What Determines the Spectrum of a Climate Variable at Zero Frequency?

JIN-SONG VON STORCH

Institute of Meteorology, University of Hamburg, Hamburg, Germany

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ABSTRACT

In order to understand the spectrum $\Gamma_x(\omega)$ of a climate variable x_t, the relation between $\Gamma_x(\omega)$ and its forcing has to be considered. If the evolution of x_t over (discretized) time is determined by f_t, that is, $\Delta x/\Delta t \equiv (x_t - x_{t-1})/\Delta t = f_t$, the only existing relation is the one between $\Gamma_x(\omega)$ and the spectrum $\Gamma_f(\omega)$ of f_t. The gain function $G(\omega)$ of the difference operator $\Delta/\Delta t$, which acts as a high-pass filter, controls the relation between $\Gamma_x(\omega)$ and $\Gamma_f(\omega)$. For $\Gamma_x(\omega)$, which is bounded at zero frequency, $G(\omega)$ completely suppresses the variations of f_t at zero frequency, so that $\Gamma_x(0)$ cannot be related to $\Gamma_f(0)$. In practice, the efficiency of the difference operator as a high-pass filter can make the detection of the low-frequency spectral relation between x_t and f_t difficult.

1. Introduction

Climate systems can be described by trajectories of climate state variables over time. The equation describing how a state variable x evolves over time is generally a first-order differential equation in time. If time is discretized, the corresponding difference equation may be written as

$$\frac{\Delta x_t}{\Delta t} \equiv \frac{x_t - x_{t-1}}{\Delta t} = f_t, \tag{1}$$

where, without loss of generality, dx/dt is discretized using $\Delta x/\Delta t \equiv (x_t - x_{t-1})/\Delta t$ with Δt being the time increment.[1] Function f_t, which is a function of x and the other state variables of the system, represents the deterministic processes controlling the time evolution of x. This note concentrates on the discrete representation (1). The continuous case is briefly discussed at the end of this note.

It is generally believed that if f_t as a function of the state variables is exactly known, x_t for $t \in [-\infty, \infty]$ is in principle also known. By assuming that x_t and f_t in Eq. (1) can be considered as random variables, the spectra of x_t and f_t, which can be derived from the time series of x_t and f_t, are then also known. Because of relation (1), the two spectra are related to each other in such a way that the squared coherency spectrum, $\kappa_{xf}(\omega)$, between x_t and f_t [for the definition of $\kappa_{xf}(\omega)$, see, e.g., Jenkins and Watts 1968] is exactly one, implying a perfect spectral relation between x_t and f_t. Since this situation appears to be trivial, the spectral relation between x_t and f_t has barely been seriously studied. Almost all theoretical studies are devoted to the understanding of the deterministic processes constituting f_t.

This note questions the usefulness of knowledge about f_t in determining the spectrum of x_t at zero frequency. The consideration is motivated by the squared coherency spectrum $\kappa_{MF}(\omega)$ between the global atmospheric axial angular momentum M_t and the global axial torque F_t derived from a numerical integration of a climate model. The M_t and F_t are related to each other through an equation of type (1), so that the squared coherency between the two is expected to be one at any frequency. Figure 1 shows $\kappa_{MF}(\omega)$, as derived from 200 years of 12-hourly data generated by the coupled ECHAM1/LSG atmosphere–ocean general circulation model (GCM; von Storch et al. 1997). The ‘‘chunk’’ method is used to estimate the squared coherency spectrum. At extremely high frequencies (i.e., $\omega > 10^{-1}$), the low coherence is likely caused by sampling errors. Significant coherence is only obtained for high frequencies with $10^{-1} > \omega > 10^{-3}$. At low frequencies with $\omega < 10^{-3}$, no significant relationship between M_t and F_t is found. Note that for 12-h data considered, the frequency of one cycle per year corresponds to $\omega = 1/720 = 1.39 \times 10^{-3}$, and one cycle per 10 yr corresponds to $\omega = 1.39 \times 10^{-4}$.

