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Numerical Simulation of Precipitation Development in Supercooled Cumuli—Part I

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ABSTRACT—Based on numerical experiments in droplet collection with a stochastic model similar to Berry's, a new quantitative definition of autoconversion is discussed. The new formulation of autoconversion is compared with Kessler's and with Berry's. The new formulation has the decisive advantage over Berry's model of being directly compatible with Kessler's accretion model.

1. INTRODUCTION

Recent numerical experiments in cumulus convection by Das (1964), Takeda (1966), Srivastava (1967), Weinstein and Davis (1968), Weinstein (1968), Liu and Orville (1969), and Simpson and Wiggert (1969) have demonstrated quite clearly the importance of the interaction between dynamical and microphysical processes in individual cumulus clouds.

This investigation has as its objective the development of a microphysical system capable of simulating the dominant processes of precipitation formation in supercooled cumuli as well as the interaction between the microphysical system and cloud dynamics. Because the processes of collision and coalescence of liquid droplets (warm-cloud processes) play an important role in setting the stage for the microphysical interactions in supercooled portions of cumuli, these processes will first be considered here. Symbols used in this paper are defined in table 1.

TABLE 1.—*Definitions of symbols*

A	geometric cross-section defined in eq (9)
a', h', k'	coefficients in eq (17)
D_0	initial cloud droplet radius dispersion
$dM/dt\vert_{auto}$	rate of formation of rainwater by autoconversion
$dM/dt\vert_{Kess}$	rate of formation of rainwater by Kessler accretion model
$dM/dt\vert_T$	total rate of rainwater formation
$E(x\vert x')$	collection efficiency relative to drops of mass x and x'
$F(J)$	computational spectral density function
$f(r)$	concentration of cloud droplets of radius r to $r\pm dr/2$
J_{gain}, J_{loss}	collection integrals defined in eq (6) and (7)
M	rainwater density or liquid water content
m	cloud water density or water content
N_o	initial cloud droplet concentration
N_0	intercept coefficient in eq (2)
$N(D)$	concentration of raindrops of diameter $D\pm\delta D/2$
$q(x)$	concentration of cloud droplets of mass x to $x\pm dx/2$
r	cloud droplet radius
$\bar{r}$	average cloud droplet radius
t	time
$V(x\vert x')$	collection kernel relative to droplets of mass x and x'
v_x	terminal velocity of droplet of mass x
x	mass of cloud droplets
$\bar{x}$	mean mass of cloud droplets
α, β	coefficients in eq (10)
λ	slope parameter in eq (2)
ν_r	cloud distribution radius dispersion
σ	standard deviation of cloud droplet distribution

2. "WARM"-CLOUD PRECIPITATION MECHANISMS

It is assumed that all water vapor in excess of the saturation mixing ratio with respect to water is immediately condensed out. Aside from associated thermodynamic effects, it is furthermore assumed that the primary role of condensation processes is to "shape" the cloud droplet spectra by nucleation of, or vapor condensation on, existing active aerosol particles. This process of distribution shaping is visualized as taking place within several hundred meters above cloud base and, thereafter, a relatively constant cloud droplet concentration and radius dispersion is maintained until the rate of collision between cloud droplets becomes significant.

The collision and coalescence process of precipitation formation is treated by a simple parameterization procedure under the assumption that the rates of production of precipitation are determined solely by the initial condensed cloud droplet spectra and the rate of production of condensed liquid water.

3. PARAMETERIZED WARM-CLOUD MICROPHYSICS

The so-called parameterization approach to modeling of microphysics was spearheaded by the work of Kessler (1967). According to his argument, various measurements have demonstrated that precipitation generation in warm clouds is normally associated with liquid water contents, m, greater than 1.0 g·m^{-3}. He then hypothesized a model describing the rate of conversion of cloud water to precipitation water as a linear function of m. Thus,

$$\frac{dM}{dt}=-\frac{dm}{dt}=K_1(m-a)\begin{cases}K_1>0 \text{ when } m>a\\ K_1=0 \text{ when } m\leq a\end{cases} \qquad (1)$$

where M is the precipitation-water content (g·m^{-3}) and a is the threshold value below which cloud conversion does not occur.

Kessler further hypothesized that the water converted to precipitation-sized droplets is size distributed in the inverse exponential distribution formulated by Marshall and Palmer (1948). Thus,

$$N(D)=N_0e^{-\lambda D} \tag{2}$$

where $N(D)$ represents the number of droplets per unit volume of diameter $D\pm\delta D/2$.

