind and hydrostatic balance, superimposed by a secondary circulation in the radius–height cross section. From the perspective of balanced dynamics, the heating and momentum sources spin up the primary circulation through driving a radial inflow in the lower troposphere that draws high absolute angular momentum inward (Montgomery and Smith 2014). This secondary circulation can be determined by solving the Sawyer–Eliassen (SE) equation based on the gradient wind and hydrostatic balance, which often is alternatively referred to as balanced dynamics and has been applied widely to diagnose the intensification processes of a simulated TC. On the basis of the diagnosis with balanced dynamics, Fudeyasu and Wang (2011) found that the spinup of a midtropospheric outer-core primary circulation is contributed principally by diabatic heating in the outer rainbands, which suggests that diabatic heating in spiral rainbands is fundamental to continued growth of a storm-scale circulation. Möller and Shapiro (2002) diagnosed the secondary circulation and the intensity change of Hurricane Opal (1995) using the SE equation. Their results indicated that although the spinup of the inner-core vortex due to symmetric heating and friction was much greater, the role of eddy processes could not be ignored. Ohno and Satoh (2015) used balanced dynamics to investigate the upper-level warming and showed that the upper-level warming in the eye is determined by both the diabatic heating of cloud microphysics and the upper-level inertial stability. Furthermore, the balanced dynamics approach is also applicable for investigating the secondary eyewall formation and eyewall replacement cycle (e.g., Rozoff et al. 2012; Sun et al. 2013; Chen 2018).

The balance framework still has certain limitations and shortcomings. First, because the gradient wind balance assumed in the derivation of the SE equation is invalid in the boundary layer, the diagnostic effectiveness of the SE equation near the boundary layer remains controversial (Bui et al. 2009; Heng et al. 2017). Second, although the inertial stability ($2\omega/r + f)(\omega r + \omega \omega r + f$) has been widely investigated using analytic methods (e.g., Schubert and Hack 1982;
the effect of gradient wind imbalance on both the secondary circu-
section 3b. Subsequently, in section 4, we examine the balanced 
dynamics or balanced vortex from the perspective of a simulated or observed TC also remains controversial (Montgomery and Persing 2021).

Recently, Ji and Qiao (2023) derived an extended SE equation with relaxation of the gradient wind balance and demonstrated that their unbalanced solution could capture the boundary layer inflow reasonably well, especially in comparison with both the “balanced” solution presented in Bui et al. (2009) and the “pseudobalanced” solution presented in Heng et al. (2017). We regard this as the unbalanced dynamics of a TC. On the basis of the work of Ji and Qiao (2023), the objective of this study was to address the remaining unanswered questions regarding TC dynamics. The remainder of this paper is organized as follows. In section 2a, we propose a derivation of the SE equation that differs from that of Ji and Qiao (2023) by considering the temporal and spatial disturbance in the vortex. In section 2b, we define the balanced vortex in the unbalanced formulation and thus attribute the gradient wind imbalance to the agradient wind. The balanced and unbalanced idealized vortices are prescribed in section 3a, and a new algorithm to overcome the hyperbolicity near the boundary layer is proposed in section 3b. Subsequently, in section 4, we examine the effect of gradient wind imbalance on both the secondary circulation and the advection of the radial wind. Finally, a discussion and our conclusions are presented in section 5.

2. Theory

a. Perturbation analysis

In TCs, the nonhydrostatic effects are small (Zhang et al. 2000), and therefore, we consider an idealized unbalanced vortex in which the axisymmetric primitive equations in cylindrical $r$–$z$ coordinates are as follows:

\[-\frac{1}{\rho} \frac{\partial p}{\partial r} + C - U(r, z, t) = 0,\]
\[
\frac{\partial u}{\partial t} + w \frac{\partial u}{\partial z} - F_{\text{radial}} = U,\]
\[
\frac{\partial v}{\partial t} + u(\xi + f) + w \frac{\partial v}{\partial z} = F_{\lambda},\]
\[
\frac{\partial \rho}{\partial t} - \frac{\rho}{\rho z} = g = 0,\]
\[
\frac{\partial \chi}{\partial t} + u \frac{\partial \chi}{\partial r} + w \frac{\partial \chi}{\partial z} = -\chi^2 Q,\]
\[
\frac{1}{r} \frac{\partial ru}{\partial r} + \frac{\partial pw}{\partial z} = 0,\]

where $r$ and $z$ are the radius and height, respectively, $u$ and $w$ are the radial and vertical velocity, respectively, $p$ is pressure, $g$ is gravitational acceleration, $\rho$ is density, $f$ is the Coriolis parameter, $C = \nu^2/r + f\nu$ (in which $\nu$ is tangential wind), $\chi$ is $1/\theta$ (where $\theta$ is potential temperature), $U$ is the agredient force that quantifies the degree of the gradient wind imbalance, $F_{\text{radial}}$ is radial friction, $\zeta = \partial w \partial r + \nu/r$ is the vertical component of the relative vorticity, and $Q$ and $P_f$ represent diabatic heating and tangential friction, respectively.

Here we give an alternative derivation of the SE equation, by considering the temporal and spatial disturbance in the vortex. First, we consider a fluid parcel that is displaced radially outward from $X_a(t_a, r, z, t)$ to $X_b(t_b, r, z, t)$, where $r_b = r_a + \Delta r$. The perturbations of the sum of the centrifugal and Coriolis forces acting on the displaced parcel at $X_b$ are

\[ F_{r,\Delta r,c} = \frac{v^2}{r} + f v' - \frac{v^2}{r} - f v_b, \]

where $v_r$ and $v_b$ represent the tangential velocity at $X_a$ and $X_b$, respectively, and $v'_r$ represents the tangential velocity of the displaced parcel at $X_b$. Assuming the absence of tangential friction, the displaced parcel conserves its absolute angular momentum such that

\[ v'_r r_a + \frac{1}{2} f r_a^2 = v'_r r_b + \frac{1}{2} f r_b^2. \]

Based on Eq. (8), Eq. (7) can be recast as follows:

\[ F_{r,\Delta r,c} = -\frac{1}{\sqrt{r}} \frac{\partial M^2}{\partial r} \Delta r = \frac{1}{\sqrt{r}} (\xi + f) \Delta r, \]

where $M$ is the absolute angular momentum and $\xi = 2\nu/r + f$.