At the first glance, Fig. 1 suggests that something is wrong with the angular momentum budget in the model, that is, with the representation of the processes consti-

[1] The result of this note can be easily generalized for $\Delta x/\Delta t \equiv (x_{t+n} - x_{t-m})/[(n + m)\Delta t]$, where n and m can be any finite integers, except $n = m = 0$. For $n = 1$ and $m = 1$, the above expression represents the centered difference.

Corresponding author address: Dr. Jin-Song von Storch, Institute of Hydrophysics, GKSS Research Center, 21502 Geestacht, Germany.
E-mail: jin@gkss.de

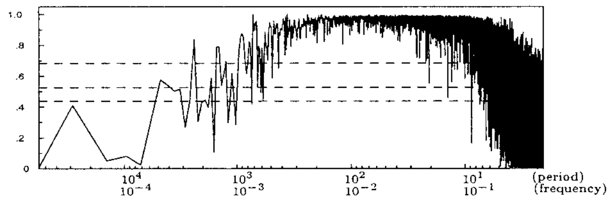

FIG. 1. Squared coherency spectrum between the global atmospheric angular momentum M_t and the global torque F_t. The spectrum is estimated using the chunk method. The number of the chunks is 5. Here, 200 yr of 12-h data obtained from an integration with the coupled ECHAM1/LSG model are used for the calculation. Frequency (period) is in units of cycles per half day (half day). The three dashed lines indicate the 90%, 95%, and 99% confidence levels for the null hypothesis that the coherence is zero.

tuting F. The purpose of this note is to show that the low coherence at low frequencies as shown in Fig. 1 has little to do with the representation of processes constituting F. Instead, it is caused by the constraint that F_t must be, by construction, $M_t - M_{t-1}$ divided by Δt.

It is noted that M_t is a state variable whose forcing can be, at least in a GCM, exactly specified. For many other variables of the climate system, such as velocity, temperature, or pressure at a given grid point, or time series of a given spatial pattern or mode, the forcing f has generally a complicated functional form and can therefore not be easily estimated. Without a precise estimation of f_t, the expected spectral relation between x_t and f_t cannot be tested. Thus, Fig. 1 is, to some extent, the only spectral estimation ever obtained at such low frequencies for a climate variable and its forcing. If indeed Fig. 1 reflects a general property of Eq. (1), the lack of coherence at low frequencies would be expected to also be found for other pairs of climate variables and their forcings. The general belief that the knowledge of f_t is sufficient for determining *everything* of x (i.e., not only any value of x_t at a given time t, but also the variance and spectrum of x_t) would therefore become questionable.

2. Consequences of the constraint $f_t = (x_t - x_{t-1})/\Delta t$

In order to demonstrate that the lack of coherence shown in Fig. 1 reflects a general property of Eq. (1), an arbitrary climate variable x_t and its forcing f_t, which determines the time evolution of x_t and satisfies (1), are considered. It is assumed that f_t is exactly known. With this assumption, f_t must be exactly equal to $(x_t - x_{t-1})/\Delta t$. The consequence of this constraint is discussed below. For the sake of simplicity, the constant time increment Δt is set to 1. Equation (1) then reduces to

$$\frac{\Delta x}{\Delta t} = x_t - x_{t-1} = f_t; \qquad (2)$$

$x_t - x_{t-1}$ is known as the first difference of x_t.

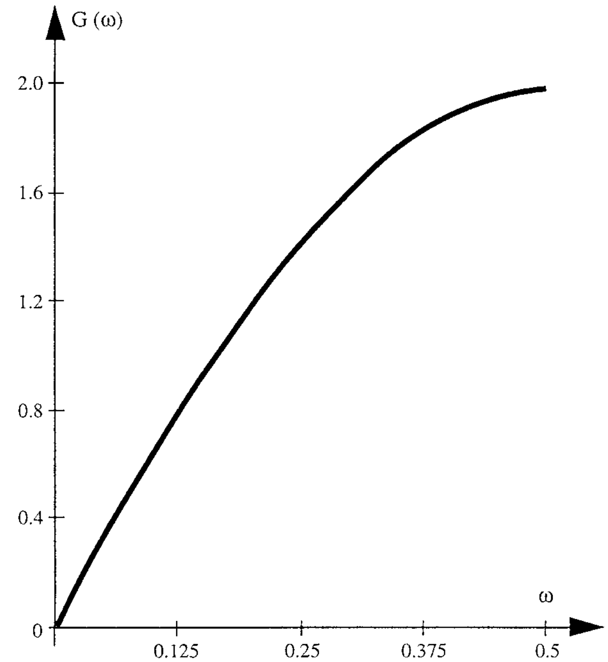

FIG. 2. Gain function $G(\omega) = 2\sin(\pi\omega)$ of the first difference operator with $0 \le \omega \le 0.5$. This is a reproduction of Fig. 1.4 in Jenkins and Watts (1968).

