Symmetry Invariant Solutions in Atmospheric Boundary Layers

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ABSTRACT: The symmetries of the governing equations of atmospheric flows constrain the solutions. The present study applies those symmetries identified from the governing equations to the atmospheric boundary layers under relatively weak stratifications (stable and unstable). More specifically, the invariant solutions are analyzed, which conserve their forms under possible symmetry transformations of a governing equation system. The key question is whether those invariant solutions can rederive the known vertical profiles of both vertical fluxes and the means for the horizontal wind and the potential temperature. The mean profiles for the wind and the potential temperature in the surface layer predicted from the Monin–Obukhov theory can be recovered as invariant solutions. However, the consistent vertical fluxes both for the momentum and heat no longer remain constant with height, as assumed in the Monin–Obukhov theory, but linearly and parabolically change with height over the dynamic sublayer and the above, respectively, in stable conditions. The present study suggests that a deviation from the constancy, though observationally known to be weak, is a crucial part of the surface-layer dynamics to maintain its symmetry consistency.

SIGNIFICANCE STATEMENT: The atmospheric flows are governed by a differential equation system, which is often difficult to solve in any satisfactory manner, either analytically or numerically. However, without solving them explicitly, many insights can be obtained by examining the "symmetries" of the governing equations. The study suggests that basic vertical profiles of the mean state of the atmospheric boundary layer is more strongly constrained by the symmetry consistency than suggested by standard similarity theories.

KEYWORDS: Atmosphere; Turbulence; Boundary layer; Differential equations

1. Introduction

The most canonical formulation in the dynamic meteorology may be considered the quasigeostrophic system. It permits a very concise description of the large-scale flows in the atmosphere based on a conservation law of the potential vorticity. Under the quasigeostrophic approximation, the potential vorticity is defined by a linear operator of the pressure. Thus, the whole system is described by a single equation for the pressure, and all the other variables including the velocity and the temperature can be diagnosed from the pressure (cf. Holton and Hakim 2013).

However, such a simple description of the dynamics of the atmosphere becomes less feasible as we shift our focus to the smaller scales, and also closer to the surface, because the nonlinearity of the flow increases. The boundary layer dynamics may be considered such an opposite limit. In the atmospheric boundary layer, there is no known concise description of a system in terms of a simple closed set of equations, in a similar manner as the quasigeostrophy can provide for the large-scale flows, so that we can have analytical insights to the system easily.

Here, the closest option provided in the literature is the so-called turbulence closure theories (cf. Mellor and Yamada 1974): the turbulent boundary layer can be described by a series of moments of the governing equation system under truncations. With certain additional approximations, some analytical solutions are possible, and, for example, similarity solutions of the Monin–Obukhov theory (see immediately below) can be obtained (e.g., Mellor 1973; Łobocki and Porretta-Tomaszewska 2021). However, those solutions are less self-constrained than by those from the quasigeostrophic theories, due to the additional closures introduced.

A different approach is required for describing the flows in atmospheric boundary layers. The most standard is the similarity theories, which are constructed solely based on dimensional consistencies of governing variables of a given system, but without directly referring to a set of governing equations of the system. The core of the celebrated Monin–Obukhov theory (Monin and Obukhov 1954), as presented in today's literature (e.g., Wyngaard 2010, chapter 10; Sun et al. 2016), is the best-known example that is constructed from a dimensional analysis. The advantage of similarity theories is their capacity of deriving solutions of a system without actually solving the governing equation system. However, the advantage also becomes a disadvantage, because a choice of wrong dimensional parameters can lead to totally wrong conclusions. Only good physical intuitions to the problem can avoid them by choosing these dimensional parameters in a sensible manner (cf. Batchelor 1954). Here, the "intuition" is not a simple guesswork, but based on physical considerations, yet in a nondeductive manner. Certainly, those

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intuitions are derived from observations and also verified against observations and simulations. Also, the choice of dimensional parameters are preferably chosen from conserved variables, for example, the potential temperature rather than the temperature. However, from a theoretical point of view, a procedure for identifying the key dimensional parameters is hardly deductive.

For this reason, the present authors are pursuing alternative approaches for describing the atmospheric boundary layer flows. In Yano and Wacławczyk (2022), we have attempted to derive the Obukhov length by directly nondimensionalizing the governing equation system of the boundary layer.

The present paper attempts yet a different approach based on Lie-group transformations: we refer to the textbooks by Bluman and Kumei (1989), Ibragimov (1993), and Grigoriev et al. (2010) for the fundamentals. The basic premise behind this approach is the fact that once all the symmetries of the governing equation system are identified, the given system can be completely described in terms of those identified symmetries. One of the most basic concepts in this symmetry analysis are the so-called invariant solutions, which are the solutions invariant under those symmetry transformations. By adopting this basic concept, the present study asks the question of whether this symmetry-based approach can reproduce a standard description of the atmospheric boundary based on the Monin–Obukhov similarity theory. The obtained results are rather promising. However, if we accept the results at face value, they also demand a revision in the standard boundary layer theory.

The paper begins by introducing a well-established phenomenological description of the atmospheric boundary layer in the next section. The symmetry analysis based on the Lie-group theory is introduced in section 3, and applied in section 4 to derive, as invariant solutions, the well-known description of the boundary layer presented in section 2. Results are further discussed in section 5, and the paper is concluded in section 6.

2. Basic boundary layer descriptions: Basic boundary layer theories

This section presents basic vertical profiles of the variables, as expected from the standard Monin–Obukhov theory and its extension. Recall that this standard theory is based on the so-called dimensional analysis. Here, an attempt is made to reproduce it for the profiles of vertical eddy fluxes by a heuristic, phenomenological argument that provides a physical interpretation to the standard description. The presentation of this section is partially inspired by Calder (1966) and Mahrt (1998). However, the presentation is of our own. The standard description of the mean profiles follows an eddy-diffusion hypothesis. The following description is restricted to the dry atmosphere under the Boussinesq approximation for simplicity. More specifically, we focus on the vertical profiles of the horizontal velocity u and the potential temperature θ. In the following, the latter is alternatively referred to as the buoyancy, considering its role in the dynamics. The standard description presented in this section is to be formally verified by the symmetry analysis in section 4.

a. Vertical eddy fluxes

The budget equations for the horizontally averaged velocity and potential temperature in atmospheric boundary layers may be presented by

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial z}\overline{u'w'} = -\frac{\partial \phi}{\partial x}, \tag{2.1a}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial}{\partial z}\overline{\theta'w'} = Q. \tag{2.1b}$$

Here, the bar and the prime designate the horizontal average and the deviation from it; x and z are the horizontal and vertical coordinates; w the vertical velocity, ϕ the pressure divided by the density (geopotential); and Q is the source term for the potential temperature, principally due to the radiation. Here, for simplifying the presentation, the Coriolis term is dropped from the momentum Eq. (2.1a). This simplification also justifies considering only one velocity component. Under this formulation, the geopotential ϕ is, strictly speaking, considered an ageostrophic component, after subtracting a component balancing with the Coriolis force. Furthermore, both molecular viscosity and thermal diffusions are neglected, because those terms do not contribute significantly to the budget above the viscous layer, which is excluded from the present analysis.

1) Strict equilibrium: Homogeneous layer

The *theoretically* simplest description of the system (2.1a) and (2.1b) is obtained by assuming a strict equilibrium state, i.e.,

$$\frac{\partial \overline{u}}{\partial t} = 0,$$
$$\frac{\partial \overline{\theta}}{\partial t} = 0.$$

We expect that such a state is realized close to the surface: the flux components at the surface, due to eddies, the radiation, and that from Earth below, are much larger than the temporal tendency, with the surface state quickly adjusted against those surface forcings. By extrapolating this state upward, we also expect that the atmosphere close to the surface quickly adjusts to the change of the surface state. Thus, it follows that

$$\frac{\partial}{\partial z}\overline{u'w'} = -\frac{\partial \phi}{\partial x},$$
$$\frac{\partial}{\partial z}\overline{\theta'w'} = Q.$$

Furthermore, one may *heuristically* assume that the evolution of a layer immediately above the surface is primarily controlled by the vertical fluxes, which must be in comparable magnitudes as their surface values. Thus, the source terms on the right-hand sides may be dropped out in good approximations, and we obtain

$$\frac{\partial}{\partial z}\overline{u'w'} = 0, \tag{2.2a}$$

$$\frac{\partial}{\partial z}\overline{\theta'w'} = 0. \tag{2.2b}$$

Thus, the vertical eddy fluxes become constant with height (i.e., constant flux approximation; Kramm and Herbert 2009). A layer that satisfies the conditions (2.2a) and (2.2b) is called the surface layer, or more specifically, the homogeneous flux layer.

