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Scalar and Vector Partitions of the Ranked Probability Score[1]

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ABSTRACT—Scalar and vector partitions of the ranked probability score, RPS, are described and compared. These partitions are formulated in the same manner as the scalar and vector partitions of the probability score, PS, recently described by Murphy. However, since the RPS is defined in terms of cumulative probability distributions, the scalar and vector partitions of the RPS provide measures of the reliability and resolution of scalar and vector *cumulative* forecasts, respectively. The scalar and vector partitions of the RPS provide similar, but not equivalent (i.e., linearly related), measures of these attributes. Specifically, the reliability (resolution) of cumulative forecasts according to the scalar partition is equal to or greater (less) than their reliability (resolution) according to the vector partition. A sample collection of forecasts is used to illustrate the differences between the scalar and vector partitions of the RPS and between the vector partitions of the RPS and the PS.

Several questions related to the interpretation and use of the scalar and vector partitions of the RPS are briefly discussed, including the information that these partitions provide about the reliability and resolution of forecasts (as opposed to cumulative forecasts) and the relative merits of these partitions. These discussions indicate that, since a one-to-one correspondence exists between vector and vector cumulative forecasts, the vector partition of the RPS can also be considered to provide measures of the reliability and resolution of vector forecasts and that the vector partition is generally more appropriate than the scalar partition.

1. INTRODUCTION

Scalar and vector partitions of the probability score, PS, (Brier 1950) have recently been described and compared (Murphy 1972*a*, 1972*b*). We have demonstrated that these partitions, which are based upon expressions for the PS in which forecasts are considered to be scalars and vectors, respectively, provide similar, but not equivalent (i.e., linearly related), measures of the reliability and resolution of the forecasts. Specifically, the scalar and vector partitions of the PS provide measures of the reliability and resolution of individual *probabilities* and sets of probabilities, or *forecasts*, respectively. Measures of these attributes[3] are of interest to meteorologists who are concerned with the scientific or inferential aspects of the evaluation of probability forecasts (Winkler and Murphy 1968, Murphy and Winkler 1970).

The PS is a particularly appropriate measure for the evaluation of forecasts of *unordered* variables, while the ranked probability score, RPS, (Epstein 1969)[4] is a particularly appropriate measure for the evaluation of forecasts of *ordered* variables (Murphy 1970).[5] The purpose of this paper is to describe and compare scalar and vector partitions of the RPS. These partitions are formulated in the same manner as the scalar and vector partitions of the PS. However, the RPS is defined in terms of cumulative probability distributions. Specifically, the RPS represents the average of the sum of the squared differences between the forecast and observed cumulative probability distributions, while the PS represents the average of the sum of the squared differences between the forecast and observed probability distributions (Murphy 1971). The scalar and vector partitions of the RPS, therefore, provide measures of the reliability and resolution of scalar and vector *cumulative* forecasts, respectively.

In section 2, we define both scalar and vector forecasts and observations and scalar and vector cumulative forecasts and observations, introduce notation to identify these quantities, and briefly consider the relationships between forecasts and cumulative forecasts. The RPS is defined and the expressions upon which the scalar and vector partititions of the RPS are based are presented in section 3. The scalar and vector partitions of the RPS are described in sections 4 and 5, respectively, and these partitions are compared in section 6. In section 7, we illustrate the differences between the scalar and vector partitions of the RPS and between the vector partitions of the RPS and the PS. Several questions related to the interpretation and use of the scalar and vector partitions of the RPS are briefly discussed in section 8, including the information that these partitions provide about the reliability and resolution of forecasts (as opposed to cumulative forecasts) and the relative merits of these partitions. Section 9 consists of a brief summary and conclusion.

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[3] We use the terms originally proposed by Sanders (1958) to describe these attributes. (See Murphy 1972*a*.)

[4] The RPS was formulated independently by Epstein and Thompson. (See Murphy 1971.)

[5] Ordered and unordered variables are variables for which the concept of *distance* between values, or states (sec. 2), is and is not meaningful, respectively. Examples of ordered variables are temperature and precipitation amount; an example of an unordered variable is the three-state (N=3) variable, "no precipitation," "rain," and "snow."

2. SCALAR AND VECTOR FORECASTS AND OBSERVATIONS

The variable of concern is assumed to be an ordered variable, such as temperature or precipitation amount, the range of which has been divided into a set of N mutually exclusive and collectively exhaustive states $\{s_1, \ldots, s_N\}$.

Forecasts and Observations

When the probability assigned to each state on each occasion is assumed to constitute a separate forecast, we donote the forecast by a scalar, r $(0 \leq r \leq 1)$, and the relevant observation by a scalar, d, where $d=1$ if the state of concern occurs and $d=0$ otherwise. On the other hand, when the set of probabilities assigned to the set of N states on each occasion is assumed to constitute a forecast, we denote the forecast by a row vector $\mathbf{r}=(r_1, \ldots, r_N)$ $(r_n \geq 0, \sum_n r_n=1; n=1, \ldots, N)$, where r_n is the probability assigned to state s_n on the occasion of concern. Similarly, we denote the relevant observation by a row vector $\mathbf{d}=(d_1, \ldots, d_N)$, where $d_n=1$ if state s_n occurs and $d_n=0$ otherwise. Note that the vectors $\mathbf{r}$ and $\mathbf{d}$ represent probability mass functions, the discrete counterparts of probability *density* functions. The vector $\mathbf{d}$ is, in reality, a *degenerate* probability mass function.