Similarly, the perturbations of the radially inward pressure gradient force acting on the displaced parcel at $X_b$ are as follows:

\[ F_{r,\Delta r,p} = \frac{1}{\rho_a} \frac{\partial P_a}{\partial r} + \frac{1}{\rho_b} \frac{\partial P_b}{\partial r}, \]

where $\rho_a$ and $\rho_b$ are the density at $X_a$ and $X_b$, respectively, $P_a$ and $P_b$ are the pressure at $X_a$ and $X_b$, respectively, and $\rho'_a$ is the density of the displaced parcel at $X_b$. Assuming the absence of diabatic heating, the displaced parcel conserves its potential temperature such that

\[ \rho'_a \sim \chi_a P_b^{1-k}, \quad \rho_b \sim \chi_b P_b^{1-k}, \]

where $\chi_a$ and $\chi_b$ are the inverse of the potential temperature at $X_a$ and $X_b$, respectively, and $k$ is the Poisson constant. On the basis of Eq. (11), Eq. (10) can be recast as follows:

\[ F_{r,\Delta r,p} = -\frac{1}{\chi \partial r} \left( \frac{\partial P}{\partial r} \right) \Delta r = \frac{1}{\chi \partial r} C \Delta r + \frac{1}{\chi \partial r} U \Delta r. \]

Through combination of Eqs. (9) and (12), we can obtain the radial restoring force derived from the radial disturbance:
\[ xF_{r \Delta r} = xF_{r \Delta r, C} + xF_{r \Delta r, P} \]
\[ = - \left[ \xi(x + f) + \frac{\partial x}{\partial r} \right] \Delta r + \frac{U}{\partial r} \frac{\partial x}{\partial r} \Delta r. \]  
(13)

Second, we consider a considering the fluid parcel that is displaced vertically from \( X_c(r, z_c, t) \) to \( X_c(r, z, t) \), where \( z_c = z_a + \Delta z \). The conservation of absolute angular momentum implies that \( \nu'_a = \nu'_c \); therefore, the vertical disturbance can give rise to a perturbation of the sum of the centrifugal and Coriolis forces:
\[ F_{r \Delta r, C} = \frac{v'^2}{r} + \frac{v''}{r} - \frac{v}{r} - \frac{2v}{r} = -\left( \frac{2v}{r} + f \right) \frac{\partial v}{\partial z} \Delta z = -\frac{\partial C}{\partial z} \Delta z, \]  
(14)

where \( v_c \) represent the tangential velocity at \( X_c \).

Similarly, the conservation of potential temperature indicates that \( \rho'_a - \chi \rho'_c - \omega \) and \( \rho'_a - \chi \rho'_c - \omega \), where \( \chi \) is the inverse of the potential temperature at \( X_c \), \( \rho_c \) is the density at \( X_c \), \( P_r \) is the pressure at \( X_c \), and \( \rho'_c \) is the density of the displaced parcel at \( X_c \). Therefore, the perturbations of the radially inward pressure gradient force acting on the displaced parcel at \( X_c \) due to the vertical disturbance are
\[ F_{r \Delta r, P} = \frac{1}{\rho'_a} \frac{\partial \rho_a}{\partial r} + \frac{1}{\rho'_c} \frac{\partial \rho_c}{\partial r} = -\frac{1}{\rho_c} \frac{\partial (\rho P)}{\partial z} \Delta z \]
\[ = -\frac{\partial C}{\rho_c} \Delta z + \frac{\partial C}{\rho_c} \Delta z. \]  
(15)

Through combination of Eqs. (14) and (15), we can obtain the radial restoring force derived from the vertical disturbance:
\[ xF_{r \Delta r} = xF_{r \Delta r, C} + xF_{r \Delta r, P} = \frac{\partial (\chi)}{\partial z} \Delta z + \frac{\partial C}{\rho_c} \Delta z. \]  
(16)

Finally, we consider the temporal disturbance of a fluid parcel from \( X_c(r, z, t) \) to \( X'_c(r, z, t') \), where \( t' = t + \Delta t \). Based on Eq. (3), we can obtain \( \Delta v = F_{r \Delta t} \). Therefore, the perturbations of the sum of the centrifugal and Coriolis forces due to the temporal disturbance are
\[ F_{r \Delta t, C} = \left( \frac{2v}{r} + f \right) \Delta v = \xi F_{x} \Delta t. \]  
(17)

Similarly, on the basis of Eq. (5), we have \( \rho'_a - \chi \rho'_c - \omega \) and \( \rho'_a - \chi \rho'_c - \omega \), and \( \Delta X_c = X'_c - X_a = -\chi x^2 \Delta Q \Delta t \), where \( \rho'_a \) and \( \rho'_c \) are the density and the inverse of the potential temperature, respectively, at location \( X_a \) and time \( t' \). The perturbations of the radially inward pressure gradient force due to the temporal disturbance can be expressed as follows:
\[ F_{r \Delta t, P} = \frac{1}{\rho'_a} \frac{\partial P_a}{\partial r} + \frac{1}{\rho'_c} \frac{\partial P_c}{\partial r} = -\chi x^2 \frac{1}{\rho_c} \Delta Q \Delta t \]
\[ = -\frac{1}{\chi} C x^2 \Delta Q \Delta t + \frac{1}{\chi} x^2 \Delta Q \Delta t. \]  
(18)

On the basis of Eq. (1), the perturbations of the sum of the supergradient and subgradient forces due to the temporal disturbance can be expressed as follows:
\[ F_{r \Delta t, U} = -\frac{\partial U}{\partial t} \Delta t. \]  
(19)

Through combination of Eqs. (17)–(19), we can obtain the radial restoring force due to the temporal disturbance:
\[ xF_{r \Delta t} = xF_{r \Delta t, C} + xF_{r \Delta t, P} + xF_{r \Delta t, U} \]
\[ = \chi x^2 \Delta Q \Delta t + \frac{\partial x}{\partial t} \Delta t. \]  
(20)

Through combination of Eqs. (13), (16), and (20), the total radial restoring force is thus expressed as follows:
\[ xF_{r \Delta t} = xF_{r \Delta t, C} + xF_{r \Delta t, P} + xF_{r \Delta t, U} \]
\[ = -\left[ \xi(x + f) + \frac{\partial x}{\partial r} \frac{\partial (\chi)}{\partial z} \Delta z + \chi x^2 \Delta Q \Delta t \right] - \frac{\partial U}{\partial t} \Delta t \]
\[ = -\xi(x + f) + \chi x^2 \Delta Q \Delta t - \frac{\partial U}{\partial t} \Delta t. \]  
(21)

Following similar steps, the total vertical restoring force can easily be obtained:
\[ xF_{z \Delta t} = g \frac{\partial x}{\partial r} \Delta r + g \frac{\partial x}{\partial z} \Delta z + g x^2 \Delta Q \Delta t. \]  
(22)

By multiplying Eq. (3) by \( g x \) and using Eq. (5) by \( C \), Eqs. (21) and (22) can be recast as follows:
\[ xF_{r \Delta t} = \chi \frac{\partial (x C)}{\partial t} \Delta t, \]  
(23)
\[ xF_{z \Delta t} = \left( -g \frac{\partial x}{\partial t} \Delta t \right). \]  
(24)

Following Ji and Qiao (2023), eliminating pressure from Eqs. (1) and (4) gives the extended thermal wind equation:
\[ g \frac{\partial x}{\partial z} = \frac{\partial}{\partial r} \left( \chi \frac{\partial x}{\partial r} \right) \frac{\partial (x U)}{\partial z}. \]  
(25)