2) LESS STRICT EQUILIBRIUM: "LINEAR" LAYER

However, in moving to the upper levels, it gradually becomes difficult to maintain a state of such strict equilibrium. This is because the boundary layer is constrained by different vertical flux values from the top and the bottom. For this reason, above the surface layer, vertical fluxes can no longer be considered constant with height; hence, the state becomes less strict at equilibrium. Yet, we may still expect a certain vertical homogeneity. The simplest assumption is to set that the tendencies of the mean state values $\overline{u}$ and $\overline{\theta}$ become constant with height:

$$\frac{\partial \overline{u}}{\partial t} = \text{const.}, \tag{2.3a}$$

$$\frac{\partial \overline{\theta}}{\partial t} = \text{const.} \tag{2.3b}$$

Especially, when a given layer is initialized with a vertically homogeneous state, the layer grows by maintaining vertical homogeneity with time. That is what happens with convective situations of boundary layers, which tend to grow by maintaining vertical homogeneity of the potential temperature $\overline{\theta}$. Under nonconvective regimes, on the other hand, a stable stratification is expected to develop, although we may still expect that Eq. (2.3b) is a good approximation so long as the stratification is weak enough. It may also be remarked that in moving away from the surface layer, it becomes less justifiable to neglect the source terms on the right-hand sides of Eqs. (2.1a) and (2.1b). Nevertheless, by following the same idea, we may assume $\partial\phi/\partial x = \text{const.}$ and $Q = \text{const.}$

As a result, we obtain the state with the gradients of the vertical eddy fluxes also constant with height:

$$\frac{\partial}{\partial z}\overline{u'w'} = \text{const.},$$
$$\frac{\partial}{\partial z}\overline{\theta'w'} = \text{const.}$$

By solving the above, we obtain

$$\overline{u'w'} \simeq (\overline{u'w'})_0\left[1 + \frac{\alpha' z}{L}\right], \tag{2.4a}$$

$$\overline{\theta'w'} \simeq (\overline{\theta'w'})_0\left[1 + \frac{\beta' z}{L}\right]. \tag{2.4b}$$

Here, the subscript 0 suggests the surface values, L is a characteristic vertical scale of the system to be specified, and α' and β' are constants.

In spite of even less than a heuristic argument invoked here, those profiles appear, at least partially, to agree with some observations. Nieuwstadt [1984; see also Sorbjan (1986) for further references] shows by observation that weakly stratified boundary layer tends to follow the profiles with Eqs. (2.4a) and (2.4b) by setting $L = h$, the boundary layer height. The thermal profile (2.4b) with $\beta'/L = -(1 + \beta_T)/h$ is a prediction from basic theories of convectively well-mixed boundary layers, with empirically, $\beta_T \simeq 0.2$ (Carson 1973; Tennekes 1973). Since there is no proper name for this state, we propose to call it the "linear" layer in the following.

3) MONIN–OBUKHOV THEORY

From a point of view of the dimensional analysis, the essence of the Monin–Obukhov theory is to describe all the boundary layer variables in terms of the surface values and universal profiles, called similarity functions, say, φ, given in terms of a vertical coordinate z/L nondimensionalized by a length scale L.

Here, a length scale L more specifically adopted in the Monin–Obukhov theory is the Obukhov length [cf. Eq. (7.13) in Monin and Yaglom 1971; see also Yano and Wacławczyk 2022]. On the other hand, the present study keeps the scale L general and unspecified, because the symmetry analysis in section 4 will equally be applicable to any definition of the length scale, so long as it is taken to be constant with height. Nevertheless, we retain the standard sign convention adopted by the Obukhov, in which the length scale is set negative under the unstable stratifications so that $L < 0$. We refer to Sorbjan (2010) for a summary for possible alternative definitions for the length scales (but see also, e.g., Zilitinkevich and Calanca 2000; Zilitinkevich and Esau 2005) in context of local similarity theories. Those definitions can easily be translated in the present context of "global" similarities by replacing the local values in their definitions by surface values.

By applying these principles to the vertical eddy fluxes, considered in phenomenological manner in last two subsections, we find

$$\overline{u'w'} = (\overline{u'w'})_0\varphi'_m(z/L), \tag{2.5a}$$

$$\overline{\theta'w'} = (\overline{\theta'w'})_0\varphi'_h(z/L). \tag{2.5b}$$

Here, $\varphi'_m(z/L)$ and $\varphi'_h(z/L)$ are the nondimensionalized vertical profiles (similarity functions) for momentum and heat fluxes, respectively. Especially, the solutions, Eqs. (2.4a) and (2.4b), are recovered by setting

$$\varphi'_m(z/L) = 1 + \frac{\alpha' z}{L}, \tag{2.6a}$$

$$\varphi'_h(z/L) = 1 + \frac{\beta' z}{L}. \tag{2.6b}$$

Here, however, it is important to keep in mind that these two regimes of the boundary layer are identified by a type of asymptotic arguments: in reality, there is no layer where the vertical fluxes are constant with height in any strict sense, and the similar restriction also applies to the "linear" layer.

b. Mean profiles

Contrary to just being reviewed in the last subsection, the original paper by Monin and Obukhov (1954) focuses on defining the mean vertical profiles of the horizontal wind $\overline{u}$ and the potential temperature $\overline{\theta}$ but applying the same principle of the dimensional analysis outlined. The similarity solutions for those mean profiles are given by

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{\kappa z}\varphi_m(z/L), \tag{2.7a}$$

$$\frac{\partial \overline{\theta}}{\partial z} = \frac{\theta_*}{\kappa z}\varphi_h(z/L), \tag{2.7b}$$

in terms of the similarity functions φ_m and φ_h. Here, κ is the von Kármán constant; $u_* = (\overline{-u'w'})_0^{1/2}$ and $\theta_* = -(\overline{\theta' w'})_0/u_*$ are the scales for the velocity and the potential temperature, respectively, based on surface flux values. Here, the factor $1/\kappa z$ is applied for a historical reason.

Extensive observational data analyses (Dyer 1974; Yaglom 1977; Högström 1988; and references therein) suggest the forms

$$\varphi_m(z/L) = 1 + \frac{\alpha z}{L}, \tag{2.8a}$$

$$\varphi_h(z/L) = 1 + \frac{\beta z}{L}, \tag{2.8b}$$

when the boundary layer is only weakly stably stratified, and z/L (>0) is relatively small (say, $z/L < 1$). Coincidentally, they take the identical forms as those for the vertical eddy fluxes [cf. Eqs. (2.6a) and (2.6b)]. Consequently, in analogy with the vertical flux profiles (cf. section 2a), these mean profiles can be classified into the homogeneous and "linear" layers in the following manner.

1) HOMOGENEOUS LAYER: DYNAMICAL SUBLAYER

Over the lowest part of the surface layer with $z/L \ll 1$, called the dynamic sublayer, we may set $\varphi_m(z/L) \simeq 1$ and $\varphi_h(z/L) \simeq 1$; thus,

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{\kappa z},$$
$$\frac{\partial \overline{\theta}}{\partial z} = \frac{\theta_*}{\kappa z},$$

which lead to logarithmic vertical profiles:

$$\overline{u} = \frac{u_*}{\kappa}\log\left(\frac{z}{z_*}\right), \tag{2.9a}$$

$$\overline{\theta} = \frac{\theta_*}{\kappa}\log\left(\frac{z}{z_*}\right), \tag{2.9b}$$

where z_* is the roughness length.