Cumulative Forecasts and Observations

We simply denote a scalar cumulative forecast by a scalar, R, where R represents the sum of the appropriate scalar forecasts (i.e., the sum of the appropriate rs). The relevant scalar cumulative observation is denoted by a scalar, D, where D represents the sum of the appropriate scalar observations (i.e., the sum of the appropriate ds).[6] For example, if on a particular occasion the forecast probabilities (i.e., the scalar forecasts) for a three-state ($N=3$) variable are 0.1, 0.7, and 0.2 and state s_2 occurs (in which case the scalar observations are 0, 1, and 0), then the corresponding scalar cumulative forecasts and observations are 0.1, 0.8, and 1.0 and 0, 1, and 1, respectively.

We denote the cumulative forecast, which corresponds to the vector forecast $\mathbf{r}$, by the row vector $\mathbf{R}=(R_1, \ldots, R_N)$, where

$$R_n=\sum_{m=1}^{n} r_m \tag{1}$$

$(n=1, \ldots, N)$. Similarly, we denote the cumulative observation, which corresponds to the vector observation $\mathbf{d}$, by the row vector $\mathbf{D}=(D_1, \ldots, D_N)$, where

$$D_n=\sum_{m=1}^{n} d_m \tag{2}$$

$(n=1, \ldots, N)$. Thus, when the vector forecast and observation are (0.1, 0.7, 0.2) and (0, 1, 0), respectively, the cumulative vector forecast and observation are (0.1, 0.8, 1.0) and (0, 1, 1), respectively. Note that the vectors $\mathbf{R}$ and $\mathbf{D}$ represent *cumulative* probability mass functions, the discrete counterparts of probability *distribution* functions. The vector $\mathbf{D}$ is, in reality, a *degenerate* cumulative probability mass function.

[6] We would have to introduce more complex notation to describe scalar cumulative forecasts and observations more precisely.

Collections of Forecasts and Observations

The partitions of the RPS of concern in this paper are based upon the assumption that the probabilities that constitute the forecasts can assume only a finite set of values. Specifically, we assume that the collection of forecasts of concern consists of M scalar or K $(=M/N)$ vector forecasts and that the probabilities can assume only S distinct values. When the forecasts are considered to be scalars, we can identify S distinct forecasts and, as a result, S subcollections of the collection of M forecasts, where subcollection s consists of the M^s forecasts for which $r_m=r^s$ $(m=1, \ldots, M^s; \sum_s M^s=M; s=1, \ldots, S)$. For the collection and subcollections of concern, we denote the relevant observations by d_m and d_m^s $(m=1, \ldots, M^s)$, respectively.

When the forecasts are considered to be vectors, we can identify T distinct forecasts, where

$$T=\sum_{s=1}^{S}\binom{N+s-4}{s-1}(S-s+1), \tag{3}$$

in which $\binom{x}{y}=x!/[y!(x-y)!]$ for $0 \leq y \leq x$, $\binom{x}{y}=1$ for $x=-1$ and $y=0$, and $\binom{x}{y}=0$ otherwise.[7] Thus, we can identify T subcollections of the collection of K forecasts, where subcollection t consists of the K^t forecasts for which $\mathbf{r}_k=\mathbf{r}^t$ $(k=1, \ldots, K^t; \sum_t K^t=K; t=1, \ldots, T)$, in which $\mathbf{r}_k=(r_{1k}, \ldots, r_{Nk})$ and $\mathbf{r}^t=(r_1^t, \ldots, r_N^t)$. For the collection and subcollections of concern, we denote the relevant observations by $\mathbf{d}_k$ and $\mathbf{d}_k^t$ $(k=1, \ldots, K^t)$, respectively, where $\mathbf{d}_k=(d_{1k}, \ldots, d_{Nk})$ and $\mathbf{d}_k^t=(d_{1k}^t, \ldots, d_{Nk}^t)$.

Forecasts and Cumulative Forecasts

As previously indicated, the partitions of the RPS provide measures of the reliability and resolution of *cumulative* forecasts (secs. 4 and 5). However, meteorologists are primarily concerned with the reliability and resolution of forecasts. Thus, the relationships between forecasts and cumulative forecasts are of considerable interest.

With regard to the relationship between scalar and scalar cumulative forecasts, we are concerned, in particular, with the relationship between the S subcollections of scalar forecasts, where subcollection s consists of the M^s forecasts for which $r_m=r^s$ $(m=1, \ldots, M^s; \Sigma_s M^s=M; s=1, \ldots, S)$, and the S^* subcollections of scalar cumula-

[7] Equation (3) is valid only if the set of S distinct scalar forecasts r^s $(s=1, \ldots, S)$ includes the values zero and one and if the difference between adjacent probability values is constant. For example, if this difference is 0.1 in a three-state ($N=3$) situation, then $S=11$ and $r^s=0.0(0.1)1.0$ and, as a result, $T=66$ and $\mathbf{r}^t=(1.0, 0.0, 0.0), (0.9, 0.1, 0.0), \ldots, (0.0, 0.0, 1.0)$.

tive forecasts, where subcollection s^* consists of the M^{s^*} cumulative forecasts for which $R_m=R^{s^*}$ ($m=1, \ldots, M^{s^*}$; $\sum_{s^*} M^{s^*}=M$; $s^*=1, \ldots, S^*$). Note that the number of distinct forecasts and cumulative forecasts, S and S^*, respectively, are generally not equal. Further, even if S and S^* are equal, neither the forecasts and cumulative forecasts, r^s and R^{s^*}, respectively, which define the subcollections of scalar and scalar cumulative forecasts, nor the number of forecasts and cumulative forecasts, M^s and M^{s^*}, respectively, in the subcollections of scalar and scalar cumulative forecasts are generally equal.[8] For example, if on a particular occasion the scalar forecasts for a three-state ($N=3$) variable are 0.2, 0.6, and 0.2 (in which case the scalar cumulative forecasts are 0.2, 0.8, and 1.0), then $S=2$, $r^1=0.2$ and $r^2=0.6$, and $M^1=2$ and $M^2=1$ for scalar forecasts and $S^*=3$, $r^1=0.2$, $r^2=0.8$, and $r^3=1.0$, and $M^1=M^2=M^3=1$ for scalar cumulative forecasts. Thus, a one-to-one correspondence does not generally exist between the subcollections of scalar and scalar cumulative forecasts.