Once these large precipitation particles have formed, they can grow very rapidly by accretion of liquid water content (LWC). Kessler derived the following equation describing the accretion rate of cloud water by rainwater:

$$\left.\frac{dM}{dt}\right|_{\text{accr}} =\frac{130\pi}{4}\left[\frac{\pi\rho_l\Gamma(4)}{6}\right]^{-0.875} N_0^{0.125}\overline{E(D|C)}\Gamma(3.5)mM^{0.875} \tag{3}$$

where ρ_l is the density of water and $\overline{E(D|C)}$ represents an average collection efficiency between the rainwater distribution and cloud droplets.

The question now arises, how well do eq (3) and (1) simulate the complex process of cloud droplet collection? The more sensitive equation is, of course, eq (1). Whether or not precipitation is generated at all in cumulus clouds is determined by the conversion rate. It is well known that the rate of initial formation of precipitation is considerably different depending on whether the clouds are of tropical-maritime origin or of continental origin. Adjusting the coefficients in eq (1) does not provide us with the flexibility necessary to simulate the extreme differences in the colloidal stability of clouds formed in these different air masses.

A reasonable approach to the autoconversion simulation of the early stages of cloud droplet collection was formulated by Berry (1968). Based on his numerical experiments on cloud droplet collection (Berry 1965), he developed an autoconversion formula of the form

$$\frac{dM}{dt}=\frac{m^2}{60\left[2+\left(\frac{0.0266}{D_0}\right)\right]\frac{N_o}{m}} \qquad (\text{g}\cdot\text{m}^{-3}\cdot\text{s}^{-1}) \tag{4}$$

where D_0 is the initial cloud droplet radius dispersion and N_o is the concentration (cm^{-3}).

Equation (4) was devised by calculating the flux of water passing a minimum precipitation droplet size of $r=40\mu$m. These calculations were later repeated for a minimum precipitation droplet size of 100-μm radius (Simpson and Wiggert 1969). The form of eq (4) remained unchanged, whereas the values of the coefficients were modified slightly. Qualitatively, at least, eq (4) is much more satisfying than Kessler's formulation. The autoconversion rate is roughly proportional to the cube of the LWC, proportional to the initial droplet dispersion, and inversely proportional to the initial cloud droplet concentration. Since maritime air masses generally produce cumulus clouds with initial droplet concentrations of 50–100 cm^{-3} whereas continental cumuli exhibit concentrations of 300–500 cm^{-3}, eq (4) provides us with the capability of differentiating between these air masses. Similarly, the effect of the radius dispersion provides an additional distinguishing parameter. Unfortunately, numerical experiments with eq (4) or the 100-μm radius cutoff version (Simpson and Wiggert 1969) in the cumulus-dynamics model described by Weinstein and Davis (1968) demonstrate that the equation appears to overestimate the autoconversion rate. That is, higher rainwater contents are generated than are deduced from radar evaluations or than would be expected from clouds of a given size and duration.

One possible qualitative explanation for the discrepancy between eq (4) and observation is that neither eq (4) nor (1) specifies any time dependence for the rate of autoconversion whereas experience has shown that, regardless of the LWC, initial droplet dispersion, concentration, etc., a certain "aging" time is required before the distribution can be expected to form precipitation particles. In fact, numerical experiments in cloud droplet collection by Berry (1965) also demonstrate this period of "quiescence" before precipitation forms. It was, therefore, decided to repeat Berry's numerical collection experiments to determine a parameterization more applicable to cloud modeling.

4. CLOUD-DROPLET COLLECTION MODEL

The basic theoretical formulation and numerical framework of the so-called stochastic collection model is identical with that described by Berry (1965). Thus, if $f(x)$ represents the concentration (number per unit volume) of droplets of mass x to $x+dx$, then the rate of change of $f(x)$ due to cloud droplet collection is

$$\left.\frac{\partial f(x)}{\partial t}\right|_{\text{coll}}=J_{\text{gain}}-J_{\text{loss}} \tag{5}$$

where J_{gain} represents the rate of change in $f(x)$ due to droplet collisions forming a droplet of mass x and J_{loss} represents the rate of change in $f(x)$ due to droplet collisions destroying a droplet of mass x.

The collection integrals are defined as follows:

$$J_{\text{gain}}=\int_{x_0}^{x/2} dx'f(x_c)V(x_c|x')f(x') \tag{6}$$

and

$$J_{\text{loss}}=\int_{x_0}^{\infty} dx'f(x)V(x|x')f(x') \tag{7}$$

where $x_c=x-x'$ and x_o is the mass of the smallest droplet considered.