2) MEAN "LINEAR" LAYER

Mean profiles over the "linear" layer, which begins in an upper part of the surface layer in standard terminologies, are obtained by setting $z/L \gg 1$ in Eqs. (2.6a) and (2.6b):

$$\frac{\partial \overline{u}}{\partial z} = \frac{\alpha u_*}{\kappa L}, \tag{2.10a}$$

$$\frac{\partial \overline{\theta}}{\partial z} = \frac{\beta \theta_*}{\kappa L}, \tag{2.10b}$$

leading to linear dependencies on height both for $\overline{u}$ and $\overline{\theta}$.

3) UNSTABLE CONDITIONS

Finally, in unstable conditions, observations suggest

$$\varphi_m = (1 - \gamma_1 z/L)^{-p}, \tag{2.11a}$$

$$\varphi_h = (1 - \gamma_2 z/L)^{-q} \tag{2.11b}$$

for the mean-profile similarity functions with positive constants, γ_1, γ_2, p, and q (cf. Högström 1996). Fitting by observation generally suggests $p = 1/4$ and $q = 1/2$ (cf. list in Högström 1988); thus, a "linear" layer is no longer identified in the limit of $-z/L \gg 1$, recalling that $L < 0$ under the sign convention following the definition of the Obukhov length.

c. Eddy-diffusion formulation

Although Monin and Obukhov (1954) focus on determining the mean profiles by a similarity theory, reading through Obukhov (1948), one realizes that their main motivation of developing a boundary layer theory for the mean profiles was to diagnose the vertical eddy fluxes from these mean profiles based on an eddy diffusion formulation. For this purpose, the mean profiles are required. Although flux values are more important than the mean profiles for predicting the actual evolution of the latter with time, it is harder to measure accurately than mean values, back then, and even today (cf. Hicks and Baldocchi 2020). We interpret that it was the main reason that Monin and Obukhov (1954) focus on mean profiles in their study.

Under an eddy-diffusion formulation, the vertical fluxes are defined by

$$\overline{u'w'} = -K_m\frac{\partial \overline{u}}{\partial z}, \tag{2.12a}$$

$$\overline{\theta' w'} = -K_h\frac{\partial \overline{\theta}}{\partial z}, \tag{2.12b}$$

where K_m and K_h are eddy-diffusion coefficients for these two vertical fluxes. By following Obukhov (1948), those coefficients may be defined by

$$K_m = \kappa u_* z\phi_m(z/L), \tag{2.13a}$$

$$K_h = \kappa u_* z\phi_h(z/L) \tag{2.13b}$$

in terms of similarity functions, ϕ_m and ϕ_h.

By substituting Eqs. (2.7a), (2.7b) and (2.13a), (2.13b) into Eqs. (2.12a), (2.12b), and comparing them with Eqs. (2.5a) and (2.5b), we find the relations between those similarity functions:

$$\varphi'_m(z/L) = \phi_m(z/L)\varphi_m(z/L), \tag{2.14a}$$

$$\varphi'_h(z/L) = \phi_h(z/L)\varphi_h(z/L). \tag{2.14b}$$

Obukhov (1948) notes that in the limit of weak stratifications, with $z/L \ll 1$, it is expected to be $\phi_m(z/L) \simeq \phi_h(z/L) \simeq 1$, being consistent with the mixing-length theory originally proposed by Prandtl (1932); thus, two sets of similarity functions for vertical profiles become identical, i.e., $\varphi'_m(z/L) \simeq \varphi_m(z/L)$ and $\varphi'_h(z/L) \simeq \varphi_h(z/L)$. Otherwise, the two sets of similarity functions differ by factors of $\phi_m(z/L)$ and $\phi_h(z/L)$.

Over the surface layer, by definition, the vertical fluxes are approximately constant with height, as assumed by Monin and Obukhov (1954); thus, $\varphi'_m(z/L) \simeq 1$ and $\varphi'_h(z/L) \simeq 1$. It also follows that $\phi_m(z/L) \simeq 1/\varphi_m(z/L)$ and $\phi_h(z/L) \simeq 1/\varphi_h(z/L)$. It also means that the "linear" layer for the vertical fluxes will be found above the surface layer, whereas the same corresponds to a layer above the dynamic sublayer for the mean profiles. This distinction becomes important in our attempt of reproducing those features as invariant solutions in section 4.

3. Symmetry analysis: Methodology

Symmetries of a given set of differential equations are inferred by identifying the possible transformations that conserve the form of the differential equations. This approach is mathematically based on the fact that these transformations constitute a so-called Lie group. However, the knowledge of the Lie group does not directly concern us, when we only focus on the applications, as in the present case. Readers are referred to the textbooks by Bluman and Kumei (1989), Ibragimov (1993), and Grigoriev et al. (2010) for the fundamentals.

a. Symmetries

The symmetries of the governing equations constrain the solutions: there is no way for the solutions to escape any constraints arising from the given equations.

As a simple example, take the Galilean invariance: if a given governing equation is conserved under a Galilean transform, any solution of this equation can lead to another solution of a given governing equation by any Galilean transformations. If a *proposed solution* does not satisfy this "constraint," i.e., if it does not lead to another solution after a Galilean transform, it is *actually not* a solution of this equation. In this manner, solutions of a given equation are constrained by symmetries of the given equation.

b. Basic mathematical formulation

For any fluid dynamic systems, the symmetries to be considered include stretching of space and time, as well as translation of the independent variables. For a horizontal component of the velocity u, those possible stretching transformations can be stated as

$$z^* = \lambda z, \tag{3.1a}$$

$$t^* = \lambda' t, \tag{3.1b}$$

$$u^* = \frac{\lambda}{\lambda'} u, \tag{3.1c}$$

where the transformed variables are indicated by the superscript asterisk; λ and λ' are stretching rates of the height z and the time t.

Here, it may be worthwhile to note that Monin and Obukhov (1954) invoke the spatial stretching, (3.1a), as well as additional translation symmetry of the mean velocity $\overline{u}^* = \overline{u} + u_0$ for deriving the log profile for the mean wind, (2.9a). Although those authors do not address explicitly the Lie symmetries of the underlying equations, they use the related concept of an "invariant," which is a quantity which remains unchanged after certain transformation of variables.

As it turns out, in general, it is useful to rewrite the above transformation in a differential form: consider a limit of small transformation rate ε so that we can develop Taylor series of the transformations up to the terms of order ε. Thus, a general form of transformations read to the leading order

$$z^* = z + \varepsilon\xi_z, \tag{3.2a}$$

$$t^* = t + \varepsilon\xi_t, \tag{3.2b}$$

$$u^* = u + \varepsilon\xi_u, \tag{3.2c}$$

where

$$\xi_z(u, z, t) = \left.\frac{dz^*}{d\varepsilon}\right|_{\varepsilon=0}, \quad \xi_t(u, z, t) = \left.\frac{dt^*}{d\varepsilon}\right|_{\varepsilon=0},$$
$$\xi_u(u, z, t) = \left.\frac{du^*}{d\varepsilon}\right|_{\varepsilon=0} \tag{3.3}$$

are the infinitesimal transformation rates. By taking the limit of $\varepsilon \to 0$, the relations (3.2a), (3.2b), and (3.2c) reduce to a differential form:

$$\frac{dz}{\xi_z} = \frac{dt}{\xi_t} = \frac{du}{\xi_u}, \tag{3.4}$$

where dz, dt, and du designate infinitesimal additives to the transformation. For example, the transformation rules (3.1a), (3.1b), and (3.1c) are rewritten into differential form by setting $\lambda = e^{\varepsilon a_z}$, $\lambda' = e^{\varepsilon a_t}$, and $u_0 = e^{\varepsilon a_u}$ with a_z, a_t, and a_u as constants:

$$\frac{dz}{a_z z} = \frac{dt}{a_t t} = \frac{du}{(a_z - a_t)u},$$

where $a_u = a_z - a_t$.