With regard to the relationship between vector and vector cumulative forecasts, we are particularly concerned with the relationship between the T subcollections of vector forecasts, where subcollection t consists of the K^t forecasts for which $\mathbf{r}_k=\mathbf{r}^t$ ($k=1, \ldots, K^t$; $\sum_t K^t=K$; $t=1, \ldots, T$) and the T^* subcollections of vector cumulative forecasts, where subcollection t^* consists of the K^{t^*} cumulative forecasts for which $\mathbf{R}_k=\mathbf{R}^{t^*}$ ($k=1, \ldots, K^{t^*}$; $\sum_{t^*} K^{t^*}=K$; $t^*=1, \ldots, T^*$). Note that the number of distinct forecasts and cumulative forecasts, T and T^*, respectively, is equal. Further, note that, while the forecasts and cumulative forecasts, $\mathbf{r}^t$ and $\mathbf{R}^{t^*}$, respectively, which define the subcollections of vector and vector cumulative forecasts, are generally not equal, we can identify for each subcollection, t, of K^t forecasts for which $\mathbf{r}_k=\mathbf{r}^t$, a subcollection, t^*, of K^{t^*} cumulative forecasts for which $\mathbf{R}_k=\mathbf{R}^{t^*}$, where, from eq (1),

$$R_n{}^{t^*}=\sum_{m=1}^{n} r_m^t. \tag{4}$$

Finally, note that the number of forecasts and cumulative forecasts, K^t and K^{t^*}, respectively, in the subcollections of vector and vector cumulative forecasts is equal. For example, if on two occasions the vector forecasts for a three-state ($N=3$) variable are (0.1, 0.7, 0.2) and (0.2, 0.6, 0.2), in which case the vector cumulative forecasts are (0.1, 0.8, 1.0) and (0.2, 0.8, 1.0), then $T=2$, $\mathbf{r}^1=(0.1, 0.7, 0.2)$ and $\mathbf{r}^2=(0.2, 0.6, 0.2)$, and $K^1=K^2=1$ for vector forecasts and $T^*=2$, $\mathbf{R}^1=(0.1, 0.8, 1.0)$ and $\mathbf{R}^2=(0.2, 0.8, 1.0)$, and $K^1=K^2=1$ for vector cumulative forecasts. Thus, a one-to-one correspondence exists between the subcollections of vector and vector cumulative forecasts. Similarly, a one-to-one correspondence exists (does not exist) between the subcollections of vector (scalar) and vector (scalar) cumulative observations.

[8] The subcollections s and s^* represent any arbitrary subcollections in their respective collections.

3. THE RPS

The RPS for a collection of K vector forecasts $\mathbf{r}_k$ and the K relevant observations $\mathbf{d}_k$ can be defined as

$$\text{RPS}(\mathbf{r},\, \mathbf{d})=\frac{1}{K}\sum_{k=1}^{K}\sum_{n=1}^{N}\left(\sum_{m=1}^{n} r_{mk}-\sum_{m=1}^{n} d_{mk}\right)^2 \tag{5}$$

(Murphy 1971). The range of RPS($\mathbf{r}$, $\mathbf{d}$) in eq (5) is the closed interval [0, $N-1$]. In this regard, note that on each occasion the Nth term in eq (5) equals zero and the sum of the remaining $N-1$ terms attains a maximum value of $N-1$ when each term equals one.

RPS ($\mathbf{r}$, $\mathbf{d}$) in eq (5) is expressed in terms of cumulative forecasts and observations (sec. 2). Thus, the scalar and vector partitions of the RPS must be formulated in terms of *cumulative* forecasts and observations. With regard to the scalar partition, note, from eq (1) and (2), that RPS ($\mathbf{r}$, $\mathbf{d}$) in eq (5) can be expressed as

$$\text{RPS}(\mathbf{r},\, \mathbf{d})=\frac{1}{K}\sum_{k=1}^{K}\sum_{n=1}^{N}(R_{nk}-D_{nk})^2. \tag{6}$$

RPS ($\mathbf{r}$, $\mathbf{d}$) in eq (6) is the expression upon which the scalar partition of the RPS is based (sec. 4).

With regard to the vector partition, note that RPS ($\mathbf{r}$, $\mathbf{d}$) in eq (6) can be expressed in vector notation as

$$\text{RPS}\ (\mathbf{r},\, \mathbf{d})=\frac{1}{K}\sum_{k=1}^{K}(\mathbf{R}_k-\mathbf{D}_k)(\mathbf{R}_k-\mathbf{D}_k)' \tag{7}$$

where a prime denotes a column vector. RPS ($\mathbf{r}$, $\mathbf{d}$) in eq (7) is the expression upon which the vector partition of the RPS is based (sec. 5).

4. SCALAR PARTITION

In the scalar framework, we have $M(=KN)$ scalar cumulative forecasts, R_m, and the M relevant scalar cumulative observations, D_m, ($m=1, \ldots, M$).[9] We denote the RPS in this framework by RPS(R, D), where, from eq (6),

$$\text{RPS}(R,\, D)=\frac{1}{M}\sum_{m=1}^{M}(R_m-D_m)^2. \tag{8}$$

Note that the range of RPS(R, D) in eq (8) is the closed interval [0, $(N-1)/N$].