The function, $V(x|x')$, is what Berry (1965) calls the collection kernel or Twomey (1964) describes as the coagulation coefficient. Generally, the coefficient is defined in terms of geometric volume sweep-out rate relative to two droplets of mass x and x'. Strictly speaking, this geometric definition is not necessary and, in fact, is quite unrealistic when applied to collisions between droplets having small relative velocities or experiencing strong, nongravitationally induced forces such as electrical ones. Fortunately, it appears that the dominant mechanisms of cloud droplet collision can be described in terms of the geometrically defined collection kernel, which is

$$V(x|x')=\pi r_x^2E(x|x')(v_x-v_{x'}) \tag{8}$$

where r_x is the radius of a droplet of mass x and v_x and $v_{x'}$ are the terminal velocities of droplets of mass x and x', respectively. The coefficient, $E(x|x')$, is the well-known collection efficiency defined appropriate to the geometric cross-section,

$$A=\pi r_x^2. \tag{9}$$

The computations described here employ the approximations to Shafrir and Neiberger's collision efficiencies formulated by Berry (1965) for collector droplets greater than 30 μm in radius and the collision efficiencies calculated by Davis and Sartor (1967) for the collector droplets less than 30 μm in radius. The coalescence efficiency is taken to be unity.

To complete specification of the collection kernel, we calculated the terminal velocity of cloud droplets less than 50 μm in radius under the assumption that they are spherical. The empirical expression found by Foote and DuToit (1969) to represent the Gunn and Kinzer (1949) data is used for larger droplet sizes.

Given a proper specification of the collection kernal and of the initial distribution, we can solve eq (5) numerically. Because numerical integration schemes are extremely sensitive to small-scale perturbations present in raw data, we decided to approximate "observed" droplet distributions by a smooth, unimodal distribution-generating function. The initial droplet distribution is therefore taken to be a gamma distribution of the form

$$f(r)=\frac{N_o r^{\alpha-1}e^{-r/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \qquad 0<r<\infty \tag{10}$$

and N_o is the total concentration.

The mean and variance of such a distribution can be shown to be $\bar{r}=\alpha\beta$ and $\sigma^2=\alpha\beta^2$ (Hogg and Craig 1965). Therefore, the radius dispersion specifies the parameter α by

$$\nu_r=\frac{\sigma}{\bar{r}}=\alpha^{-1/2}. \tag{11}$$

The coefficient β can be found from the mean radius or, better still, from the mean mass of the distribution. If eq (10) is transformed into a mass distribution such that

$$q(x)=f(r)\left|\frac{dr}{dx}\right|=\frac{N_o\left(\frac{3}{4\pi}\right)^{\frac{\alpha}{3}-1}x^{\frac{\alpha}{3}-1}e^{\frac{-(3x/4\pi)^{1/3}}{\beta}}}{4\pi\Gamma(\alpha)\beta^{\alpha}} \tag{12}$$

with

$$0<x<\infty,$$

then the mean mass of the distribution is

$$\bar{x}=\frac{m}{N_o}=\frac{1}{N_o}\int_0^{\infty}xq(x)dx=\frac{4\pi\beta^3(\alpha+2)(\alpha+1)\alpha}{3} \tag{13}$$

where m is the total water content. Thus,

$$\beta=\left[\frac{3\bar{x}}{4\pi\alpha(\alpha+1)(\alpha+2)}\right]^{1/3}. \tag{14}$$

If we know the radius dispersion, droplet concentration, and LWC, then eq (10) completely specifies the distribution. Levin (1954) has shown that gamma distributions are good approximations to natural distributions. The above form of gamma distribution approximates actual distributions even better, since the coefficients α and β are defined in terms of the first three moments of the observed distributions. With the initial distribution so defined, eq (5) is solved numerically with the transformed computational spectral density function described by Berry (1965). The new value of each discrete spectral function at time $t+\Delta t$ is predicted by using a simple, first-order time integration.