c. Invariant transformations

A complete set of invariant transformations can be identified in the following manner. We first substitute the infinitesimal transformations given in forms, e.g., Eqs. (3.2a), (3.2b), and (3.2c), into a given equation system. Let us describe the given system, symbolically, as

$$F(z, t, u, u_t, u_z, \ldots) = 0, \tag{3.5}$$

where F is a function of the arguments partially indicated inside the parenthesis. By substituting the given infinitesimal transformations into Eq. (3.5), we obtain a form

$$F(z, t, u, u_t, u_z, \ldots) + \varepsilon\,\delta F = 0, \tag{3.6}$$

in which the second term represents a modification of the equations due to the infinitesimal transformation, which must vanish under an invariant transformation, i.e., $\delta F = 0$ at $F = 0$. After relatively lengthy reductions (cf. section 1.3 of Grigoriev et al. 2010), this condition is found to take a form

$$\zeta_0 + \zeta_z \frac{\partial F}{\partial z} + \zeta_t \frac{\partial F}{\partial t} + \zeta_u \frac{\partial F}{\partial u} + \cdots = 0, \tag{3.7}$$

where ζ with varying subscripts are functions of the same variables as for F. By solving Eq. (3.7) above, treating the arguments in the function F as independent variables, we can obtain the transform coefficients, ξ_z, ξ_t, ξ_u, ... in Eq. (3.4). The procedure is straightforward in principle, although it may be tedious in practice.

d. *Invariant solutions*

1) Introduction

The invariant solutions are those solutions that satisfy invariant conditions (or some of them) by themselves. Take the Galilean invariance as an example: a Galilean-invariant solution is one that satisfies the Galilean invariance by itself. That would be a spatially homogeneous, time-independent solution, in short, a constant solution. Although it may look trivial by itself, a physical intuition suggests that such a constant solution is important to define a basic state of a given system, and in that manner, it has its own values.

Importantly, we show in the next section that the invariant solutions are often much less trivial, worthwhile more attention. Applying the same interpretation, we expect that those invariant solutions, in general, also constitute basic solutions of given systems. Since the domain-averaged profiles of the mean state and the vertical fluxes are such basic features in the atmospheric boundary layers, it is also reasonable to expect that those may be described as invariant solutions of the atmospheric-dynamic system. The next section pursues this possibility. This anticipation is supported by the fact that Oberlack and Rosteck (2010) derived the logarithmic wind profile for the neutral boundary layer as an invariant solution. Importance of the invariant solutions is also widely recognized in the literature (e.g., Bluman and Kumei 1989, especially sections 4.2–4.4).

2) Local nature of the analysis

A major limitation of the invariance analysis performed in the present study is that those invariant rules are identified solely from a given set of differential equations, without explicitly taking into account the boundary conditions of a system. It also follows that the obtained invariant solutions do not necessarily satisfy physically sensible boundary conditions of a given system.

For this reason, in the present study, we adopt an interpretation that those invariant solutions are only locally valid. Here, as a common practice in atmospheric science, however, we do not define the *locality* in any strict sense by an analytical limiting procedure. Rather, this concept remains a very loose notion that is defined only in an asymptotic sense: the important element of the asymptotic methods is that although we introduce a small parameter for a purpose of the asymptotic expansion, this parameter is not necessarily small numerically (cf. section 5.2 of Yano 2015).

It is fairly common to apply these only locally valid solutions rather globally. A notable example is Roundy and Janiga (2011), in which vertically propagating wave solutions are adopted to explain equatorial waves observed over a full depth of the free troposphere. Of course, these solutions do not satisfy the surface boundary condition of the waves, although the study shows a good consistency of these solutions with observations.

3) Mathematical statements

Mathematically speaking, the invariant solutions are those $u(z, t)$ of the given system (3.5) which remain unchanged when written in new variables u^*, z^*, and t^* determined by Eqs. (3.2a), (3.2b), and (3.2c). In general, to obtain them we need to solve the invariance condition (3.4).

In the following, we will focus on defining the height dependence of the system, because that is also an exclusive focus on the similarity theories reviewed in the last section. In this case, the problem reduces to that of defining the height dependencies that satisfy the invariant transformations in the z direction. We will leave the remaining dependencies of the system implicit, also for considering the impracticality of deriving such full solutions analytically.

Slightly more detailed discussions are provided in the following subsubsections for those reader who wish to follow these principle slightly better and refer to the aforementioned textbooks for the full details.

4) Derivation of the invariant solutions

As outlined in the last subsection, the invariant transformation rule is defined in terms of a differential form given by Eq. (3.4) by solving Eq. (3.7). Once the infinitesimal transformations are given, in turn, we can obtain the finite transformation rules in an equivalent form as given by, e.g., (3.1a), (3.1b), and (3.1c), by solving the Lie's differential equations

$$\xi_z(u^*, z^*, t^*) = \frac{dz^*}{d\varepsilon}, \quad \xi_t(u^*, z^*, t^*) = \frac{dt^*}{d\varepsilon},$$
$$\xi_u(u^*, z^*, t^*) = \frac{du^*}{d\varepsilon}, \tag{3.8}$$

with the conditions $u^* = u$, $z^* = z$, and $t^* = t$ at $\varepsilon = 0$. Thus, the infinitesimal transformation rules are equivalent to their finite counterparts.

As an intermediate step for obtaining the invariant solutions, the constraints between the variables in concern, z, t, and u, are first obtained by directly integrating Eq. (3.4). These (or some of those) constraints can be directly substituted into the differential equation, Eq. (3.5), so that solutions to this equation that are invariant under the transformations can be obtained.

5) Height dependence of the invariant solutions

When we focus on determining the height dependence of a system, by these first integrals of Eq. (3.4), as just described in section 3d(3), all the dependent variables, such as u, are constrained as a function of z by the invariant condition.

These invariant constraints becomes a solution set of a system by themselves, because their substitution into the original differential equation system leaves in the latter only an algebraic dependence on z. Moreover, by satisfying the invariance of the equation system after substitution of those relations, the resulting z dependence constitutes a common algebraic factor in the governing equation, and it is only left to define the dependence on the other independent variables (time and horizontal coordinates in the next section). Thus, a set of these first invariant integrals automatically constitute a solution of the original system in the one-variable case.

e. PDE and statistical symmetries

In the next section, we are going to examine the symmetries of a governing equation system for the atmospheric boundary layers, to which we adopt the Boussinesq approximation. The symmetries of this system is examined by following the procedure outlined in section 3d by adopting a system with the Boussinesq approximation as the system (3.5) to be examined.

We are going to designate the symmetries obtained in this manner the PDE symmetries, because we additionally consider the so-called statistical symmetries. The PDE and statistical symmetries are distinguished in the following manner: the PDE symmetries directly concern the symmetries (invariant transformations) of a given PDE system that governs a physical system. On the other hand, the statistical symmetries concern those associated with a certain set of statistical equations derived from an original set of PDEs, say, by ensemble averaging. By definition, the derived transformations keep the given statistical equations invariant.

However, the original PDE system is not necessarily (and usually not) conserved by these statistical transformations. Nevertheless, we expect that these invariant transforms still have certain relevance in the original PDE system, because those statistical equations represent statistics of the original system. Thus, it is reasonable to expect that the original system is *statistically* constrained by those statistical symmetries. It also follows that the invariant solutions derived under the statistical symmetries should have certain physical relevance. This interpretation is supported by the fact that the logarithmic velocity profile observed in the neutral boundary layer is derived as an invariant solution under statistical symmetry, as shown by Oberlack and Rosteck (2010).

However, we do not believe that all the statistical symmetries are equally relevant to the original PDE system, but the relevance varies from one given symmetry to another. For this reason, more specifically, we exclude the statistical freedom of adding arbitrary functions from the considerations of the present study, with more specific reasons explained in the appendix section c.