The scalar partition of the RPS is based upon the assumption that a finite set of S distinct scalar cumulative forecasts, R^s ($s=1, \ldots, S$), exists and that S distinct subcollections of these forecasts can be identified, where subcollection s consists of the M^s cumulative forecasts for which $R_m=R^s$ ($m=1, \ldots, M^s$). See section 2. We denote the M and M^s relevant cumulative observations by D_m and D_m^s, ($m=1, \ldots, M^s$), respectively. Then, since the expression for RPS(R, D) in eq (8) is, in essence, identical to the expression upon which the scalar partition of the

[9] The scalar cumulative forecast R_m and observation D_m represent the sums of one or more scalar forecasts r_m and observations d_m, respectively (sec. 2).

PS is based (Murphy 1972*b*),[10] the scalar partition of the RPS can be expressed as

$$\mathrm{RPS}(R,D)=\frac{1}{M}\sum_{s=1}^{S}M^s(R^s-\overline{D}^s)^2+\frac{1}{M}\sum_{s=1}^{S}M^s\overline{D}^s(1-\overline{D}^s)$$

where (9)

$$\overline{D}^s=\frac{1}{M^s}\sum_{m=1}^{M^s}D_m^s.$$

The two terms on the right-hand side (RHS) of eq (9) represent measures of the reliability and resolution, respectively, of scalar cumulative forecasts. (See Murphy 1972*a*, footnote 11.)

5. VECTOR PARTITION

In the vector framework, we have $K(=M/N)$ vector cumulative forecasts, $\mathbf{R}_k$, and the K relevant vector cumulative observations, $\mathbf{D}_k$, $(k=1, \ldots, K)$. We denote the RPS in this framework by RPS($\mathbf{R}$, $\mathbf{D}$), where, from eq (7),

$$\mathrm{RPS}(\mathbf{R},\mathbf{D})=\frac{1}{K}\sum_{k=1}^{K}(\mathbf{R}_k-\mathbf{D}_k)(\mathbf{R}_k-\mathbf{D}_k)'. \tag{10}$$

Note that the range of RPS($\mathbf{R}$, $\mathbf{D}$) in eq (10) is the closed interval $[0, N-1]$.

The vector partition of the RPS is based upon the assumption that a finite set of T distinct vector cumulative forecasts, $\mathbf{R}^t$ $(t=1, \ldots, T)$, exists and that T distinct subcollections of these forecasts can be identified, where subcollection t consists of the K^t cumulative forecasts for which $\mathbf{R}_k=\mathbf{R}^t$ $(k=1, \ldots, K^t)$. We denote the K and K^t relevant cumulative observations by $\mathbf{D}_k$ and $\mathbf{D}_k^t$ $(k=1, \ldots, K^t)$, respectively. Then, since the expression for RPS($\mathbf{R}$, $\mathbf{D}$) in eq (10) is, in essence, identical to the expression upon which the vector partition of the PS is based (Murphy 1972*b*), the vector partition of the RPS can be expressed as

$$\mathrm{RPS}(\mathbf{R},\mathbf{D})=\frac{1}{K}\sum_{t=1}^{T}K^t(\mathbf{R}^t-\overline{\mathbf{D}}^t)(\mathbf{R}^t-\overline{\mathbf{D}}^t)'+\frac{1}{K}\sum_{t=1}^{T}K^t\overline{\mathbf{D}}^t(\mathbf{U}-\overline{\mathbf{D}}^t)' \tag{11}$$

where

$$\overline{\mathbf{D}}^t=\frac{1}{K^t}\sum_{k=1}^{K^t}\mathbf{D}_k^t$$

and $\mathbf{U}$ is a row vector with N elements all equal to one; that is, $\mathbf{U}=(1, \ldots, 1)$. The two terms on the RHS of eq (11) represent the measures of the reliability and resolution, respectively, of vector cumulative forecasts.

6. SCALAR AND VECTOR PARTITIONS: A COMPARISON

Hereafter, for comparative purposes, we denote the vector partition of the RPS by RPS*($\mathbf{R}$, $\mathbf{D}$), where RPS*($\mathbf{R}$, $\mathbf{D}$)$=(1/N)$RPS($\mathbf{R}$, $\mathbf{D}$). Note that the range of RPS*($\mathbf{R}$, $\mathbf{D}$) is the closed interval $[0, (N-1)/N]$.

[10] Of course, the expression for the PS relates to forecasts and observations; the expression for the RPS relates to cumulative forecasts and observations.

The expressions for the scalar and vector partitions of the RPS, RPS(R, D), and RPS*($\mathbf{R}$, $\mathbf{D}$), respectively, are identical to the expressions for the scalar and vector partitions of the PS (Murphy 1972*b*), except that the former relate to *cumulative* forecasts and observations while the latter relate to forecasts and observations. We compare the terms in the expressions for RPS(R, D) and RPS*($\mathbf{R}$, $\mathbf{D}$) in the same manner as that in which we compared the terms in the expressions for the scalar and vector partitions of the PS.

The scalar partition of the RPS, RPS(R, D) [eq (9)], can be expressed as

$$\mathrm{RPS}(R,D)=\frac{1}{M}\left[\sum_{s=1}^{S}M^s(R^s)^2-2\sum_{s=1}^{S}M^sR^s\overline{D}^s+\sum_{s=1}^{S}M^s(\overline{D}^s)^2\right]+\frac{1}{M}\left[\sum_{s=1}^{S}M^s\overline{D}^s-\sum_{s=1}^{S}M^s(\overline{D}^s)^2\right]. \tag{12}$$