It can be shown (von Storch and Zwiers 1998), that the spectrum $\Gamma_x(\omega)$ of x_t and the spectrum $\Gamma_f(\omega)$ of $f_t = (x_t - x_{t-1})$ are related to each other via

$$\Gamma_f(\omega) = [2 - 2\cos(2\pi\omega)]\Gamma_x(\omega) = G^2(\omega)\Gamma_x(\omega), \qquad (3)$$

with

$$G(\omega) = 2\sin(\pi\omega), \qquad (4)$$

where $\Gamma_x(\omega)$, $\Gamma_f(\omega)$, and $G(\omega)$ are defined for $\omega \in [0, ½]$.

The $G(\omega)$ results from the first difference operator and is known as the gain function of the first difference operator (Jenkins and Watts 1968). Because of the form of $G(\omega)$ (Fig. 2), the first difference operator is often used as a high-pass filter (Jenkins and Watts 1968; von Storch and Zwiers 1998).

Following Eq. (3), $\Gamma_f(\omega)$ is simply a filtered version of $\Gamma_x(\omega)$. The filter $G(\omega)$ acts to respectively enhance and damp, rather than to alter, the spectrum of f_t in relation to the spectrum of x_t. The enhancement occurs at high frequencies with $\frac{1}{2} \ge \omega > \frac{1}{6}$, while the damping occurs at low frequencies with $0 \le \omega < \frac{1}{6}$. At any nonzero low frequency, the damping can never completely diminish the amplitude of $\Gamma_f(\omega)$ [as long as $\Gamma(\omega) \ne 0$]. Since the squared coherency is a normalized quantity, it will remain one, no matter how efficient the filter is.

On the other hand, Fig. 2 also clearly demonstrates the efficiency of the first difference operator in suppressing the variations of f_t at low frequencies. The efficiency appears more dramatic in Fig. 2, where both $G(\omega)$ and ω are plotted in a linear fashion, rather than in a logarithm representation. For low frequencies, $\omega < 10^{-3}$, at which the squared coherency in Fig. 1 is low, the values of $G(\omega)$ appear to be located at the origin where $\omega = 0$; that is, $G(\omega)$ is approximately zero for small ω. This is because that, for $\pi\omega \ll 1$, $G(\omega)$ can be approximated by

$$G(\omega) = 2\pi\omega, \tag{5}$$

which is, in a linear representation, approximately zero. Together with Eq. (3), Eq. (5) implies that, as frequency goes to zero, more and more variations of f_t are filtered out by $G(\omega)$. The amplitude of $\Gamma_f(\omega)$ relative to that of $\Gamma_x(\omega)$ drastically decreases. At zero frequency, if the spectral value of x_t is finite, which is a reasonable assumption for a climate variable, $G(\omega)$ would completely suppress the variations of f_t. The $\Gamma_f(0)$ would be zero regardless of $\Gamma_x(0)$. Consequently, $\Gamma_x(0)$ cannot be determined from $\Gamma_f(0)$. In this sense, Eq. (3) is ill at zero frequency.[2]

The efficiency of the first difference operator as a high-pass filter can cause practical difficulty in detecting the spectral relation between x_t and f_t. This is particularly true when $\Gamma_x(\omega)$ remains essentially constant as ω goes to zero. The values of $\Gamma_f(\omega)$ with $\omega \to 0$ are then extremely small and can therefore not be easily estimated. This fact makes the detection of the spectral relation between x_t and f_t difficult.

