CORRIGENDUM

JOHANNES M. L. DAHL[a] AND JANNICK FISCHER[b]

[a] *Department of Geosciences, Texas Tech University, Lubbock, Texas*
[b] *Karlsruher Institut für Technologie, Germany*

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This corrigendum concerns two equations in Dahl and Fischer's (2023) derivation: one erroneously omitted term in their Eq. (13), which, however, will be shown not to affect the downstream mathematical development, and one typo in Eq. (30).

1. Equation (13)

In the following, the correct form of the time integral of the momentum equation, Eq. (13) in Dahl and Fischer (2023), is presented, and it is demonstrated that an additional term that appears in that equation is irrotational and hence does not contribute to the vorticity integral, Dahl and Fischer's Eq. (14). The momentum equation is given by

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}, \tag{1}$$

where $\mathbf{v}$ is the velocity vector and $\mathbf{f}$ is the net force acting on the fluid parcel. Using the same notation as in Dahl and Fischer (2023), this may be written as

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(u_\alpha \mathbf{g}^\alpha) = \frac{du_\alpha}{dt}\mathbf{g}^\alpha + u_\alpha \frac{d\mathbf{g}^\alpha}{dt} = \mathbf{f}. \tag{2}$$

The term on the rhs involving du_α/dt describes the time rate of change of the coordinate velocity u_α, which includes the effect of the force as well as the effect of the time-dependent coordinate basis vectors. The second term, involving $d\mathbf{g}^\alpha/dt$, is the geometric term that corrects for the effect of the time-dependent coordinate basis. In the present case, the basis vectors change because the material volume, including the coordinate system materially attached to it, is deformed by the flow.[1] To obtain an expression for the time evolution of $\mathbf{g}^\alpha$, we first recognize that

$$\mathbf{g}^\alpha = \nabla \xi^\alpha. \tag{4}$$

Now, for any parcel property Φ, we may write $\Phi(t) = \Phi[\mathbf{r}(t), t]$, where $\mathbf{r}$ is the parcel's location. Using the chain and product rules, we find

[1] In many applications, the coordinate basis is constant in time but nonuniform in space, such that the geometric term is given by (e.g., Simmonds 1994, p. 57)

$$u_\alpha \frac{d\mathbf{g}^\alpha}{dt} = u_\alpha u^\beta \frac{\partial \mathbf{g}^\alpha}{\partial \xi^\beta} = -u_\alpha u^\beta \Gamma^\alpha_{\beta\gamma}\mathbf{g}^\gamma \tag{3}$$

where $u^\beta\, \partial \mathbf{g}^\alpha/\partial \xi^\beta$ is the time rate of change of $\mathbf{g}^\alpha$ due to "advection" [see also appendix A of Dahl and Fischer (2023)] and $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol.

Corresponding author: Johannes Dahl, johannes.dahl@ttu.edu

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$$\nabla\left(\frac{d\Phi}{dt}\right) = \nabla\left(\frac{\partial\Phi}{\partial t} + \mathbf{v}\cdot\nabla\Phi\right) = \frac{d}{dt}(\nabla\Phi) + (\nabla\mathbf{v})\cdot\nabla\Phi. \tag{5}$$

With $\Phi = \xi^\alpha$ and $d\xi^\alpha/dt = 0$, we see that

$$\frac{d\mathbf{g}^\alpha}{dt} = \frac{d}{dt}(\nabla\xi^\alpha) = -(\nabla\mathbf{v})\cdot\nabla\xi^\alpha. \tag{6}$$

The rhs is equivalent to the 3D vector frontogenesis function [vector frontogenesis in 2D is discussed by Keyser et al. (1988)], which connects the evolution of the coordinate curves to the flow field: Like isentropes in the usual frontogenesis case, material coordinate curves are deformed and rotated by the flow; the contravariant basis vectors correspond to the gradients of the isentropes. This implies that the geometric term $\mathbf{G}$ may be written as

$$\mathbf{G} = u_\alpha \frac{d\mathbf{g}^\alpha}{dt} \tag{7}$$

$$= -u_\alpha(\nabla\mathbf{v})\cdot\nabla\xi^\alpha \tag{8}$$

$$= -(\nabla\mathbf{v})\cdot\mathbf{g}^\alpha u_\alpha \tag{9}$$

$$= -(\nabla\mathbf{v})\cdot\mathbf{v}. \tag{10}$$

With this, the momentum equation becomes

$$\frac{\delta u_\beta}{\delta t} = \frac{du_\beta}{dt} - \mathbf{g}_\beta\cdot[(\nabla\mathbf{v})\cdot\mathbf{v}] = f_\beta. \tag{11}$$