Let $S1$ and $S2$ denote the two sets of terms on the RHS of eq (12) and let $S11$, $S12$, and $S13$ and $S21$ and $S22$, respectively, denote the terms that constitute these sets. Then,

$$S1=S11+S12+S13 \tag{13}$$

and

$$S2=S21+S22, \tag{14}$$

and, since $S13=-S22$,

$$\mathrm{RPS}(R,D)=S11+S12+S21. \tag{15}$$

The vector partition of the RPS, RPS*($\mathbf{R}$, $\mathbf{D}$)[eq (11)], can be expressed as

$$\mathrm{RPS}^*(\mathbf{R},\mathbf{D})=\frac{1}{NK}\left[\sum_{t=1}^{T}K^t\sum_{n=1}^{N}(R_n^t)^2-2\sum_{t=1}^{T}K^t\sum_{n=1}^{N}R_n^t\overline{D}_n^t+\sum_{t=1}^{T}K^t\sum_{n=1}^{N}(\overline{D}_n^t)^2\right]+\frac{1}{NK}\left[\sum_{t=1}^{T}K^t\sum_{n=1}^{N}\overline{D}_n^t-\sum_{t=1}^{T}K^t\sum_{n=1}^{N}(\overline{D}_n^t)^2\right]. \tag{16}$$

Let $V1$ and $V2$ denote the two sets of terms on the RHS of eq (16) and let $V11$, $V12$, and $V13$ and $V21$ and $V22$, respectively, denote the terms that constitute these sets. Then,

$$V1=V11+V12+V13 \tag{17}$$

and

$$V2=V21+V22, \tag{18}$$

and, since $V13=-V22$,

$$\mathrm{RPS}^*(\mathbf{R},\mathbf{D})=V11+V12+V21. \tag{19}$$

A comparison of the terms in the expressions for the scalar and vector partitions of the RPS indicates that $S11=V11$, $S12=V12$, and $S21=V21$ and that $S13\leq V13$ (appendix). Thus, from eq (15) and (19),

$$\mathrm{RPS}(R,D)=\mathrm{RPS}^*(\mathbf{R},\mathbf{D}) \tag{20}$$

TABLE 1.—*A sample collection of forecasts and the relevant observations for a three-state* (N=3) *variable when the forecasts and observations are considered to be scalars*

Forecast/ observation number	Forecast	Observation	Cumulative forecast	Cumulative observation
m	r_m	d_m	R_m	D_m
1	0.1	0	0.1	0
2	.3	0	0.4	0
3	.6	1	1.0	1
4	.1	0	0.1	0
5	.7	1	0.8	1
6	.2	0	1.0	1
7	.3	0	0.3	0
8	.5	1	0.8	1
9	.2	0	1.0	1
10	.5	0	0.5	0
11	.4	1	0.9	1
12	.1	0	1.0	1
13	.7	1	0.7	1
14	.3	0	1.0	1
15	.0	0	1.0	1
16	.6	0	0.6	0
17	.1	0	0.7	0
18	.3	1	1.0	1
19	.5	1	0.5	1
20	.4	0	0.9	1
21	.1	0	1.0	1
22	.1	0	0.1	0
23	.8	1	0.9	1
24	.1	0	1.0	1
25	.1	0	0.1	0
26	.6	0	0.7	0
27	.3	1	1.0	1
28	.1	0	0.1	0
29	.7	0	0.8	0
30	.2	1	1.0	1

while, from eq (13) and (17),

$$V1-S1\geqq 0 \tag{21}$$

and, from eq (14) and (18),

$$V2-S2\leqq 0. \tag{22}$$

Therefore, the value of the reliability term for scalar forecasts is generally less than that for vector forecasts while the value of the resolution term for scalar forecasts is generally greater than that for vector forecasts. That is, if a collection of forecasts is considered to consist of scalar forecasts, then the collection will appear, in general, to have more reliability and less resolution than if the collection is considered to consist of vector forecasts.

7. SCALAR AND VECTOR PARTITIONS: A SAMPLE COLLECTION OF FORECASTS

To illustrate the differences between (1) the scalar and vector partitions of the RPS and (2) the vector partitions of the RPS and the PS, we consider a sample collection of probability forecasts for a three-state ($N=3$) variable.[11]

[11] The RPS and the PS are equivalent for two-state ($N=2$) variables. (See Murphy 1970, p. 918.)

TABLE 2.—*A sample collection of forecasts and the relevant observations for a three-state* (N=3) *variable when the forecasts and observations are considered to be vectors*

Forecast/ observation number	Forecast	Observation	Cumulative forecast	Cumulative observation
k	$\mathbf{r}_k$	$\mathbf{d}_k$	$\mathbf{R}_k$	$\mathbf{D}_k$
1	(0.1, 0.3, 0.6)	(0, 0, 1)	(0.1, 0.4, 1.0)	(0, 0, 1)
2	(0.1, 0.7, 0.2)	(0, 1, 0)	(0.1, 0.8, 1.0)	(0, 1, 1)
3	(0.3, 0.5, 0.2)	(0, 1, 0)	(0.3, 0.8, 1.0)	(0, 1, 1)
4	(0.5, 0.4, 0.1)	(0, 1, 0)	(0.5, 0.9, 1.0)	(0, 1, 1)
5	(0.7, 0.3, 0.0)	(1, 0, 0)	(0.7, 1.0, 1.0)	(1, 1, 1)
6	(0.6, 0.1, 0.3)	(0, 0, 1)	(0.6, 0.7, 1.0)	(0, 0, 1)
7	(0.5, 0.4, 0.1)	(1, 0, 0)	(0.5, 0.9, 1.0)	(1, 1, 1)
8	(0.1, 0.8, 0.1)	(0, 1, 0)	(0.1, 0.9, 1.0)	(0, 1, 1)
9	(0.1, 0.6, 0.3)	(0, 0, 1)	(0.1, 0.7, 1.0)	(0, 0, 1)
10	(0.1, 0.7, 0.2)	(0, 0, 1)	(0.1, 0.8, 1.0)	(0, 0, 1)

TABLE 3.—*The scalar partition of the RPS for the sample collection of cumulative forecasts and observations presented in table 1*