4. Symmetry analysis of the boundary layer

a. Governing equation system: Boussinesq approximation

The methodology outlined in the last sections has been extensively applied to the fluid mechanics systems, but without stratification, as reviewed in Oberlack et al. (2015): see especially their section 6. In the present paper, their analysis is generalized by including the stratification presented by the buoyancy to the system under the Boussinesq approximation. Although this extension is relatively straightforward, we find qualitatively different consequences as will be seen below. Note that the Coriolis force is still to be included in the analysis, but we believe that its contribution can partially be taken into account by defining the pressure to be an ageostrophic component.

In interpreting the invariant solutions under stratifications presented in the following, it may be useful to keep in mind that the system with stratifications also include the neutral case as its subset. It also follows that the invariant solutions obtained for the neutral case are also included in those under stratifications as special case with a zero-buoyancy solution. However, we will exclude such a trivial solution in the following analysis.

Some readers may feel it rather misleading to call a system with a Boussinesq approximation to be "stratified." We should emphasize a special feature of the present analysis, in which a fully nonlinear buoyancy equation is examined without explicitly specifying the background stratification. Rather, the background stratification (i.e., the mean profile of the potential temperature $\overline{\theta}$) is going to be determined as an invariant solution.

b. General symmetry rules

Here, we focus on the height dependences of the velocity and the buoyancy (potential temperature), although the invariant transformation rules, (4.1) and (4.2), presented below are derived by identifying all the transformations including the time and the horizontal coordinates. Details of the derivation are provided in the appendix. With direct applications of the result to the atmospheric boundary layer in mind, we present the symmetry of the system in terms of a truncated moment expansion of variables, namely, the horizontal averages of the wind and the potential temperature, and their total vertical fluxes. Note that considerations of higher-order moments do not alter the symmetry relations of those variables. Note further that, in the following, turbulence statistics is assumed depending only weakly on time.

The identified transformation rules are

$$\frac{dz}{a_z z + a_0} = \frac{d\overline{u}}{(a_z - a_t + a_s)\overline{u} + a_1} = \frac{d\overline{uw}}{[2(a_z - a_t) + a_s]\overline{uw} + a_2} = \frac{d\overline{\theta}}{(a_\theta + a_s)\overline{\theta} + a_3} = \frac{d\overline{\theta w}}{(a_z - a_t + a_\theta + a_s)\overline{\theta w} + a_4}. \tag{4.1}$$

Here, a with various subscripts designate constants for symmetric transformations. Note that the parameters a_s a_1, a_2, a_3, and a_4 represent a statistical symmetries originally identified by Oberlack and Rosteck (2010). Wacławczyk et al. (2014) *interpret* that the parameter a_s measures a degree of the intermittency of a given system, although this interpretation is under debate (Frewer et al. 2015; Wacławczyk and Oberlack 2015).

Note that for an ease of seeking symmetric transformations, the vertical fluxes $\overline{uw}$ and $\overline{\theta w}$ are defined in terms of the full variables, rather than those of the eddy components as usual. However, when the vertical velocity vanishes in average, i.e., $\overline{w} = 0$, we can reset them as $\overline{uw} = \overline{u'w'}$ and $\overline{\theta w} = \overline{\theta' w'}$, where the prime suggests a deviation from a horizontal average.

Here, the infinitesimal transformation parameters ξ with varying subscripts, explicitly presented in Eq. (4.1), represent a clear pattern of a change from one variable to another. Consider the derivation in the appendix carefully to trace the origin of the patterns. The conditions for the invariant solutions to be derived in the following reflect the given pattern in the invariant rules, (4.1).

Furthermore, when the potential temperature acts in the momentum equation as a buoyancy force, the above transformation relations (4.1) are further constrained by

$$a_z - 2a_t = a_\theta \tag{4.2}$$

[cf. appendix section a(2)].

Our main goal in the following is to identify the boundary layer states outlined in section 2 as particular invariant solutions. Oberlack and Rosteck (2010) have already shown that the logarithmic velocity profile expected for the dynamic sublayer can be obtained as an invariance solution for the neutral case. Thus, the natural question to ask is whether the other profiles of the boundary layer can also be obtained as invariant solutions when the stratification is further included. That is the key question addressed in the following.

The analysis is separated into two major parts: when the potential temperature (buoyancy) is treated purely as a passive scalar and when the potential temperature acts in the momentum equation as a buoyancy force. In the former case (sections 4c and 4d), the transformation relations (4.1) can be considered in standalone manner. In the latter case (sections 4e and 4f), the above transformation relations are further constrained by Eq. (4.2). For each case, by following the order of the presentation in section 2, the vertical fluxes are considered first (sections 4c and 4e), and the mean profiles the next (sections 4d and 4f), then consistencies with standard theories and observations are discussed. The presentation will be focused on the solutions in stable conditions. Results in unstable conditions are briefly summarized in the last subsection (section 4g) without repeating the same details.

In the following, the analysis of each subsection is further divided into two parts, in which the homogeneous and linear layers are considered separately. The separate consideration is along the line of the interpretation of the invariant solutions presented as local solutions in section 3d(2). Note that in this manner, the present analysis covers the full planetary boundary layers.

c. *Vertical eddy flux profiles*

The general relation between z and $\overline{uw}$ can be rewritten into the form

$$\frac{d\overline{uw}}{\overline{uw} - (\overline{uw})_0} = \mu_u \frac{dz}{z + z_0}$$

by setting $z_0 = a_0/a_z$, $\overline{uw}_0 = -a_2/a_z\mu_u$, and $\mu_u = [2(a_z - a_t) + a_s]/a_z$. By solving the above, we find

$$\overline{uw} = \overline{uw}^*(z + z_0)^{\mu_u} + (\overline{uw})_0, \tag{4.3a}$$

where $\overline{uw}^*$ is an integral constant.

Similarly, the vertical buoyancy flux is generally given by

$$\overline{\theta w} = \overline{\theta w}^*(z + z_0)^{\mu_\theta} + (\overline{\theta w})_0, \tag{4.3b}$$

where $(\overline{\theta w})_0 = -a_4/a_z\mu_\theta$, $\mu_\theta = [a_z - a_t + a_\theta + a_s]/a_z$, and $\overline{\theta w}^*$ is an integral constant.

1) Homogeneous layer

To obtain a vertically homogeneous state for the vertical momentum flux, we need to set for the vertical momentum flux in (4.3a)

$$\mu_u = 0 \quad \text{or} \quad 2a_z - 2a_t + a_s = 0. \tag{4.4a}$$

However, if a_2 remains finite as $\mu_u \to 0$, the solution actually asymptotically approaches to a logarithmic profile. To avoid this consequence, thus we also need to set $a_2 = 0$, i.e., $(\overline{uw})_0 = 0$ in Eq. (4.3a).

A similar argument also follows for the vertical buoyancy flux in (4.3b):

$$\mu_\theta = 0 \quad \text{or} \quad a_z - a_t + a_\theta + a_s = 0, \tag{4.4b}$$

as well as $a_4 = 0$, which implies $(\overline{uw})_0 = 0$ in Eq. (4.3b). From conditions (4.4a) and (4.4b), it further follows that

$$a_z = a_t + a_\theta. \tag{4.4c}$$

2) "Linear" layer

A linear dependence of the vertical flux with height is obtained by setting $\mu_u = 1$ and $\mu_\theta = 1$ or

$$a_z - 2a_t + a_s = 0, \tag{4.5a}$$

$$-a_t + a_\theta + a_s = 0, \tag{4.5b}$$

respectively, for the momentum and the buoyancy.

d. *Mean profiles*

Transformation rules for the means in Eq. (4.1) can be rewritten as

$$\mu_1 \frac{dz}{z+z_0} = \frac{d\overline{u}}{\overline{u}-u_0}, \quad (4.6a)$$

$$\mu_2 \frac{dz}{z+z_0} = \frac{d\overline{\theta}}{\overline{\theta}-\theta_0}, \quad (4.6b)$$

where

$$\mu_1 = (a_z - a_t + a_s)/a_z, \quad (4.7a)$$

$$\mu_2 = (a_\theta + a_s)/a_z, \quad (4.7b)$$

$$u_0 = -a_1/(a_z - a_t + a_s), \quad (4.7c)$$

$$\theta_0 = -a_3/(a_\theta + a_s). \quad (4.7d)$$

1) $\overline{u}$-logarithmic profile

For the mean velocity to follow a logarithmic profile as expected over the dynamic sublayer, it is required to assume

$$\mu_1 = 0 \quad \text{or} \quad a_z - a_t + a_s = 0, \quad (4.8)$$

but keeping $\mu_1 u_0$ nonvanishing; thus, $a_1 \neq 0$. As a result, the vertical wind profile is defined from the differential relation,

$$d\overline{u} = u^* \frac{dz}{z+z_0},$$

with $u^* = a_1/a_z$, which leads to

$$\overline{u} = u^* \log(z+z_0) + \overline{u}_0,$$

where $\overline{u}_0$ is an integral constant. Here, note that a logarithmic profile is a special limit of a homogeneous profile with an additional constraint of $a_1 \neq 0$.