Based on Eq. (25), it is easy to obtain the relationship between \( F_{r \Delta t} \) and \( F_{z \Delta t} \):
\[ \frac{\partial}{\partial r} \left( xF_{r \Delta t} \right) = \frac{\partial}{\partial z} \left( xF_{z \Delta t} \right). \]  
(26)
and thus, the extended SE equation can be expressed as follows:

\[
\frac{\partial}{\partial r} \left[ \frac{\partial \chi}{\partial z} \frac{1}{\rho} \frac{\partial \psi}{\partial r} + \frac{g}{\rho} \frac{\partial \chi}{\partial r} \frac{1}{\rho} \frac{\partial \psi}{\partial z} \right]
+ \frac{\partial}{\partial z} \left[ \xi(\chi + f) + \frac{C}{\rho} \frac{\partial \chi}{\partial r} \frac{1}{\rho} \frac{\partial \psi}{\partial z} + \frac{g}{\rho} \frac{\partial \chi}{\partial r} \frac{1}{\rho} \frac{\partial \psi}{\partial z} \right]
= \frac{g}{\rho} \frac{\partial}{\partial r} \left( C \chi^2 Q \right) + \frac{\partial}{\partial z} \left( C \chi^2 Q \right) - \frac{\partial}{\partial z} (\chi \xi F_{r}) + \frac{\partial^2}{\partial oz^2} (\chi \xi U),
\]
(27)

where \(\psi\) is the transverse streamfunction from which the radial and vertical velocities are obtained as \(u = (-1/\rho) \partial \psi / \partial z\) and \(w = (1/\rho) \partial \psi / \partial r\), respectively, and \(\Delta r = u \Delta t, \Delta z = w \Delta t\).

Previous studies conventionally employed the thermal wind equation \(g(\partial \chi / \partial r) = -(\partial \psi / \partial z)(\chi C)\) to simplify the SE equation, which is invalid in an unbalanced vortex and introduces an additional imbalance (Ji and Qiao 2023). Here, we provide a brief explanation of the additional term \((\partial^2 \psi / \partial oz^2)(\chi U)\), which can be recast as follows:

\[
\frac{\partial}{\partial t} (\chi U) = U \frac{\partial \chi}{\partial t} + \chi \frac{\partial u_{nf}}{\partial t},
\]
(28)

where \(u_{nf}\) represents the ageostrophic wind, \(|u_{nf}| \ll |F_{r}|, U < C\), and \(|\chi \partial \psi / \partial z| \ll |\chi \xi Q|\). Therefore, the magnitude of this additional term is always negligible in comparison with that of other forcing terms derived from heating and friction, which is verified by Ji and Qiao (2023). Consequently, the SE Eq. (29) can always provide an approximate but sufficiently accurate diagnosis of the secondary circulation:

\[
\frac{\partial}{\partial r} \left[ \frac{\partial \chi}{\partial z} \frac{1}{\rho} \frac{\partial \psi}{\partial r} + \frac{g}{\rho} \frac{\partial \chi}{\partial r} \frac{1}{\rho} \frac{\partial \psi}{\partial z} \right]
+ \frac{\partial}{\partial z} \left[ \xi(\chi + f) + \frac{C}{\rho} \frac{\partial \chi}{\partial r} \frac{1}{\rho} \frac{\partial \psi}{\partial z} + \frac{g}{\rho} \frac{\partial \chi}{\partial r} \frac{1}{\rho} \frac{\partial \psi}{\partial z} \right]
= \frac{g}{\rho} \frac{\partial}{\partial r} \left( C \chi^2 Q \right) + \frac{\partial}{\partial z} \left( C \chi^2 Q \right) - \frac{\partial}{\partial z} (\chi \xi F_{r}) + \frac{\partial^2}{\partial oz^2} (\chi \xi U).
\]
(29)

Specifically, the SE Eqs. (27) and (29) imply that radial friction and tangential friction play different roles in TC dynamics. The tangential friction acts as momentum forcing, whereas the radial friction leads to the subgradient wind and thus deforms the coefficients that appear in SE Eqs. (27) and (29). It is also noteworthy that the additional term \(\partial^2 \psi / \partial oz^2\) is nonnegligible for diagnosing the TC intensification rate, and its contribution will be investigated in the next work.

b. Balanced and unbalanced vortex

An unbalanced vortex (i.e., the distributions of tangential wind, density, pressure, and potential temperature) can be obtained from simulation or observation directly, and thus is well-defined. However, a balanced vortex exists only as an idealized concept. In this section, we consider a balanced vortical system with gradient wind and hydrostatic balance and an unbalanced vortical system with only hydrostatic balance. Subsequently, we define a balanced vortex in the unbalanced vortical system, which is beneficial in understanding the evolution of the unbalanced vortex.

First, we consider the balanced system, and thus, Eqs. (1) and (25) should be revised as follows:

\[
- \frac{1}{\rho} \frac{\partial p}{\partial r} + C_{g} = 0
\]
(30)

and

\[
\frac{g}{\rho} \frac{\partial \chi}{\partial r} = -\frac{\partial}{\partial z} (C \xi U),
\]
(31)

where \(C_{g} = (u_{nf}^2 + v_{nf}^2)\), in which \(v_{nf}\) is the gradient wind. Following Smith (2006), Eq. (31) can be recast as follows:

\[
\frac{\partial}{\partial r} \ln p + \frac{C_{g}}{g} \frac{\partial}{\partial z} \ln = -\frac{1}{g} \frac{\partial}{\partial z} (C_{g}).
\]
(32)

The characteristics of Eqs. (31) and (32) should satisfy the following:

\[
d \frac{d}{dr} \ln = \frac{1}{g} \frac{\partial C_{g}}{\partial z},
\]
(33)

and along these characteristics, we have

\[
d \frac{d}{dr} \ln = \frac{1}{g} \frac{\partial C_{g}}{\partial z},
\]
(34)

where the characteristics are the isobaric surfaces. Given the distribution of \(u_{nf}\), one can calculate \(p, \rho, \) and \(\chi\) in the thermal wind balance with \(v_{nf}\) using Eqs. (33)–(35), respectively. The detailed procedure can be found in section 2.1 of Smith (2006). The value of \(\rho \xi\) is clearly constant along the characteristics (i.e., isobaric surfaces). Consequently, we have \(p = p_{0}, \rho \chi = \rho_{0}, \) where \(p_{0}, \rho_{0}, \) and \(\chi_{0}\) are the ambient pressure, density, and the inverse of potential temperature at some large radius on the same isobaric surface with \((p, \rho, \chi)\). Because \(p_{0}, \rho_{0}, \) and \(\chi_{0}\) satisfy the ideal gas law, the calculated \(p, \rho, \) and \(\chi\) should also satisfy the ideal gas law.

The derivation of the balanced SE equation in section 2.2 of Bui et al. (2009) shows that the diagnostic \(u\) and \(w\) must ensure the following:

\[
\frac{g}{\rho} \frac{\partial \chi}{\partial t} + \frac{\partial}{\partial z} \left[ \frac{\partial (C \xi U)}{\partial t} \right] = 0,
\]
(36)

which indicates that the evolving balanced system must always satisfy the thermal wind equation.

The balanced system thus evolves as described in the following steps:

(i) \(v_{nf}^2, p_{0}^2, \rho_{0}^2, \) and \(\chi^2\) satisfy the thermal wind balance and the ideal gas law;

(ii) \(u\) and \(w\) are diagnosed using the SE Eq. (29);

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(iii) \( v'^{+1} = v' + (\partial v'/\partial t)\Delta t \) and \( \chi'^{+1} = \chi' + (\partial \chi'/\partial t)\Delta t \), where \( \partial v'/\partial t \) and \( \partial \chi'/\partial t \) are calculated using Eqs. (3) and (5), respectively;
(iv) \( p'^{+1} \) and \( p'^{+1} \) are calculated using Eqs. (33) and (34) with \( v'_{g} \);
(v) \( v'^{+1} \), \( p'^{+1} \), \( p'^{+1} \), and \( \chi'^{+1} \) remain in the thermal wind balance and the ideal gas law, where the thermal wind balance is the sufficient and necessary condition of the gradient wind and hydrostatic balance.