5. QUANTITATIVE DEFINITION OF AUTOCONVERSION

The major purpose of the following numerical collection experiments was to develop an improved parameterization of the cloud droplet collection process. It would be especially convenient to formulate a parameterization compatible with Kessler's accretion rate [eq (3)]. Thus, if $dM/dt|_T$ represents the total change in hydrometeor water content due to collection and $dM/dt|_{\text{Kess}}$ represents the change predicted by eq (3), then the autoconversion rate, $dM/dt|_{\text{auto}}$, can be defined as

$$\left.\frac{dM}{dt}\right|_{\text{auto}}=\left.\frac{dM}{dt}\right|_T-\left.\frac{dM}{dt}\right|_{\text{Kess}}. \tag{15}$$

The numerical procedure is then to compute the change in droplet spectra with eq (5). At the end of each time step, the total change in hydrometeor water content, ΔM_T, is computed, where

$$M_T=\int_{x_r=100\mu m}^{x_{\max}}xf(x)dx. \tag{16}$$

The rate of autoconversion is then calculated by means of eq (15) with the approximation that the differential change in total rainwater content is equal to the finite-difference calculated value.

6. RESULTS OF NUMERICAL COLLECTION EXPERIMENTS

The results of the numerical cloud droplet collection experiments are displayed with the droplet mass density function described by Berry (1965). Two sets of numerical experiments were performed. The first set represents a maritime-tropical cloud having an initial concentration of 100 droplets per cubic centimeter (100 cm^{-3}) and a radius dispersion of 0.25. The second set represents a cloud formed in a moderate continental air mass having an initial concentration of 300 cm^{-3} and a radius dispersion of 0.25.

The results of the experiments with maritime-tropical conditions are displayed in figures 1A–1C, representing total water contents of 2.0, 1.5, and 1.0 g·m^{-3}. The experiments with 2.0- and 1.5-g·m^{-3} water contents are characterized by the rapid motion of a pronounced "wall" of water through log-radius space in the droplet size range 20 μm $< r <$ 100 μm. Once the wall of water passes the

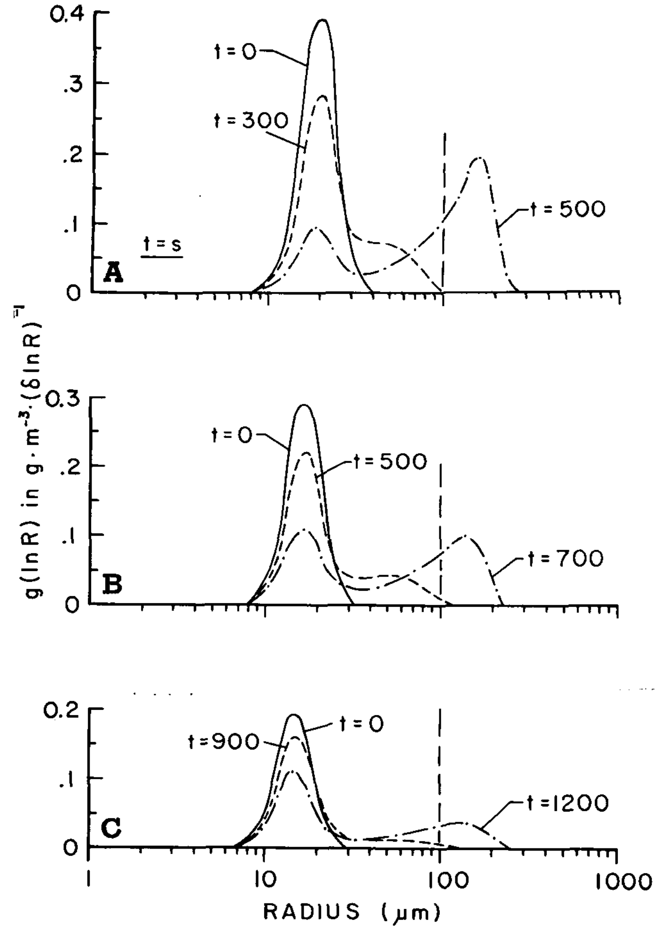

FIGURE 1.—Computed variation in the distribution of water mass density for an initial concentration of 100 cm^{-3}, a radius dispersion of 0.25, and LWCs of (A) 2.0, (B) 1.5, and (C) 1.0 $g \cdot m^{-3}$.

minimum-sized precipitation particle (100-μm radius), it grows quite rapidly in amplitude, eventually destroying the initial water-content mode. This "mature" stage of precipitation development, which occurs after 400 s at 2.0 $g \cdot m^{-3}$ and after 600 s at 1.5 $g \cdot m^{-3}$ water contents, respectively, cannot be considered a truly physical modeling because of the lack of precipitation advection. That is, once precipitation-sized particles are formed, the lack of simulation of precipitation particle advection into or out of the closed system makes the computations quite unrealistic. Reducing the LWC to 1.0 $g \cdot m^{-3}$ or 0.85 $g \cdot m^{-3}$ (not shown) produces a reduction in the relative intensity of the pronounced wall of water. Instead, a shallow "wave front" of water moves through log-radius space in the size range 20 μm $< r <$ 100 μm, eventually building a secondary mode of water after about 1,200 s (at 1.0 $g \cdot m^{-3}$ LWC).