2) $\overline{\theta}$-logarithmic profile

Similarly, the condition for obtaining a logarithmic profile for the potential temperature is

$$\mu_2 = 0 \quad \text{or} \quad a_\theta + a_s = 0. \quad (4.9)$$

The resulting differential relation is

$$d\overline{\theta} = \theta^* \frac{dz}{z+z_0},$$

with $\theta^* = a_3/a_z$, which leads to

$$\overline{\theta} = \theta^* \log(z+z_0) + \overline{\theta}_0,$$

with $\overline{\theta}_0$ an integral constant.

An important requirement to satisfy both the constant vertical flux and the logarithmic mean profiles simultaneously is that the statistical symmetry must vanish, i.e., $a_s = 0$ from conditions (4.4a) and (4.8). It further follows from the condition (4.9) that $a_\theta = 0$. Under these conditions, the power exponents defining the vertical fluxes in Eqs. (4.3a) and (4.3b) reduce to

$$\mu_u/2 = \mu_\theta = 1 - a_t/a_z. \quad (4.10)$$

Thus, the constant vertical fluxes with height are obtained by further setting $a_t = a_z$.

3) Linear velocity regime

A linear velocity, as expected over the mean "linear" layer, is obtained from a differential form

$$\frac{dz}{z+z_0} = \frac{d\overline{u}}{\overline{u}-u_0},$$

with u_0 defined by Eq. (4.7c). To obtain this particular invariant relation from the general invariant conditions in (4.1), we need to assume

$$\mu_1 = 1 \quad \text{or} \quad a_t = a_s. \quad (4.11)$$

Alternatively, a linear solution can be obtained from

$$\frac{dz}{a_0} = \frac{d\overline{u}}{a_1}$$

by assuming less asymmetry constraints with $a_z = a_t = a_s = 0$.

4) Linear potential temperature regime

A similar consideration suggests that to obtain a linear profile for the potential temperature is

$$\mu_2 = 1 \quad \text{or} \quad a_\theta = a_z - a_s. \quad (4.12)$$

Under the conditions of (4.11) and (4.12), the power exponents defining the vertical fluxes in Eqs. (4.3a) and (4.3b) reduce to

$$\mu_u = \mu_\theta = 2 - a_s/a_z. \quad (4.13)$$

This condition becomes more comparable to Eq. (4.10) by substituting the additional condition, $a_s = a_t$. However, also keep in mind that condition (4.10) for the logarithmic profile is associated with a different additional constraint, $a_s = 0$.

Now we seek for consistency conditions for homogeneous and "linear" layers of vertical fluxes with the linear mean profiles in the following two subsubsections. Keep in mind that in the dynamic sublayer, in which logarithmic mean profiles are found, the vertical fluxes are expected to be homogeneous with height, as already reviewed in section 2.

5) Homogeneous vertical flux layer

By substituting condition (4.13) into (4.4a) and (4.4b), we find

$$a_s = 2a_z, \quad a_t = a_s, \quad a_\theta = -a_z.$$

Thus, a consistent invariant solution is available for this layer.

6) "Linear" vertical flux layer

By substituting condition (4.13) into (4.5a) and (4.5b), we obtain $a_s = a_z$ as a condition to be $\mu_u = \mu_\theta = 1$. Here, importantly, nonvanishing statistical symmetry a_s becomes a key condition to obtain a nontrivial solution in the surface layer.

e. When the potential temperature acts upon the dynamics as buoyancy force: Vertical fluxes

We now add, to the previous results, the constraint (4.2) to see the consequences of considering an active role of the buoyancy in the dynamics. We first apply this constraint in this subsection to the vertical flux profiles and examine the consequences to the mean profiles. In the next subsubsection, we conversely consider the consequences to the vertical flux profiles when the constraint (4.2) is applied to the mean profiles. Please note that the deductions proceed in a parallel manner exactly as in the last two subsections; thus, the following presentation will be more succinct.

1) Constant flux profiles

When constant flux profiles are assumed, as expected for a surface layer, the system is further constrained by a relation, $a_\theta = a_z - a_t$ from Eq. (4.4c). By combining it with the condition (4.2), we find $a_t = 0$, leading to $a_\theta = a_z$. By substituting those results into Eqs. (4.4a) and (4.4b), we further conclude $\mu_u = \mu_\theta = (2a_z + a_s)/a_z = 0$, or $a_s = -2a_z$. By substituting those results into Eqs. (4.7a) and (4.7b), the mean profiles are found to take the forms

$$\overline{u} = \frac{u^*}{z + z_0} + u_0,$$

$$\overline{\theta} = \frac{\theta^*}{z + z_0} + \theta_0.$$

Thus, we have to conclude that constant fluxes and logarithmic mean profiles are not mutually compatible as invariant solutions. It further suggests that either of the assumed profiles is not literally a solution. Here, we may note that the above profiles are qualitatively close to the logarithmic profiles by sharing a basic property of the singularity toward $z \to -z_0$. Nevertheless, the quantitative difference is drastic enough to accept it as a feasible alternative profile. It follows that the constant flux condition must be abandoned in the dynamic sublayer to be consistent with symmetry constraints of the system. As an exception, when we further assume $a_z = 0$, the means follow linear profiles. However, realize that this is rather a special solution, in which the vertical coordinate only satisfy the translation invariances.

2) Linear profiles

Recall that the vertical fluxes of the linear profile layer are constrained by Eqs. (4.5a) and (4.5b). Using the constraint (4.2) to eliminate a_θ from Eq. (4.5b), these two conditions lead to $a_t = 0$ and $a_z + a_s = 0$. Substitution of those results into Eqs. (4.7a) and (4.7b), we further conclude $\mu_1 = \mu_2 = 0$, or $a_s = -2a_z$. Thus, the mean profiles must become constant with height for consistency, implicitly assuming $a_1 = a_3 = 0$. In a convectively well-mixed boundary layer, the potential temperature is expected to be well mixed into a state constant with height, and also with a less rigorous argument, the mean momentum profile may also be expected to be vertically well mixed. Thus, the obtained result appears to be reasonably consistent with unstable conditions about the surface layer. However, less is obvious for the stable conditions.

f. When the potential temperature acts upon the dynamics as buoyancy force: Mean profiles

1) Logarithmic profiles

To obtain the logarithmic profiles for $\overline{u}$ and $\overline{\theta}$, we need to assume Eqs. (4.8) and (4.9) as before, in addition to Eq. (4.2). We find $a_\theta = -a_s$ from the constraint (4.9). Its substitution into (4.2) yields

$$a_z - 2a_t + a_s = 0.$$

Comparing it with (4.8), we conclude $a_t = 0$ and $a_z + a_s = 0$. We consequently find $\mu_u = \mu_\theta = 1$. Thus, vertical fluxes follow linear profiles in the dynamic sublayer, although their slopes may be very weak.