Then we consider the unbalanced system, and the extended thermal wind Eq. (25) can also be rewritten as follows:
\[
\frac{\partial}{\partial r} \ln \rho + \frac{C_{g} \partial}{\partial z} \ln \rho = -\frac{1}{g} \frac{\partial}{\partial z} (C_{g}),
\]
(37)

where \( C_{g} = C - \dot{U} \). Following the same procedures as described previously, \( p \), \( \rho \), and \( \chi \) can be computed using Eqs. (33)–(35) with the gradient wind \( v_{g} \), and will always satisfy the ideal gas law.

Similarly, the derivation of the unbalanced SE equation in section 3.1 of Ji and Qiao (2023) indicates that the evolving unbalanced system must always satisfy the following:
\[
g \frac{\partial \chi}{\partial r} = -\frac{\partial}{\partial z} (x C_{g}),
\]
(38)

which in this study is called the “thermal gradient wind balance” (between \( v_{g} \) and \( \chi \)) in the unbalanced vortex, which differs from the thermal wind balance (between \( v \) and \( \chi \)).

The difference between the balanced system and the unbalanced system is that the gradient wind \( v_{g} \) in the unbalanced system cannot be directly obtain from simulation [i.e., \( v'^{+1} = v' + (\partial v'/\partial t)\Delta t \) and \( \chi'^{+1} = \chi' + (\partial \chi'/\partial t)\Delta t \)]. Note that Eqs. (1) and (2) can be rewritten as follows:
\[
C - C_{g} = v \frac{\partial U}{\partial r} + w \frac{\partial U}{\partial z} - F_{\text{radial}},
\]
(39)

\((\rho, \chi)\) is determined only by the distribution of \( v_{g} \) and the parameterization of diabatic heating and friction can be prescribed, and therefore, Eq. (39) and the SE Eq. (27) are closed. Consequently, \( v_{g} \) is determined by the tangential wind \( v \) [obtained from \( v'^{+1} = v' + (\partial v'/\partial t)\Delta t \)] and parameterization of the boundary layer, and thus, the gradient wind imbalance is introduced.

The unbalanced system evolves as described in the following steps:

(i) \( v'_{g}, p', \rho', \chi' \) satisfy the thermal gradient wind balance and ideal gas law, and \( v'_{g} \) and \( v' \) satisfy Eq. (39) and the SE Eq. (27);
(ii) \( v'^{+1} = v' + (\partial v'/\partial t)\Delta t \) and \( \chi'^{+1} = \chi' + (\partial \chi'/\partial t)\Delta t; \)
(iii) \( v'^{+1} \) can be obtained using Eq. (39) and the SE Eq. (27) with \( v'^{+1} \);
(iv) \( p'^{+1} \) and \( p'^{+1} \) are calculated using Eqs. (33) and (34) with \( v'_{g} \);
(v) \( v'^{+1} \), \( p'^{+1} \), \( p'^{+1} \), and \( \chi'^{+1} \) satisfy the thermal gradient wind balance and the ideal gas law, and \( u^{+1} \) and \( u^{+1} \) satisfy Eq. (39) and the SE Eq. (27).

Notably, the aforementioned steps constitute a reduced conceptual model for TC intensification, which needs more elaboration iteration for it to be achievable in practice, especially for \( U \) and \( \dot{U} \). For this reason, the tangential wind can be decomposed into balanced portion (gradient wind) and unbalanced portion (agradiant wind), and the agradient wind is derived from the advection of radial velocity and radial friction. We define \( v_{g}, p, \rho, \chi \) as the balanced vortex corresponding to the unbalanced vortex \( (v, p, \rho, \chi) \), and thus, the thermal gradient wind balance (between \( v_{g} \) and \( \chi_{g} \)) always remains when the unbalanced vortex evolves. Here, \( (v_{g}, p, \rho, \chi) \) constitutes the balanced thermodynamics of a TC, and \( v_{g} \) is associated with \( v \) through the dynamics of the boundary layer.

c. Balanced and unbalanced dynamics

For an unbalanced vortex \( (v, p, \rho, \chi) \), the secondary circulation is determined by the extended SE equation:
\[
\frac{\partial}{\partial r} (\bar{A} \frac{\partial \psi}{\partial r} + \bar{B}_{1} \frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial z} (\bar{C} \frac{\partial \psi}{\partial z} + \bar{B}_{1} \frac{\partial \psi}{\partial r}) = \bar{T},
\]
(40)

where
\[
\bar{A} = -g \frac{\partial \chi}{\partial z} \frac{1}{\rho r},
\]
(41)
\[
\bar{B}_{1} = -\frac{\partial}{\partial z} (x C_{g}) \frac{1}{\rho r},
\]
(42)
\[
\bar{B}_{2} = \frac{\partial \chi}{\partial z} \frac{1}{\rho r},
\]
(43)
\[
\bar{C} = \left[ \hat{\xi} (\hat{\chi} + f) + C_{g} \frac{\partial \chi}{\partial r} \right] \frac{1}{\rho r},
\]
(44)
\[
\bar{T} = \frac{\partial}{\partial r} (x^{2} \bar{Q}) + \frac{\partial}{\partial z} (C \bar{\chi} \bar{Q}) - \frac{\partial}{\partial z} (x \hat{\xi} F_{1}),
\]
(45)

and the additional term \( \delta^{2} (\hat{\chi} U) \hat{\alpha} \hat{\beta} \) is omitted for diagnosing secondary circulation.

For a balanced vortex \( (v_{g}, p, \rho, \chi) \), the extended SE equation degenerates into the traditional SE equation that is used in both Bui et al. (2009) and Montgomery and Persing (2021):
\[
\frac{\partial}{\partial r} (\bar{A} \frac{\partial \psi}{\partial r} + \bar{B}_{1} \frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial z} (\bar{C} \frac{\partial \psi}{\partial z} + \bar{B}_{1} \frac{\partial \psi}{\partial r}) = \bar{T}_{g},
\]
(46)
3. Method

a. Idealized vortices

The above section indicates that the gradient wind imbalance could be primarily attributed to the tangential wind field rather than to thermodynamic fields. In comparison with the balanced SE equation, the gradient wind deforms the fields of the coefficients that appear in the unbalanced SE equation. With the following definitions of balanced and unbalanced vortices, it is natural to examine how the gradient wind imbalance affects the secondary circulation. First, we prescribe a gradient wind profile, and then, the density and potential temperature fields in the thermal wind balance with the gradient wind are computed following the procedures described in Smith (2006), which constitute the balanced vortex. Finally, a prescribed gradient wind is superposed upon the gradient wind, which constitutes the unbalanced vortex, together with the previous thermodynamic fields.