Subcollection number	Cumulative forecast	Number of forecasts	Cumulative observed relative frequency	Subcollection reliability	Subcollection resolution
s	R^s	M^s	$\overline{D}^s$	$M^s(R^s-\overline{D}^s)^2$	$M^s\overline{D}^s(1-\overline{D}^s)$
1	0.1	5	0.00	0.05	0.00
2	.3	1	.00	.09	.00
3	.4	1	.00	.16	.00
4	.5	2	.50	.00	.50
5	.6	1	.00	.36	.00
6	.7	3	.33	.40(3)	.66(7)
7	.8	3	.67	.05(3)	.66(7)
8	.9	3	1.00	.03	.00
9	1.0	11	1.00	.00	.00
Total		30		1.14(7)	1.83(3)
Average				0.038(2)	0.061(1)

TABLE 4.—*The vector partition of the RPS for the sample collection of cumulative forecasts and observations presented in table 2*

Subcollection number	Cumulative forecast	Number of forecasts	Cumulative observed relative frequency	Subcollection reliability	Subcollection resolution
t	$\mathbf{R}^t$	K^t	$\overline{\mathbf{D}}^t$	$K^t(\mathbf{R}^t-\overline{\mathbf{D}}^t)\times(\mathbf{R}^t-\overline{\mathbf{D}}^t)'$	$K^t\overline{\mathbf{D}}^t(\mathbf{U}-\overline{\mathbf{D}}^t)'$
1	(0.1, 0.4, 1.0)	1	(0.0, 0.0, 1.0)	0.17	0.00
2	(0.1, 0.7, 1.0)	1	(0.0, 0.0, 1.0)	.50	.00
3	(0.1, 0.8, 1.0)	2	(0.0, 0.5, 1.0)	.20	.50
4	(0.1, 0.9, 1.0)	1	(0.0, 1.0, 1.0)	.02	.00
5	(0.3, 0.8, 1.0)	1	(0.0, 1.0, 1.0)	.13	.00
6	(0.5, 0.9, 1.0)	2	(0.5, 1.0, 1.0)	.02	.50
7	(0.6, 0.7, 1.0)	1	(0.0, 0.0, 1.0)	.85	.00
8	(0.7, 1.0, 1.0)	1	(1.0, 1.0, 1.0)	.09	.00
Total		10		1.98	1.00
Average*				0.066	0.033(3)

*This average is computed on the basis of $NK=30$ forecasts (sec. 6).

The forecasts and cumulative forecasts and the relevant observations and cumulative observations are presented in tables 1 and 2 as scalars and vectors, respectively.

Scalar and Vector Partitions of the RPS

The scalar partition of the RPS for these cumulative forecasts is presented in table 3. Note that the values of the terms $S1$ (reliability) and $S2$ (resolution) are 0.038(2) and 0.061(1), respectively, and that their sum [i.e., RPS(R, D)] equals 0.099(3). The vector partition of the

Table 5.—*The vector partition of the PS for the sample collection of forecasts and observations presented in table 2*

Subcollection number	Forecast	Number of forecasts	Observed relative frequency	Subcollection reliability	Subcollection resolution
t	$\mathbf{r}^t$	K^t	$\bar{\mathbf{d}}^t$	$K^t(\mathbf{r}^t-\bar{\mathbf{d}}^t)\times(\mathbf{r}^t-\bar{\mathbf{d}}^t)'$	$K^t\bar{\mathbf{d}}^t(\mathbf{u}-\bar{\mathbf{d}}^t)'$
1	(0.1, 0.3, 0.6)	1	(0.0, 0.0, 1.0)	0.26	0.00
2	(0.1, 0.6, 0.3)	1	(0.0, 0.0, 1.0)	.86	0.00
3	(0.1, 0.7, 0.2)	2	(0.0, 0.5, 0.5)	.28	1.00
4	(0.1, 0.8, 0.1)	1	(0.0, 1.0, 0.0)	.06	0.00
5	(0.3, 0.5, 0.2)	1	(0.0, 1.0, 0.0)	.38	0.00
6	(0.5, 0.4, 0.1)	2	(0.5, 0.5, 0.0)	.04	1.00
7	(0.6, 0.1, 0.3)	1	(0.0, 0.0, 1.0)	.86	0.00
8	(0.7, 0.3, 0.0)	1	(1.0, 0.0, 0.0)	.18	0.00
Total		10		2.92	2.00
Average*				0.097(3)	0.066(7)

*This average is computed on the basis of $NK=30$ forecasts. (See footnote 13.)

RPS for these cumulative forecasts is presented in table 4. Note that the values of the terms $V1$ (reliability) and $V2$ (resolution) are 0.066 and 0.033(3), respectively, and that their sum [i.e., RPS*(**R**, **D**)] also equals 0.099(3). Thus, as indicated in eq (21) and (22), $V1(0.066) \geq S1(0.038)$ and $V2(0.033) \leq S2(0.061)$. Note that, for this collection of cumulative forecasts, the resolution term is almost twice as large as the reliability term according to the scalar partition while the reliability term is twice as large as the resolution term according to the vector partition.[12]

Vector Partitions of the RPS and the PS

The vector partition of the PS for these forecasts is presented in table 5. Note that the values of the reliability and resolution terms are 0.097(3) and 0.066(7), respectively, and that their sum [i.e., PS*(**r**, **d**)] equals 0.164.[13] Since the vector partition of the RPS also provides measures of the reliability and resolution of vector forecasts (sec. 8), a comparison of the *relative* values of the reliability and resolution terms for the RPS and the PS is of some interest.[14] Note that the distribution of the score between these terms indicates that this collection of forecasts has relatively less reliability and more resolution according to the RPS than according to the PS.