2) Linear profiles

To obtain the linear profiles for $\overline{u}$ and $\overline{\theta}$, we need to assume Eqs. (4.11) and (4.12) as before. Their substitution into (4.2) leads to $a_s = 0$. As a result, the power exponent for the vertical fluxes becomes $\mu_u = \mu_\theta = 2$, and they follow parabolic profiles, although the associated curvatures may be small enough to justify constant approximations to some extent.

g. Unstable conditions

In unstable conditions, to be consistent with the observationally diagnosed similarity functions (2.11a) and (2.11b) with the limits of $-\gamma_1 z/L \gg 1$ and $-\gamma_2 z/L \gg 1$, the invariant transformations must satisfy the conditions

$$\mu_1 = \frac{a_z - a_t + a_s}{a_z} = -p, \tag{4.14a}$$

$$\mu_2 = \frac{a_\theta + a_s}{a_z} = -q, \tag{4.14b}$$

along with the buoyancy constraint (4.2). By solving for those three constraints, we find

$$a_s = -(1 + 2p - q)a_z, \tag{4.15a}$$

$$a_t = (-p + q)a_z, \tag{4.15b}$$

$$a_\theta = [1 + 2(p - q)]a_z, \tag{4.15c}$$

and henceforth, the exponents for the momentum and heat fluxes are estimated as

$$\mu_u = 1 - q, \tag{4.16a}$$

$$\mu_\theta = 1 + p - q. \tag{4.16b}$$

Here, the commonly accepted values are $p = 1/4$ and $q = 1/2$. By substituting those values, we find $a_s = -a_z$, $a_t = a_z/4$,

$a_\theta = a_z/2$, $\mu_u = 1/2$, and $\mu_\theta = 1/4$ from Eqs. (4.15a), (4.15b), and (4.15c) and Eqs. (4.16a) and (4.16b). It is seen that both vertical fluxes modulate with height with fractional exponents to be consistent with the observed mean profiles with given symmetry constraints.

5. Discussion

a. Summary of the obtained results

Attempt has been made to rederive the vertical profiles of the means and vertical fluxes in the atmospheric boundary layer, as expected from the Monin–Obukhov theory and its extension to the vertical fluxes, as invariant solutions under the symmetry analysis of the governing equation system. Here, an extension is made partially based on a phenomenological argument. The present study focuses only on the two variables: the horizontal wind and the potential temperature by taking a dry approximation. The latter is also considered the buoyancy in the momentum equation.

Two distinguished layers are phenomenologically identified: (i) the homogeneous layer, over which those vertical profiles are either constants or logarithmic with height, and (ii) the "linear" layer, over which vertical profiles change linearly with height. Here, recall that a logarithmic profile is a special limit of a homogeneous profile with an additional constraint, as already remarked in section 4d. For the vertical fluxes, the first layer is expected to correspond to the surface layer, whereas the second is an extrapolation into the planetary boundary layer. With the mean profiles, the homogeneous layer is commonly referred to as the dynamic sublayer, whereas the "linear" layer is identified over the surface layer in stable conditions above the dynamic sublayer. Vertical profiles in unstable conditions have also been examined separately.

The symmetry analysis reported herein is a natural extension of the previous studies for nonstratified flows as summarized by Oberlack et al. (2015), who also show that a logarithmic profile of the mean wind at a surface layer can be established as an invariant solution for such flows. A key question of the present study has been how these previous results with nonstratified flows are modified by adding a stratification (i.e., active role of the buoyancy) to the system. As a natural extension of the previous studies, we have assumed a Boussinesq approximation to the fluid. The buoyancy (potential temperature) is added to the system by taking two steps (cf. appendix section a): first, simply as a passive scalar, which leads to symmetry relations (4.1). Second, when an active role of buoyancy in the momentum equation is further taken into account, parameters, a_z, a_t, and a_θ, for those symmetric transformations are further constrained by Eq. (4.2).

It has been shown that when the buoyancy is considered only as a passive scalar, the standard description for these two distinguished layers can be recovered as invariant solutions, both for the vertical fluxes and the mean profiles, when transformation parameters are appropriately chosen. On the other hand, when an additional constraint, (4.2), due to an active role of the buoyancy in the dynamics is also taken into account, the standard descriptions are no longer simultaneously recovered both for the vertical fluxes and the mean profiles. Only either of them can satisfy the standard description, and different vertical profiles must be assumed for the other. The identified inconsistency must somehow be reconciled if the invariant solutions remain relevant to this system.

Note in general, when one of the variables of a system satisfies an invariant solution, the other variables of the system must also belong to the same family of invariant solutions, which satisfy the same transformation constraints, as given by Eqs. (4.1) and (4.2) in the present study. We expect that experimental data in the atmospheric boundary layer are consistent with these mathematical rules at least up to the second-order moments, to which the present study focuses, as a tendency of the system (cf. section 3e). However, the higher-order moments increasingly deviate from the invariant solutions. A known example is the deviation from the Kolmogorov's scaling observed for higher-order structure functions (cf. chapter 8 of Frisch 1995).

Over the surface layer, when constant vertical fluxes are assumed, the mean profiles for both the horizontal wind and the buoyancy approaches infinity over the surface layer with a rate inversely proportional to the effective height, $z + z_0$, as $z \to -z_0$. A qualitative similarity between this profile and the standard logarithmic profiles may be noted. Nevertheless, since the logarithmic profile for the mean wind is so well established, it appears to be hard enough to justify this alternative.

On the other hand, when we assume standard logarithmic profiles for the means, the profiles of vertical fluxes become linear with height. This qualitative change is likely to be less significant, so long as the resulting linear slopes are much weaker than reference constant values. However, this weak linear dependence of vertical fluxes is expected to be a crucial part of the surface-layer dynamics. A similar argument follows over the "linear" layer for the mean profiles, above the dynamic sublayer, the vertical fluxes must follow parabolic profiles for symmetry consistency. Similarly, by symmetry consistency, observationally identified mean profiles lead to modifications of vertical fluxes to fractional powers of height in unstable conditions.

b. Observational evidence

As the summary above suggests, the main conclusion from the present analysis is that a slight but a finite departure of the vertical fluxes from vertical homogeneity is a crucial feature of the surface-layer dynamics, for maintaining the symmetry consistency of the system. Thus, an important open question is whether there is any observational evidence for supporting this theoretical prediction. It is hard to judge in any definite manner from the existing observational data analyses found in the literature, partially due to the fact the accurate measurements of vertical fluxes remain difficult, even today. The consensus in the literature is that vertical fluxes must be constant with height within the errors of less than 20% (e.g., Haugen et al. 1971), or even less, over the surface layer. However, it could be worthwhile to note that Haugen et al. (1971), for example, identify a rather significant change with height in their

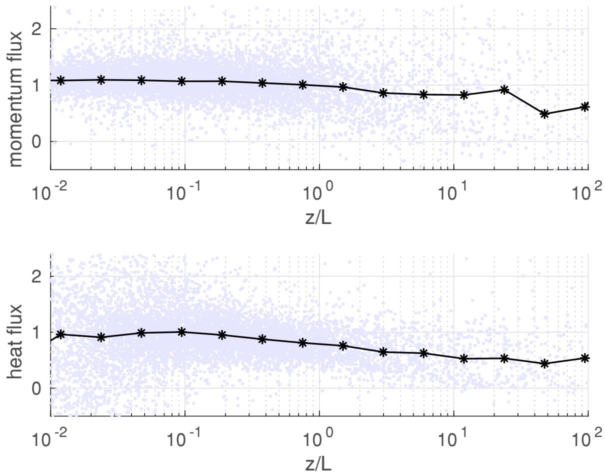

FIG. 1. Scatterplots of (a) momentum and (b) heat fluxes normalized by surface values ($\overline{u'w'}/\overline{u'w'}_0$ and $\overline{\theta'w'}/\overline{\theta'w'}_0$, respectively) against the normalized vertical coordinate z/L, superposed by a curve for bin averages of measurements.

analysis within their data accuracy: the momentum flux appears to decrease from the surface to 5-m height by about 20% (their Fig. 9), and the ratio of the heat flux between 5- and 22-m heights represent a significant Richardson number dependence (their Fig. 10).