The gradient wind profile \(v_g(r, z)\), which is based on an empirically derived sectionally continuous algebraic profile (Willoughby et al. 2006), can be defined as follows:

\[
v_g(r, z) = \begin{cases} 
V_i = V_{\text{max}}(z) \left[ \frac{r}{R_{\text{max}(z)}} \right]^n, & 0 \leq r \leq R_1(z), \\
V_o = V_{\text{max}}(z) \exp \left[ -\frac{r - R_{\text{max}(z)}}{A_1} \right], & R_1(z) \leq r \leq R_2(z), \\
V_o [1 - A(x)] + V_o A(x), & R_2(z) \leq r.
\end{cases}
\]

(50)

where \(V_i\) and \(V_o\) are the gradient wind inside the eye and beyond the transition zone [\(R_1(z) \leq r \leq R_2(z)\)], respectively, and \(V_{\text{max}}\) and \(R_{\text{max}}\) are the maximum gradient wind and the radius of maximum wind speed, respectively. The weighting function \(A(x)\) is a polynomial bell ramp function, in which \(x = (r - R_1)/(R_2 - R_1)\). The maximum wind decreases with height according to a polynomial function \(w_1 = \sigma^2(z - 2\sigma)\), in which \(\sigma = (z_{\text{top}} - z)/z_{\text{top}}\) and \(z_{\text{top}}\) is the height of the vortex. To mimic the simulated gradient wind of a TC using the Weather Research and Forecasting (WRF) Model at 62–63 h in Ji and Qiao (2023), the values of \(V_{\text{max}}(0)\) and \(z_{\text{top}}\) are set as 64 m s\(^{-1}\) and 20 km, respectively, \(R_{\text{max}}\) is 30 km at the surface and slopes outward by 40 km to the height of 20 km, and other parameters are set as follows: \(n = 1, X_1 = 140\) km, \(R_2 - R_1 = 10\) km, the bottom temperature at large radius is prescribed to 303 K, and the lapse rate is 6.43 K km\(^{-1}\).

The prescribed (green lines) and simulated (purple lines) gradient wind fields are shown in Fig. 1b, and the computed thermodynamic fields are shown in Fig. 1d. To mimic the simulated tangential wind, the superposed gradient wind \(v_g\) is prescribed as shown in Fig. 1b, and thus, the distribution of the tangential wind \(v = v_g + v_o\) (shading in Fig. 1a) of the idealized unbalanced vortex is similar to that of the simulation (purple lines in Fig. 1a).

The diabatic heating is extracted from the simulation and fitted to a function with the following form (Paull et al. 2018):

\[
Q(r, z) = Q_{\text{max}} \exp \left[ -\frac{(r - r_c - \gamma z)^n}{\beta} - \frac{(z - z_{\text{top}})^{\beta}}{\beta} \right],
\]

(51)

where \(Q_{\text{max}}\) is 66 K h\(^{-1}\), \(r_c\) is 30 km, \(\alpha\) is 3.0, \(\gamma\) is 2.0, \(\beta\) is 7 km, \(z_c\) is 4.5 km, \(\beta\) is 2.5, and \(\delta_z\) is 3 km. The prescribed (shading) and simulated heating fields (purple lines) are shown in Fig. 1c. The tangential friction \(F_t\) (green lines), extracted from the simulation directly, is also shown in Fig. 1c.

b. Solution of the SE equation

In this study, we applied the successive overrelaxation scheme (Press et al. 2007) to solve the SE equation numerically. The computational domain was 0–20 km vertically with 250-m grid spacing and 0–1000 km in the radial direction with 2.0-km grid spacing. The boundary conditions were set as \(\psi = 0\) at \(r = 0\), \(Z = 0\) and 20 km and \(\partial \psi/\partial r = 0\) at \(r = 1000\) km. To obtain the convergent numerical solution, the SE equation should satisfy the ellipticity condition:

\[
\mathbf{B} = (\mathbf{B}_1 + \mathbf{B}_2)^2 - 4\mathbf{A} \mathbf{C} < 0.
\]

(52)

All solutions of the SE equations considered in this study were convergent except Eq. (65), in which the magnitude of \(\mathbf{B}_2\) was too large to violate the ellipticity condition. Here we propose a new iterative algorithm to overcome the hyperbolicity near the boundary layer. To ensure convergence, we reduced the terms \(\mathbf{B}_1\) and \(\mathbf{B}_2\) to \(\mathbf{B}_1^{\prime}\) and \(\mathbf{B}_2^{\prime}\), respectively, by multiplying by a factor of \(\tau\) \((0 < \tau < 1)\) in the region where the baroclinicity is large. Thus, we have the following iterative equation:

\[
\frac{\partial}{\partial r} \left( A \frac{\partial \psi^{i+1}}{\partial r} + B_z \frac{\partial \psi^{i+1}}{\partial z} \right) + \frac{\partial}{\partial z} \left( C \frac{\partial \psi^{i+1}}{\partial z} + B_r \frac{\partial \psi^{i+1}}{\partial r} \right) = \mathbf{F} + \frac{\partial}{\partial r} \left( \mathbf{B}_2^\prime - \mathbf{B}_1^\prime \right) \frac{\partial \psi}{\partial r} + \frac{\partial}{\partial z} \left( \mathbf{B}_1^\prime - \mathbf{B}_2^\prime \right) \frac{\partial \psi}{\partial z},
\]

(53)

in which we set \(\psi^0 = 0\). According to our experience, Eq. (53) always quickly converges and the convergence speed is affected by \(\tau\).

4. Effect of gradient wind imbalance

a. On the secondary circulation forced by diabatic heating

The SE equation, Eq. (29), is a linear differential equation, and therefore, the effects of the gradient wind imbalance on secondary circulations forced by both diabatic heating and
tangential friction can be discussed separately. First, in Fig. 2, we compare the distributions of radial and vertical velocities forced by identical diabatic heating (Fig. 1c) in balanced and unbalanced vortices. The differences in the secondary circulations are mainly distributed near the regions where the ageostrophic wind occurs (Figs. 2e,f). In the balanced vortex, the inflow layer near the eyewall is deep (below 3 km height), and there is no obvious low-level outflow jet (Fig. 2a), which is consistent with Pendergrass and Willoughby (2009). In the unbalanced vortex, the low-level incident flow near the eyewall is significantly strengthened in comparison with that of the balanced inflow and it is concentrated below the height of 2 km (Fig. 2c). Meanwhile, a low-level outflow jet with a maximal velocity of approximately 4 m s$^{-1}$ emerges in the upper part of the region where the supergradient wind occurs, which is consistent with previous observations and simulations (Gray and Shea 1973; Kepert and Wang 2001; Zhang et al. 2001). The gradient wind imbalance also strengthens the updraft in the eyewall and the downdrafts in the exterior edge of the eyewall below the height of 3 km (Fig. 2f).

Subsequently, we provide a physical interpretation based on the restoring forces, which are defined as Eqs. (21) and (22).