8. INTERPRETATION AND USE OF SCALAR AND VECTOR PARTITIONS: DISCUSSION

A number of questions arise in connection with the interpretation and use of the scalar and vector partitions of the RPS. For example: (1) What information, if any, do these partitions provide about the reliability and resolution of forecasts (as opposed to cumulative forecasts)? (2) Which partition should a meteorologist use in forecast evaluation studies? (3) How sensitive are the results of such studies to the particular partition used? (4) What effects, if any, do the differences between scalar and vector cumulative forecasts have upon the sample size of a collection of forecasts? Several of these questions have been discussed in some detail in connection with the partitions of the PS (Murphy 1972*a*, 1972*b*) and, since the conclusions reached in those discussions also appear to be valid for the partitions of the RPS, we consider those questions only briefly in this paper.

With regard to the first question, we indicated in section 2 that a one-to-one correspondence exists between vector and vector cumulative forecasts. Thus, the vector partition of the RPS can be considered to provide measures of the reliability and resolution of vector forecasts as well as vector cumulative forecasts. For example, the subcollection reliability (0.20) and resolution (0.50) for the cumulative forecast $\mathbf{R}^{t*}=(0.1, 0.8, 1.0)$ (table 4) can also be considered to represent the subcollection reliability and resolution for the forecast $\mathbf{r}^t=(0.1, 0.7, 0.2)$.[15] On the other hand, we also indicated that a one-to-one correspondence does not exist between scalar forecasts and scalar cumulative forecasts. Thus, the scalar partition of the RPS does not necessarily provide measures of the reliability and resolution of scalar forecasts. For example, note that the subcollection of three cumulative forecasts for which $R^{s*}=0.8$ (table 3) corresponds to six forecasts from four subcollections, two for which $r^s=0.1$, one for which $r^s=0.3$, one for which $r^s=0.5$, and two for which $r^s=0.7$.

The question of which partition of the PS a meteorologist should use has recently been considered from both a scientific and an economic point of view (Murphy 1972*a*, 1972*b*). We concluded that the vector partition generally appeared to be more appropriate than the scalar partition from both points of view, and we believe that this conclusion is equally valid for the partitions of the RPS. In addition, as indicated above, the vector partition of the RPS provides measures of the reliability and resolution of vector, as well as vector cumulative, forecasts.

With regard to the third question, the scalar and vector partitions of the RPS have not been applied to any large collections of forecasts. Therefore, information relative to the differences between these measures of reliability and resolution for such collections is not presently available. However, as indicated in section 7, the differences between these measures can be substantial for small collections of forecasts.

With regard to the fourth question, consider a situation in which we have a collection of K vector or M $(=NK)$ scalar cumulative forecasts for an N-state variable and suppose that $K=100$, $N=3$, and $r^s=0.0(0.1)1.0$. Then, $M=300$ and S and T, the number of distinct scalar and vector cumulative forecasts, equal 11 and 66 [from eq (3)], respectively. Thus, in the scalar framework, we have

[12] In reality, the scalar and vector partitions of the RPS provide measures of the reliability and resolution of individual cumulative *probabilities* and sets of cumulative probabilities, or cumulative *forecasts*, respectively.

[13] The vector partition of the PS is PS(**r**, **d**), where $\mathrm{PS}(\mathbf{r},\mathbf{d})=(1/K)\sum_t K^t(\mathbf{r}^t-\bar{\mathbf{d}}^t)(\mathbf{r}^t-\bar{\mathbf{d}}^t)'+(1/K)\sum_t K^t\bar{\mathbf{d}}^t(\mathbf{u}-\bar{\mathbf{d}}^t)'(t=1,\ldots,T)$, in which $\bar{\mathbf{d}}^t=(1/K^t)\sum_t \mathbf{d}_k^t(k=1,\ldots,K^t)$ and **u** is a row vector with N elements all equal to one; that is, $\mathbf{u}=(1,\ldots,1)$ (Murphy 1972*b*). For comparative purposes, we denote the vector partition of the PS by PS*(**r**, **d**), where $\mathrm{PS}^*(\mathbf{r},\mathbf{d})=(1/N)\mathrm{PS}(\mathbf{r},\mathbf{d})$. Note that the range of PS*(**r**, **d**) is the closed interval $[0, 2/N]$.

[14] $\mathrm{RPS}^*(\mathbf{R},\mathbf{D})\leq \mathrm{PS}^*(\mathbf{r},\mathbf{d})$ for all collections of forecasts for three-state ($N=3$) variables (Murphy 1970, p. 921).

[15] Note, from eq (4), that $r_n^t=R_n^{t*}-R_{n-1}^{t*}$ $(n=1,\ldots,N;\ R_0^{t*}=0)$.

300 cumulative forecasts in 11 subcollections while in the vector framework we have 100 cumulative forecasts in 66 subcollections. Therefore, for small collections of forecasts, the number of vector cumulative forecasts may not be sufficient to obtain reasonable estimates of the reliability and resolution of certain cumulative forecasts. One possible solution to this problem would be to combine those subcollections that correspond to "adjacent" forecasts with the subcollection that corresponds to the forecast of concern. Such a procedure can be expected to provide reasonable estimates of these attributes for most, if not all, vector cumulative forecasts. We discuss this problem in greater detail for the PS in Murphy (1972*b*).[16]

9. CONCLUSIONS

In this paper, we have described and compared scalar and vector partitions of the RPS. These partitions have been formulated in the same manner as the scalar and vector partitions of the PS recently described by Murphy (1972*a*, 1972*b*). However, since the RPS is defined in terms of cumulative forecasts, the scalar and vector partitions of the RPS provide measures of the reliability and resolution of scalar and vector *cumulative* forecasts, respectively. The scalar and vector partitions of the RPS provide similar, but not equivalent (i.e., linearly related), measures of these attributes. Specifically, the reliability (resolution) of cumulative forecasts according to the scalar partition is equal to or greater (less) than their reliability (resolution) according to the vector partition. We have illustrated the differences between the scalar and vector partitions of the RPS and between the vector partitions of the RPS and the PS in terms of a sample collection of forecasts for a three-state ($N=3$) variable.