Here, a preliminary analysis has been performed by adopting the data from the Surface Heat Budget of the Arctic Ocean (SHEBA) experiment (Persson et al. 2002), as presented by Grachev et al. (2007). The scatterplots of the normalized vertical fluxes for the momentum $\overline{u'w'}/\overline{u'w'}_0$ and the heat $\overline{\theta'w'}/\overline{\theta'w'}_0$ against the normalized vertical coordinate z/L are presented in Figs. 1a and 1b, respectively: here, the subscript 0 suggests the values at the lowest measurement level (3 m), and the Obukhov length L is also evaluated by using the values at the lowest measurement level. It is furthermore superposed by a bin-averaged curve for all the measurements by four upper-level instruments. In spite of noticeable scatter, we recognize clear tendencies of both normalized fluxes decrease with the increasing z/L. However, the overall slopes appear to be gentler than a linear slope as expected for the invariance solutions for the mean linear layer. Moreover, for large z/L ($z/L > 10$), the scatter is extensive especially for the momentum flux; thus, it is no longer possible to judge any tendencies in a definite manner.

6. Conclusions

As emphasized in the introduction, the basic characteristics of the fluid flows can be inferred from the invariant symmetries of a given system. The present study has asked the question of whether the standard Monin–Obukhov boundary layer description under weak stratifications, both stable and unstable, is consistent with its characteristics expected from the symmetries of the system. Here, the main working hypothesis adopted in the present study is that those vertical profiles predicted by the Monin–Obukhov theory can be recovered as invariant solutions of the BLM system. An important role of invariant solutions is widely accepted in the literature (cf. Bluman and Kumei 1989, especially sections 4.2–4.4).

The main conclusion from the present symmetry analysis is that it is not possible to maintain a homogeneity of vertical fluxes with height, as usually assumed in the surface layer. To maintain a symmetry consistency of the governing equation system, vertical fluxes must change linearly with height over the dynamic sublayer, and parabolically over the mean "linear" layer (i.e., the surface layer above the sublayer) in stable conditions. They should change with height by fractional powers of height in unstable conditions. A major exception to this conclusion is with $a_z = a_t = a_s = 0$. Under this exception, a homogeneous flux layer coincides with the dynamic sublayer.

A preliminary observational analysis is presented to partially support this result based on the symmetry analysis. However, full data analyses should still follow. An interpretation of the result more from a physical point of view is also still missing, as it is derived purely by a mathematical consistency based on an invariant transformation of the system. However, an important implication for a finite departure of the vertical fluxes from a constant profile in the surface layer is a crucial role of transiency and horizontal inhomogeneity, which is required to maintain such a state against the homogenization tendency of the fluxes otherwise, as a phenomenological argument in section 2a has suggested.

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Data availability statement. The SHEBA data used in the present study are available from Andreas et al. (2007).

APPENDIX

Derivation of the Invariant Transformation Rules

The purpose of this appendix is to derive the invariant transformation rules adopted in the present study more explicitly, and to discuss some technical issues. The invariant transformation rules are derived for the PDE and statistical symmetries separately in the following two subsections. An important technical issue with the statistical symmetry is further discussed in the last subsection.

a. Symmetries of the governing equation system, or PDE symmetries

The symmetries of the governing equation system, or the PDE symmetries, in short, in the present study are derived by considering the full Navier–Stokes equations under the Boussinesq approximation in the inviscid limit:

$$\frac{\partial u}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{\partial \phi}{\partial x}, \quad \text{(A.1a)}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} = -\frac{\partial \phi}{\partial y}, \quad \text{(A.1b)}$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} = -\frac{\partial \phi}{\partial z} + b, \quad \text{(A.1c)}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial u\theta}{\partial x} + \frac{\partial v\theta}{\partial y} + \frac{\partial w\theta}{\partial z} = 0, \quad \text{(A.1d)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \text{(A.1e)}$$

where u, v, and w are the velocity components in the Cartesian coordinates (x, y, z). The buoyancy b is defined by $b = (\theta - \theta_0)/\theta_0$, where θ_0 is a reference constant potential temperature.

1) When the buoyancy is considered as a passive scalar

The invariant transformation rules, for example, for u and w are given in the form

$$u^* = u + \varepsilon\zeta_u, \quad \text{(A.2a)}$$

$$w^* = w + \varepsilon\zeta_w, \quad \text{(A.2b)}$$

where

$$\zeta_u = (a_z - a_t)u, \quad \text{(A.3a)}$$

$$\zeta_w = (a_z - a_t)w. \quad \text{(A.3b)}$$

When the buoyancy term (the last term on the right-hand side) is dropped from Eq. (A.1c), the invariant transformation rules for the buoyancy b are also determined from the buoyancy Eq. (A.1d) in an analogous manner. Physically speaking, this amounts to treating the buoyancy as a passive scalar. We focus on this simpler case first.

The transformation rules for the fluxes also follow, for example, by

$$u^*w^* = uw + \varepsilon\zeta_{uw}. \quad \text{(A.4)}$$

The explicit form of the infinitesimal increment ζ_{uw} is obtained by substituting Eqs. (A.2a) and (A.2b) and Eqs. (A.3a) and (A.3b) into Eq. (A.4), and we find

$$\zeta_{uw} = u\zeta_w + w\zeta_u = 2(a_z - a_t)uw. \quad \text{(A.5)}$$

Note that the invariant transformations do not change by spatial averaging operations; thus, those obtained for u, w, uw, etc., are directly applicable to the corresponding averaged quantities $\overline{u}$, $\overline{w}$, $\overline{uw}$, etc. This principle defines the PDE symmetries as

$$\frac{dz}{a_z z + a_0} = \frac{d\overline{u}}{(a_z - a_t)\overline{u}} = \frac{d\overline{uw}}{2(a_z - a_t)\overline{uw}} = \frac{d\overline{\theta}}{a_\theta \overline{\theta}} = \frac{d\overline{\theta w}}{(a_z - a_t + a_\theta)\overline{\theta w}}, \quad \text{(A.6)}$$

when the buoyancy is treated as a passive scalar. Here, in the present study, we assume $\overline{v} = 0$ for simplicity, and focus exclusively on the momentum budget in x and z directions. Note that it immediately follows $\overline{vw} = 0$ form the above assumption.

2) When the buoyancy plays an active role in the momentum equation

When the buoyancy term (the last term on the right-hand side) is explicitly taken into account in the vertical momentum equation, Eq. (A.1c), a further constraint must be introduced to the PDE symmetries, (A.6), just defined, because both the momentum and the buoyancy must be transformed in a consistent manner so that the buoyancy term also remains invariant along with all the other terms. This further constraint is given by Eq. (4.2).

b. Statistical symmetries

The statistical symmetries are derived by considering the multipoint correlation (MPC) equations. The derivation is fully described in Oberlack and Rosteck (2010). In the present study, we apply the same under the Boussinesq approximation in the inviscid limit. In this case, as clarified in section 4 of Oberlack and Rosteck, a full set of equations in infinite series is considered for identifying the invariant transformations.

The identified statistical symmetries are

$$\frac{d\overline{u}}{a_s\overline{u} + a_1} = \frac{d\overline{uw}}{a_s\overline{uw} + a_2} = \frac{d\overline{\theta}}{a_s\overline{\theta} + a_3} = \frac{d\overline{\theta w}}{a_s\overline{\theta w} + a_4}. \quad \text{(A.7)}$$

By combining the transformation rules, (A.6) and (A.7), we obtain a list of rules as (4.1) in the main text. Note especially that the intermittency parameter, a_s, applies to the all dependent variables of the system as a common rescaling factor. Also note that the statistical symmetries are applied only to the dependent variables.

c. Further possibilities with the statistical symmetries

In principle, so long as the statistical symmetries are concerned in their own standalone manner, more translational symmetries can be added to the above by any linear superpositions of the solutions, because the MPC equation set itself is linear. Here, these additional translational statistical symmetries state, more precisely, make it possible to add any arbitrary functions at will. Obviously, such a procedure can easily lead to unphysical solutions. For this reason, we do not consider this additional possibility in the present study.

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