From the perspective of dynamics, the secondary circulation is controlled by two factors: the restoring forces and mass continuity. Note that the restoring forces can also be rewritten as follows:

$$F_r = \left(-\hat{I}^2\hat{u} - b_1\hat{w} - \xi F_\chi - C\chi Q\right)\Delta t.$$  (54)

$$F_z = \left(-b_2\hat{u} - N^2\hat{w} + g\chi Q\right)\Delta t.$$  (55)

where

$$\hat{F} = \xi(\zeta + f) + C\frac{\partial \ln \chi}{\partial r},$$  (56)

$$b_1 = \frac{1}{\chi} \frac{\partial (\chi C)}{\partial z},$$  (57)

$$b_2 = -g \frac{\partial \ln \chi}{\partial r},$$  (58)

$$N^2 = -g \frac{\partial \ln \chi}{\partial z},$$  (59)

and $\hat{F}$ is the generalized inertial stability, $N^2$ is the static stability, and $b_1$ and $b_2$ are the generalized baroclinicities.
The generalized inertial stability $I^2_g = \xi_g (\xi + f) + C_g \dot{\varphi} \ln x / \partial r$ in the balanced vortex and $I^2 = \theta (\xi + f) + C \partial \ln \theta / \partial r$ in the unbalanced vortex is shown in Figs. 3a and 3c, respectively. Both $I^2_g$ and $I^2$ always remain positive, indicating that the radial restoring forces due to the generalized inertial stability always present resistance for a fluid parcel moving radially. In comparison with $I^2_g$, the magnitude of $I^2$ is considerably larger in the inner regions where the supergradient wind occurs, whereas it is much smaller in the outer part of the supergradient wind regions, which is due to the negative radial gradient of the agradient wind. Meanwhile, $I^2$ is also smaller than $I^2_g$ in the regions where the subgradient wind occurs. As shown in Fig. 3b, $b_2$ is always negative throughout the entire region. In the balanced vortex, we have $b_{1g} = (1 / \chi \partial (\chi C_g)) / \partial z = b_3$ from Eq. (31), and $b_2$ in the unbalanced vortex is shown in Fig. 3d; the differences between $b_1$ and $b_2$ are derived from the agradient wind. In the upper part of the supergradient wind regions, the negative values of $b_1$ are an order of magnitude higher than those of $b_2$, whereas in the subgradient wind regions and the lower part of the supergradient wind regions, $b_1$ is positive owing to the positive vertical gradient of the tangential wind.

In the balanced vortex, the radial and vertical restoring forces can be expressed as follows:

\[
F_{r, \text{tot}}^g = \{-2 b_3 u_r - b_4 w_r - C_g \chi \mathcal{Q}\} \Delta t, \quad (60)
\]

\[
F_{z, \text{tot}}^g = \{-b_2 u_r - N^2 w_r + g \chi \mathcal{Q}\} \Delta t, \quad (61)
\]

where $u_r$ and $w_r$ are the solution of Eq. (46) in which $F_\alpha = 0$, and $\Delta t$ is set to 1 for simplicity. Each component of Eqs. (60) and (61) is shown in Fig. 4. In the vertical direction, the vertical restoring force due to diabatic heating is primarily concentrated in the eyewall (Fig. 4f), whereas the vertical restoring force due to static stability is always present as resistance to the vertical motion (Fig. 4b). By comparison, the vertical restoring force due to radial motion is negligible (Fig. 4d). In the eyewall, the updraft is accelerated continuously until the
height of approximately 10 km, and it continues to rise quickly until the height of 12 km owing to inertia (Fig. 4h). The weak subsidence on both sides of the eyewall could be attributed primarily to mass continuity. In the radial direction, the radial restoring force due to the generalized inertial stability constantly plays the role of resistance (Fig. 4a). The radial restoring force due to vertical motion is primarily in the eyewall and oriented outward (Fig. 4c). The effect of diabatic heating is relatively weak in the radial direction (Fig. 4e). The inflow in the lower troposphere is derived from the strong updraft in the lower eyewall and it suffers resistance due to a combination of generalized inertial stability and vertical motion. In the eyewall, the updraft is accelerated continuously outward by the radial restoring force due to vertical motion within the radius of approximately 60 km, which could modulate the strength and distribution of the outflow in the upper troposphere (Fig. 4g). Beyond the radius of 60 km, the outflow could retain its strength because the local generalized inertial stability is much smaller.

In the unbalanced vortex, the radial and vertical restoring forces are as follows:

\[ F_{r,\text{tot}} = \{-F^2 u_w - b_1 w_v - C x Q\} \Delta t, \]  
\[ F_{z,\text{tot}} = \{-b_2 u_v - N^2 w_v + g x Q\} \Delta t, \]

where \( u_w \) and \( w_v \) are the solution of Eq. (40) in which \( F_h = 0 \). Figure 5 shows the distributions of the radial and vertical restoring forces in the unbalanced vortex. In comparison with Fig. 4, the most remarkable differences are the existence of a strong outward force along the radial direction in upper part of supergradient wind regions and an inward force below the height of 1 km near the eyewall (Fig. 5c). The former results in the low-level radial outflow jet, whereas the latter strengthens the inflow below the height of 1 km near the eyewall. This implies that vertical advection of agradient wind instead of agradient force itself is important in understanding the radial flow.

**Fig. 3.** Radius–height cross sections of (a) \( I^2 \) in a balanced vortex, (b) \( b_2 \) or \( b_1 \) in a balanced vortex, (c) \( I^2 \) in an unbalanced vortex, (d) \( b_1 \) in an unbalanced vortex, (e) differences between (a) and (c) below the height of 6 km, and (f) differences between (b) and (d) below the height of 6 km (unit: s\(^{-2}\)). The purple lines indicate the distribution of the gradient wind (contour interval: 8 m s\(^{-1}\)) in (a) and (b) and that of the tangential wind (contour interval: 8 m s\(^{-1}\)) in (c)–(f).
To further validate the individual effect of each coefficient in the SE equation, we explore the sensitivity of the forced secondary circulation to the coefficients $B_1$, $B_2$, and $C_1$. First, we change the generalized baroclinic coefficient $B_1$ of Eq. (40) into $B_{1g}$:

$$\frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} + B_2 \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left( C \frac{\partial \psi}{\partial z} + B_1 \frac{\partial \psi}{\partial r} \right) = F,$$  \hspace{1cm} (64)

and $u_{w1}$ and $w_{w1}$ are the solution of Eq. (64) in which $F_1 = 0$. The distributions of $u_{w1}$ and $w_{w1}$ and the corresponding restoring forces are shown in Figs. 6a–d and 7, respectively. The absence of the low-level radial outflow jet (Fig. 6a) further demonstrates that this outflow jet should be attributed primarily to the strong outward force due to the upward advection of the supergradient wind component and is controlled by the coefficient $B_1$. Meanwhile, the absence of the inward force below the height of 1 km near the eyewall (Fig. 7c) results in weaker inflow (Fig. 6b), which is also controlled by $B_1$. Notably, a “regularization procedure” is always employed in solving the SE equation to remove any regions of hyperbolicity ($\mathcal{D} > 0$) in previous studies (Wang and Smith 2019). Specifically, $B_1$ is always reduced artificially in the lower boundary layer, which might weaken the low-level inward force and thus weaken the low-level inflow near the eyewall.
Previous studies always employ the balance relation $B_1 = B_2 = B_1$ to simplify the SE equation, which is invalid in the unbalanced vortex and could lead to overestimation of the diagnosed inflow (Ji and Qiao 2023). To explore the mechanism of the dynamics in this overestimation, we change $B_2$ of Eq. (40) into $B_1$:

$$
\frac{\partial}{\partial \sigma} \left( \frac{\partial \phi}{\partial \sigma} + B_1 \frac{\partial \phi}{\partial z} \right) + \frac{\partial}{\partial z} \left( C \frac{\partial \phi}{\partial z} + B_1 \frac{\partial \phi}{\partial \sigma} \right) = F, \quad (65)
$$

and $u_{b2}$ and $w_{b2}$ are the solution of Eq. (65) in which $F_x = 0$. Figures 6e–h indicate that Eq. (65) exaggerates the inflow and the updraft near the bottom of the eyewall. Figure 8 indicates that the vertical restoring force due to the radial motion is no longer negligible. Instead, it pull the incident flow upward near the radius of 38 km below the height of 1 km (Fig. 8d), which exaggerates the updraft, generating a stronger inward force (Fig. 8c) and thus exaggerating the nearby inflow (Fig. 6f). As Heng et al. (2017, p. 2576) states, “In addition to the theoretical studies, the SE equation has also been applied to diagnose the processes of TC intensification from the simulations and observations and has been shown to be able to reproduce the secondary circulation during the intensification period, even in the boundary layer, where the balanced assumption is invalid.” Our results show that substituting $B_1$ with $B_1$ in Heng et al. (2017) might result in overestimation of the lower layer inflow, which is partly counteracted by the regularization.
To assess the effect of generalized inertial stability, we substitute $I^g$ in Eq. (40) with $I^g$:

$$\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial r} + B_z \frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z} + B_z \frac{\partial \psi}{\partial r}\right) = F,$$

and $u_{g2}$ and $w_{g2}$ are the solution of Eq. (66) in which $F_k = 0$. Figures 6–l show that Eq. (66) underestimates the inflow outside the radius of maximum wind speed in the lower boundary layer in comparison with Eq. (40), which is consistent with our assertion that larger inertial stability hinders the inflow (Fig. 3e).

b. On the secondary circulation forced by tangential friction

By only considering the secondary circulation associated with tangential friction, the distributions of radial and vertical velocities forced by identical tangential friction in the balanced and unbalanced vortices are shown in Fig. 9. In the balanced vortex, the inflow layer due to tangential friction is very shallow, with weak upward motion in the eyewall region that only reaches the height of 2 km where a slight outflow layer occurs (Figs. 9a,b). Similar to that with diabatic heating, the gradient wind also only strengthens the secondary circulation in its vicinity (Figs. 9e,f). In the unbalanced vortex, the radial and vertical restoring forces can be expressed as follows:

$$F^r_{1,\text{tot}} = (-b_1 w_j + \xi F_k) \Delta t,$$

$$F^r_{2,\text{tot}} = (-b_2 u_j - N^2 w_j) \Delta t,$$

where $u_j$ and $w_j$ are the solution of Eq. (40) in which $Q = 0$. Each component of Eqs. (67) and (68) is shown in Fig. 10. In the radial direction, the inflow in the boundary layer is driven principally by the tangential friction (Fig. 10e), and it always suffers resistance due to the generalized inertial stability (Fig. 10a). The vertical motion generates the inward radial restoring force and strengthens the inflow in the boundary layer, whereas it generates only a weak outward radial restoring force in the region where the outflow layer occurs (Fig. 10c) due to the weak eyewall updraft, which implies that this outflow layer should be attributed primarily to mass continuity. The inflow is accelerated continuously until the radius of
approximately 35 km and it then decelerates rapidly (Fig. 10g), which results in the upward motion below the height of 2 km in the eyewall. The static stability always plays the role of resistance in the vertical direction (Fig. 10b), and the vertical restoring force due to radial motion is nonnegligible (Fig. 10d).

To summarize, under either diabatic heating forcing or tangential friction forcing, the gradient wind imbalance (gradient wind) strengthens the secondary circulation in its vicinity from two aspects. The first is that the negative radial gradient of the gradient wind leads to weaker inertial stability associated with $C$ and thus weaker resistance in the outer part of supergradient wind regions. The second is that the upward advection of the gradient wind component associated with $B_1$ leads to an extra inward (outward) radial restoring force in the lower (upper) part of the supergradient wind region.

c. On the gradient wind

By considering a steady-state TC and prescribing the parameterization of subgrid-scale turbulent mixing, Eq. (39) and the SE equation, Eq. (27), are closed, and the tangential wind $v$ (gradient wind $v_g$) can be determined by the prescribed gradient wind $v_g$ (tangential wind $v$), at least theoretically. We speculate that the appositions of Eq. (39) and the SE equation, Eq. (27), are able to reproduce the primary characteristic of the axisymmetric boundary layer dynamics shown in Kepert and Wang (2001) with some appropriate iterative procedures. Because the gradient wind in this study is prescribed arbitrarily (Fig. 1b), the unbalanced vortex and the prescribed
forcing (Fig. 1c) are not dynamically consistent in Eq. (39) and the SE equation, Eq. (27). Nevertheless, it is beneficial to examine the effect of the agradient wind on $u/u_r$, because the gradient wind imbalance depends on advection of the radial wind and the radial friction [Eq. (2)]. The distributions of $u/u_r$ forced by the combination of both diabatic heating and tangential friction in the balanced and unbalanced vortices are shown in Fig. 11. Near the boundary layer, the values of $u/u_r$ in the supergradient wind regions of unbalanced vortex are always positive and much larger in comparison with those in the balanced vortex (Figs. 11a,b). Specifically, $u/u_r$ in unbalanced vortex is predominantly contributed by radial advection of radial wind $u$ below the height of 1 km (Fig. 11c) and vertical advection of the radial wind $w$ at the height between 500 and 2500 m (Fig. 11e). The latter implies that removal of the vertical advection of the radial wind would reduce both the strength and the height of the supergradient wind, as validated by the noU_BL experiment in Fei et al. (2021). As stated earlier, the upward advection of the supergradient wind component associated with $B_1$ strengthens the inflow in the lower part of the supergradient wind region and low-level outflow in the upper part of supergradient wind region, and it thus contributes positively to $w$ in the supergradient wind core, consistent with the noVa_BL experiment in Fei et al. (2021). From this perspective, even in the absence of friction, some slight agradient wind might still exist owing to the existence of advection of the radial wind. The radial

Fig. 8. As in Fig. 4, but for Eq. (65).
friction contributes to the imbalance directly, whereas the tangential friction strengthens the imbalance through strengthening the advection of radial wind.