It is noted that observations and data obtained from numerical integrations of GCMs indicate that many climate variables, in particular the atmospheric variables, have spectra that are finite at zero frequency. Moreover, the spectra tend to be flat for sufficiently small frequencies. For these variables, $\Gamma_x(0)$ represents the flat spectral level at low frequencies. The spectrum $\Gamma_f(\omega)$, which is then proportional to ω^2, has extremely small values as ω goes to zero. The smallness of $\Gamma_f(\omega)$ makes the detection of the spectral relation between x_t and f_t difficult. In the following, this difficulty is further illustrated by considering the squared coherency spectrum $\kappa_{xf}(\omega)$ between x_t and $f_t = (x_t - x_{t-1})$ for an arbitrary x_t, whose spectrum is essentially flat at low frequencies.

Figure 3 shows $\kappa_{xf}(\omega)$, where x_t is generated by a first-order autoregressive process, with the process coefficient being 0.5. As in Fig. 1, the chunk method is used to estimate $\kappa_{xf}(\omega)$. The length T of the time series varies from 10^3 to 10^6. With increasing T, lower and lower frequencies can be resolved.

For all considered Ts, $\kappa_{xf}(\omega)$ is not distinguishable from zero at low frequencies. Taking ω_{95} to be the frequency below which $\kappa_{xz}(\omega)$ is essentially lower than the 95% confidence level, Fig. 3 shows that ω_{95} decreases

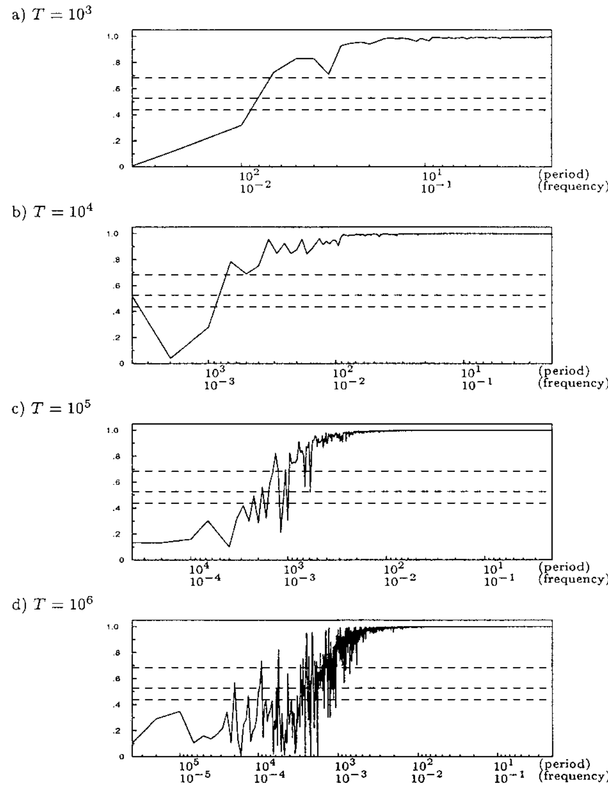

FIG. 3. Squared coherency spectrum $\kappa_{xf}(\omega)$ between x_t and $f_t = x_t - x_{t-1}$ as a function of the length of time series T. Here, x_t is generated by a first-order autoregressive process with process coefficient being 0.5. Five chunks are used to estimated $\kappa_{xf}(\omega)$. The dashed lines in each panel are, respectively, the 90%, 95%, and 99% confidence levels for the null hypothesis that the coherence is zero.

with increasing T. This suggests that if the power of the computer is increased (so that squared coherency at much lower frequencies can be resolved), the frequency range over which low coherence is found would decrease. However, Fig. 3 also shows that the rate of decrease of ω_{95} is smaller for larger T. Further, ω_{95} decreases by one order of magnitude as T increases from 10^3 to 10^4, but only by about factor of 2 or so as T increases from 10^5 to 10^6. As the lowest resolved frequency approaches zero, it becomes increasingly difficult to move ω_{95} toward $\omega = 0$ by increasing computer power. This example suggests that the attempt to detect significant coherence between x_t and f_t at frequency $\omega \to 0$ is futile. The suggestion is consistent with the result that a finite $\Gamma_x(0)$ cannot be determined from $\Gamma_f(0)$.