We have briefly discussed several questions related to the interpretation and use of the scalar and vector partitions of the RPS. In particular, we have indicated that, since a one-to-one correspondence exists between vector and vector cumulative forecasts, the vector partition of the RPS can also be considered to provide measures of the reliability and resolution of vector forecasts and that the vector partition appears to be more appropriate than the scalar partition from both a scientific and an economic point of view. Thus, the vector partition of the RPS will be of particular interest to meteorologists who are concerned with the reliability and resolution of probability forecasts of ordered variables.

APPENDIX

We present the expression for the difference between the terms $S13$ and $V13$ in the scalar and vector partitions of the RPS, respectively. This expression is obtained in the same manner as the expression for the difference between the corresponding terms in the scalar and vector partitions of the PS (Murphy 1972*b*).

TABLE 6.—*The difference between the terms S13 and V13 in the scalar and vector partitions of the RPS, respectively, for the cumulative forecasts presented in table 1. [See eq (23).]*

s	R^s	M^s	T^s	$K^{t,s}$	$N^{t,s}$	$\overline{D}_n^{t,s}$	$D1(s)$	$D2(s)$
1	0.1	5	4	1, 1, 2, 1	1, 1, 1, 1	0.0, 0.0, 0.0, 0.0	0.00	0.00
2	.3	1	1	1	1	0.0	.00	.00
3	.4	1	1	1	1	0.0	.00	.00
4	.5	2	1	1	1	0.5	.00	.00
5	.6	1	1	1	1	0.0	.00	.00
6	.7	3	3	1, 1, 1	1, 1, 1	0.0, 0.0, 1.0	.00	.66(7)
7	.8	3	2	2, 1	1, 1	0.5, 1.0	.00	.16(7)
8	.9	3	2	1, 2	1, 1	1.0, 1.0	.00	.00
9	1.0	11	8	1, 1, 2, 1, 1, 2, 1, 1	1, 1, 1, 1, 1, 1, 1, 2	1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0	.00	.00
Total		30					0.00	0.83(3)
Average							0.00	0.027(8)

Let T^s denote the number of distinct cumulative forecasts, $\mathbf{R}^t$, in the collection of vector cumulative forecasts of concern for which $R_n^t=R^s$ for some n ($t=1, \ldots, T^s$); let $N^{t,s}$ denote the number of states in $\mathbf{R}^t$ for which $R_n^t=R^s$ ($n=1, \ldots, N^{t,s}$); let $K^{t,s}$ denote the number of forecasts, $\mathbf{R}_k$, in the subcollection of K^t vector cumulative forecasts for which $\mathbf{R}_k=\mathbf{R}^t$ ($k=1, \ldots, K^{t,s}$), in which $R_n^t=R^s$ for some n; and let $\mathbf{D}_k^{t,s}$ denote an arbitrary observation in the relevant subcollection of $K^{t,s}$ vector cumulative observations in which $\mathbf{D}_k^{t,s}=(D_{1k}^{t,s}, \ldots, D_{Nk}^{t,s})$ ($k=1, \ldots, K^{t,s}$). Then, the difference between the terms $S13$ and $V13$ can be expressed as

$$V13-S13=\frac{1}{M}\Bigg[\sum_{s=1}^{S}\frac{1}{M^s}\sum_{t=1}^{T}(K^{t,s})^2\sum_{n=1}^{N^{t,s}-1}\sum_{n'=n+1}^{N^{t,s}}(\overline{D}_n^{t,s}-\overline{D}_{n'}^{t,s})^2$$
$$+\sum_{s=1}^{S}\frac{1}{M^s}\sum_{t=1}^{T^s-1}\sum_{t'=t+1}^{T^s}K^{t,s}K^{t',s}$$
$$\times\sum_{n=1}^{N^{t,s}}\sum_{n'=1}^{N^{t',s}}(\overline{D}_n^{t,s}-\overline{D}_{n'}^{t',s})^2\Bigg], \tag{23}$$

where

$$\overline{D}_n^{t,s}=\frac{1}{K^{t,s}}\sum_{k=1}^{K^{t,s}}D_{nk}^{t,s}.$$

Note, from eq (23), that

$$V13-S13\geq 0. \tag{24}$$

Further, note that equality holds in eq (24) only if either $N^{t,s}=1$ or $\overline{D}_n^{t,s}=\overline{D}_{n'}^{t,s}$ for all n, n', t, and s in $D1$, the first term on the RHS of eq (23), *and* if $\overline{D}_n^{t,s}=\overline{D}_{n'}^{t',s}$ for all n, n', t, t', and s in $D2$, the second term on the RHS of eq (23).

In table 6, we present the computation of the difference between the terms $S13$ and $V13$, as expressed in eq (23), for the collection of forecasts presented in tables 1 and 2. For example, for $s=6$, $D1(6)=0.00$ (since $N^{t,s}=1$ for $t=1$, 2, and 3) and $D2(6)=(1/3)[(1)(1)(0.0-0.0)^2+(1)(1)(0.0-1.0)^2+(1)(1)(0.0-1.0)^2]=0.66(7)$. Note that, as indicated in section 7, $V13-S13=0.027(8)$ (cf. tables 3 and 4).

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[16] The appropriate framework within which to depict vector forecasts and observations is a regular ($N-1$)-dimensional simplex (Pontryagin 1952, pp. 10-12, Murphy 1972*a*, 1972*b*); the appropriate framework within which to depict vector cumulative forecasts and observations is an N-dimensional unit hypercube.

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