Figure 11a also exhibits some oscillations of $\ddot{u} - \ddot{r}$ (the agradient wind) in the eyewall, which might give rise to the existence of the midlevel wind speed maximum in both simulated and observed TCs (Stern et al. 2020). Stern et al. (2020) systematically investigated this inertial oscillation in the agradient wind and indicated that its vertical wavelength is crucial to the existence of the midlevel wind speed maximum. Their numerical sensitivity experiments showed that the oscillation wavelength increases with larger storm size, eyewall updraft, and diffusivity. Furthermore, they proposed an expression for the oscillation wavelength as $\lambda = 2\pi \sqrt{\frac{K_v}{I^2}}$, where $K_v$ is vertical diffusivity and $I^2$ is inertial stability. Here we attempt to reestimate the oscillation wavelength based on several assumptions. First, we consider only the region between the boundary layer inflow and the outflow layer in the upper troposphere in the eyewall, where the radial velocity is relatively small. Subsequently, to simplify the estimate, we neglect the vertical gradient of the gradient wind (following Stern et al. 2020) and diffusivity [i.e., $l_y$ in Stern et al. (2020) is small]. Finally, we assume an approximate relation between the radial velocity and the vertical velocity in our considered region based on Eq. (54):

\[ -\ddot{r} u \approx \frac{\lambda (\chi C)}{\chi} \frac{\ddot{u}}{\ddot{z}} \quad w \approx \frac{\ddot{u}}{\ddot{z}}, \tag{69} \]

where $\nu_{ag}$ represents the agradient wind. Because the agradient force $\dot{U}$ is contributed principally by $\dot{w} \dot{u}$ in this region, we have the following:

\[ \dot{U} \approx \dot{w} \frac{\partial u}{\partial z} \tag{70} \]

and

\[ \dot{U} = C - C_g \approx \xi \nu_{ag}. \tag{71} \]

Through combination of Eqs. (69)–(71), we can obtain a wave equation about $\nu_{ag}$:

\[ \frac{\partial^2 \nu_{ag}}{\partial z^2} = -\frac{\dot{l}^2}{w^2} \nu_{ag}. \tag{72} \]

Consequently, we estimate the oscillation wavelength as $\lambda = 2\pi w/I$, where $w$ is vertical velocity and $\dot{l}^2$ is inertial stability.
The oscillation wavelength under our assumptions is rather different to that proposed in Stern et al. (2020) and it corresponds to the blue points shown in Fig. 19 of Stern et al. (2020), which is a finding that deserves further verification in the future.

5. Conclusions and discussion

The secondary circulation of a TC can be considered as the result of the response to diabatic heating and momentum forcings in the primary circulation, and draws high absolute angular momentum inward to spin up the primary circulation in turn. Consequently, a function that links the secondary circulation (i.e., radial and vertical circulation) with the forcing (i.e., diabatic heating and friction) and primary vortex (i.e., tangential wind, potential temperature, and density) should be constructed. We would like this function to satisfy two conditions: one is to diagnose the secondary circulation accurately, which indicates that this function is capable of capturing the predominant physical process of TC intensification, and the other is that this function should be explicit in the physical sense. Ji and Qiao (2023) has demonstrated that the SE equation, Eq. (29), is able to reproduce the secondary circulation accurately, especially in the boundary layer. However, owing to the complexity of the SE equation, it is almost impossible to obtain an accurate analytic solution without oversimplifying this equation (Schubert and Hack 1982; Vigh...
and Schubert 2009). As an alternative, section 2a in this study provided a new derivation of the SE equation from the perspective of the restoring forces, and it shows that $A$ and $C$ in the SE equation connect with the static stability and generalized inertial stability, respectively, which always play the role of resistance when a fluid parcel is moving radially and vertically, and $B_1$ measures the radial restoring force due to the vertical advection, which modulates the strength of the flow in the boundary layer and the low-level outflow jet. Sections 4a and 4b explored the effect of the agradient wind on the secondary circulation, and showed that the agradient wind strengthens the secondary circulation in its vicinity. Subsequently, we considered the secondary circulation as a consequence of the restoring forces and mass continuity, and illustrated that it is capable of speculating the qualitative characteristics of the SE solution without solving the SE equation from the perspective of the restoring forces. Consequently, the SE equation, Eq. (27), together with its new numerical solution method proposed in section 3b can provide a quantitative diagnostic tool for understanding TC dynamics (e.g., secondary circulation and intensification rate), at least in the region near the boundary layer.

By analogy with the evolution of the balanced vortex (Smith et al. 2018), section 2b defined a balanced vortex ($v_p$, $p$, $p$, $\chi$) corresponding to the unbalanced vortex ($u$, $p$, $p$, $\chi$) and demonstrated that the maintenance of the thermal gradient wind balance (between $v_p$ and $\chi$) in the evolution of the unbalanced vortex is important in understanding the unbalanced dynamics. Similar to the balanced vortex, diabatic heating and momentum forcings in the unbalanced vortex also drive the flow away from the thermal gradient wind balance, and a secondary circulation is thus derived to oppose the effects of the forcings to maintain the vortex in the thermal gradient wind balance. Unlike the situation of the balanced
vortex, \( u_{\text{g}} \) in the unbalanced vortex is an implicit variable that is connected to \( u \) through Eq. (39), which gives rise to the agradient wind and thus the gradient wind imbalance. This reduced concept model excludes high-frequency waves and nonhydrostatic effects, and attributes the gradient wind imbalance to the tangential wind field rather than to the thermodynamic fields.

On the one hand, because the thermodynamic fields \((p, \rho, \chi)\) depend only on the distribution of \( u_{\text{g}} \) through imposition of some additional restrictions, it is possible to obtain some knowledge regarding the potential maximum gradient wind, as in Emanuel (1986). On the other hand, the maximum gradient wind is associated with the maximum tangential wind through Eq. (39), as in Bryan and Rotunno (2009). Furthermore, we speculate that the combination of Eq. (39) and SE Eq. (29) could reproduce the primary characteristics of the axisymmetric boundary layer dynamics in Kepert and Wang (2001) by prescribing the parameterization of subgrid-scale turbulent mixing. As an example, section 4c showed that the noU_BL and noVa_BL experiments in Fei et al. (2021) could also be interpreted by Eq. (39) and the SE Eq. (29). Additionally, Eq. (69), which can be viewed as an approximation of the SE Eq. (29) in the eyewall region, together with Eq. (39) constitutes a wave equation of the agradient wind in the eyewall region (section 4c).

Therefore, the extended SE equation can provide an alternative approach to investigate the boundary layer dynamics, except in terms of linear theory (Kepert 2001) and the axisymmetric TC boundary layer model (Kepert and Wang 2001).

Most theoretical studies of axisymmetric TC are based on the frameworks of thermodynamic theory (e.g., Emanuel theory) and dynamic theory (e.g., SE equation). The diabatic heating (driven by the microphysical parameterization) and the friction (driven by the surface layer parameterization and turbulence parameterization) are explicitly appeared in the SE equation \((Q \text{ and } F_x)\), whereas the air–sea fluxes are explicitly appeared in the Emanuel theory \((C_D \text{ and } C_A)\). Therefore, the thermodynamic theory can tell us the global information of TC, such as the maximum potential wind speed and size, whereas the dynamic theory can tell us the local information, such as secondary circulation and agradient wind. Due to the incompatibility between the traditional SE equation and boundary layer dynamics, this study together with Ji and Qiao (2023) attempt to develop a complete dynamic theory for TC in both qualitative and quantitative perspectives.

Acknowledgments. We thank Dr. Zhuo Wang and four anonymous reviewers for comments that helped to improve this manuscript. This work was supported by the National Natural Science Foundation of China (41821004). This work is a contribution to the UN Decade of Ocean Science for Sustainable Development (2021–30) through both the Decade Collaborative Centre on Ocean-Climate nexus and Coordination among decade implementing partners in China (DCC-OCC) and the approved Programme of the Ocean to climate Seamless Forecasting system (OSF).

Data availability statement. This work did not produce or require any input data. All numerical analyses can be reproduced using the method and parameters described in section 3a.

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