The generality of the difficulty in detecting the spectral relation between x_t and f_t at frequency $\omega \to 0$ is further confirmed by the fact that the results shown in Fig. 3 are insensitive to the number of chunks (when the chunk spectral estimator is considered) and to the method used to estimate the squared coherency spectrum $\kappa_{xf}(\omega)$. The latter issue is verified by repeating the calculations using two additional methods, the Daniell spectral estimator and a method based on an autore-

[2] This result holds also for the difference operator as in defined in footnote 1. In this case, the gain function becomes $G(\omega) = \{2 - 2\cos[2\pi(n + m)\omega]\}^{1/2}$, which is, as the gain function shown in Eq. (4), a high-pass filter that completely suppresses the variations of f_t at zero frequency when $\Gamma_x(0)$ is finite.

gressive model that is fitted to the bivariate time series with one component being x_t and the other being f_t. No manner how $\kappa_{xf}(\omega)$ is estimated, it is indistinguishable from zero for sufficiently small frequencies.

For the reader who likes to redo the calculation, it is noted that the squared coherency spectrum *cannot* be estimated from the raw bivariate periodogram, since this equals unity for any two time series (see, e.g., von Storch and Zwiers 1998). Using the chunk or Daniell method, the problem is solved by choosing the number of the chunks, or the length of Daniell smoother, being larger than one.

3. Conclusions and discussion

This note considers the spectral property at and near zero frequency of a climate variable, x, whose time evolution is determined by f_t through Eq. (1). For such a variable, the forcing f_t is, by construction, the first difference of x_t divided by the time increment Δt. This has the following consequences. Theoretically, the spectral relation between x_t and f_t, as measured by the squared coherency spectrum, is perfect at any nonzero frequency but degenerated at zero frequency. Practically, it can be difficult to detect the spectral relation between x_t and f_t at sufficiently low frequencies. The perfectness is caused by the fact that the gain function of the difference operator damps, but does not alter, the low-frequency spectrum of f_t in relation to the spectrum of x_t. Since the squared coherency spectrum $\kappa_{xf}(\omega)$ is a normalized quantity, it remains one, no matter how strong the damping is. The illness at zero frequency, on the other hand, is caused by the efficiency of the difference operator in suppressing low-frequency variations of f_t. As frequency goes to zero, more and more variations of f_t are suppressed. At zero frequency, as long as the spectral value of x_t is finite, which is a reasonable assumption for a climate state variable, the spectral value of f_t must diminish to zero. The $\Gamma_x(0)$ can therefore not be determined from $\Gamma_f(0)$. The difficulty in detecting perfect coherence at low frequencies is more severe for those variables whose spectra remain essentially flat as frequency goes to zero.

The result of this note can be easily generalized for the continuous case. Equation (1) then becomes a differential equation with $\Delta x_t/\Delta t$ being replaced by $dx(t)/dt$ and x_t and f_t by $x(t)$ and $f(t)$, which are now continuous functions in time. The gain function of the differential operator d/dt is denoted by $G^*(\omega)$. The difference between $G^*(\omega)$ and $G(\omega)$ is only noticeable for large ω, in particular for ω near the frequency $1/\Delta$. At low frequencies, $G(\omega)$ approaches $2\pi\omega$, which is identical to $G^*(\omega)$. Thus, $G^*(\omega)$ acts as a high-pass filter for low-frequency variations in the same way as $G(\omega)$. The result of this note applies therefore also to a continuous state variable described by a differential equation that is the continuous representation of Eq. (1).

Altogether, one can draw the conclusion that Fig. 1 describes a general phenomenon not only for the global axial atmospheric angular momentum, M_t, but for any arbitrary climate variable, x_t, which is described by an equation of type (1) and has finite and essentially constant spectral values at sufficiently low frequencies. For such a variable x_t, the efficiency of the difference operator as a high-pass filter will ultimately make the detection of low-frequency spectral relations between x_t and its forcing difficult, even when all the processes involved in the forcing are known and have been taken into account.

Finally, it is worthwhile to emphasize that this paper does not question the ability of Eq. (1) in determining the complete evolution of x_t over time, but in determining of the spectral value of x_t at zero frequency. The constraint in form of $G(\omega)$, which makes the spectral representation of (1) degenerated at zero frequency, arises only when the forcing of the *spectrum* of x_t is considered.

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