This equation is equivalent to Eq. (12) in Dahl and Fischer (2023). When integrating this equation with respect to time, however, the geometric term needs to be carried along, so Dahl and Fischer's Eq. (13) should read

$$u_\beta(t) = u_\beta(t_0) + \int_{t_0}^{t} dt' f_\beta(t') + \int_{t_0}^{t} dt' \mathbf{g}_\beta(t')\cdot\{[\nabla\mathbf{v}(t')]\cdot\mathbf{v}(t')\}. \tag{12}$$

When this equation is inserted into the expression for the vorticity,

$$\omega^\gamma(t) = \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{G(t)}}\frac{\partial u_\beta(t)}{\partial\xi^\alpha}, \tag{13}$$

one obtains

$$\omega^\gamma(t) = \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{G(t)}}\frac{\partial u_\beta(t_0)}{\partial\xi^\alpha} + \int_{t_0}^{t} dt' \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{G(t)}}\frac{\partial f_\beta(t')}{\partial\xi^\alpha} + \int_{t_0}^{t} dt' \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{G(t)}}\frac{\partial}{\partial\xi^\alpha}\{\mathbf{g}_\beta(t')\cdot[\nabla\mathbf{v}(t')\cdot\mathbf{v}(t')]\}. \tag{14}$$

This equation corresponds to Eq. (14) in Dahl and Fischer (2023), but it includes the geometric term (the last term on the rhs). To show that this term is zero, we use Eq. (23) from Dahl and Fischer (2023) and find that

$$\int_{t_0}^{t} dt' \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{G(t)}}\frac{\partial}{\partial\xi^\alpha}\{\mathbf{g}_\beta(t')\cdot[\nabla\mathbf{v}(t')\cdot\mathbf{v}(t')]\} = -\int_{t_0}^{t} dt' \frac{\rho(t)}{\rho(t')}\frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{G(t')}}\frac{\partial G_\beta(t')}{\partial\xi^\alpha} \tag{15}$$

$$= -\int_{t_0}^{t} dt' \frac{\rho(t)}{\rho(t')}[\nabla\times\mathbf{G}(t')]^\gamma \tag{16}$$

$$= -\int_{t_0}^{t} dt' \frac{\rho(t)}{\rho(t')}\mathbf{g}^\gamma(t')\cdot[\nabla\times\mathbf{G}(t')]. \tag{17}$$

If $\nabla\times\mathbf{G} = \mathbf{0}$, this integral vanishes. To see that this is indeed the case, we first apply the product rule and see that

$$\mathbf{G} = -(\nabla \mathbf{v}) \cdot \mathbf{v} = -\frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) = -\frac{1}{2}\nabla V^2, \tag{18}$$

where $V = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ is the velocity magnitude. Then,[2]

$$\nabla \times \mathbf{G} = -\frac{1}{2}\nabla \times \nabla V^2 \equiv \mathbf{0}. \tag{19}$$

2. Equation (30)

Equation (30) in Dahl and Fischer (2023) is written in symbolic form, so there should be no index on the lhs. The correct equation is given by

$$\boldsymbol{\omega}(t) = \frac{\partial \mathbf{r}}{\partial \boldsymbol{\xi}}(t) \cdot \left[\frac{\rho(t)}{\rho(t_0)}\boldsymbol{\omega}_0 + \int_{t_0}^{t} dt' \frac{\rho(t)}{\rho(t')}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{r}}(t') \cdot \boldsymbol{\tau}(t')\right]. \tag{20}$$

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[2] Note that $(\nabla \mathbf{v}) \cdot \mathbf{v} \mathrel{\hat=} (\partial u_j/\partial x_i)u_j$ is not equal to the advection term $(\mathbf{v} \cdot \nabla)\mathbf{v} \mathrel{\hat=} u_i(\partial u_j/\partial x_i)$, whose curl is generally not zero.