Figure 2 illustrates the calculated autoconversion rates (solid lines) for LWCs of 2.0, 1.5, and 1.0 $g \cdot m^{-3}$. The curves describe a very sharp rise in the autoconversion rate, reaching an apex whose height and position in time space is determined by the LWC. The rapid "decay" of the autoconversion rate beyond the apex point is indicative of the ability of the Kessler accretion model [eq (3)] to describe the total transport of water to precipitation-sized droplets in the mature stage of precipitation formation.

Figures 3A–3C illustrate the results of a similar set of numerical experiments but with an initial concentration of 300 cm^{-3} and a radius dispersion of 0.25. Figure 3A demonstrates many features common both in relative intensity of the water density in log-radius space and in time, to figure 1B. Similarly, figures 3B and 1C

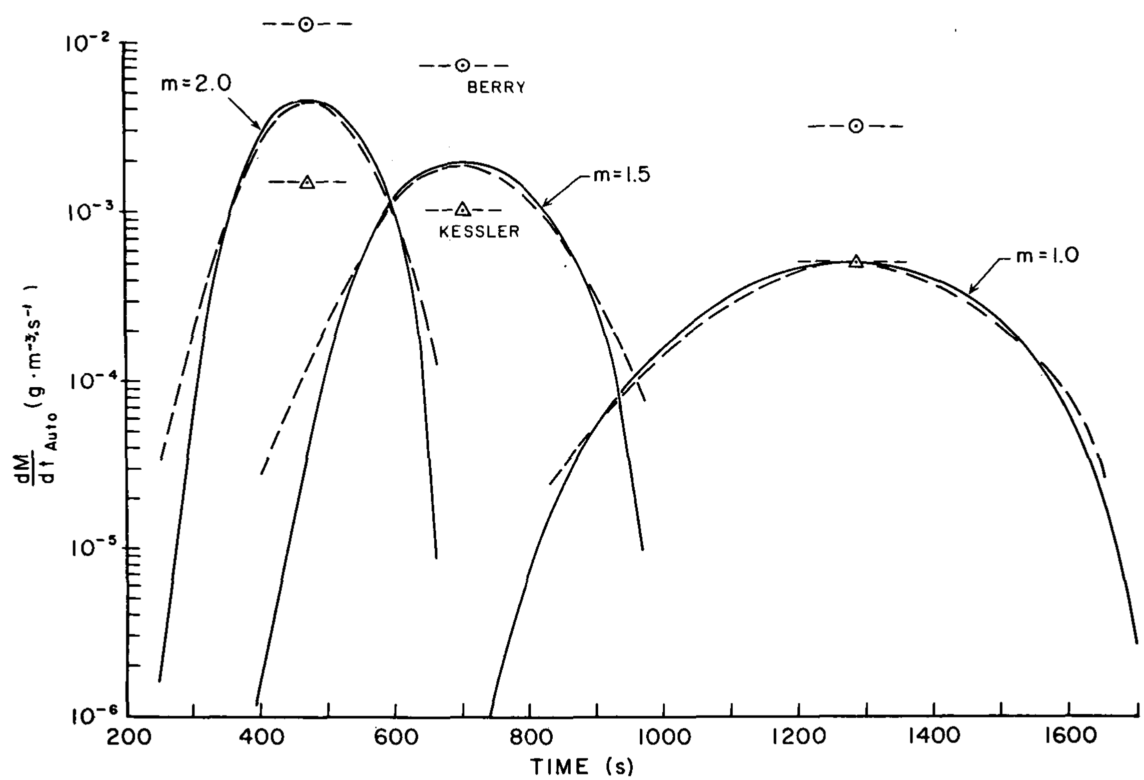

FIGURE 2.—Calculated autoconversion rates (solid lines) and regression formulation curves (dashed lines) for $N_o = 100$ cm^{-3}, $\nu_r = 0.25$, and LWCs of 2.0, 1.5, and 1.0 $g \cdot m^{-3}$. Also shown are the calculated values using Kessler's and Berry's formulations.

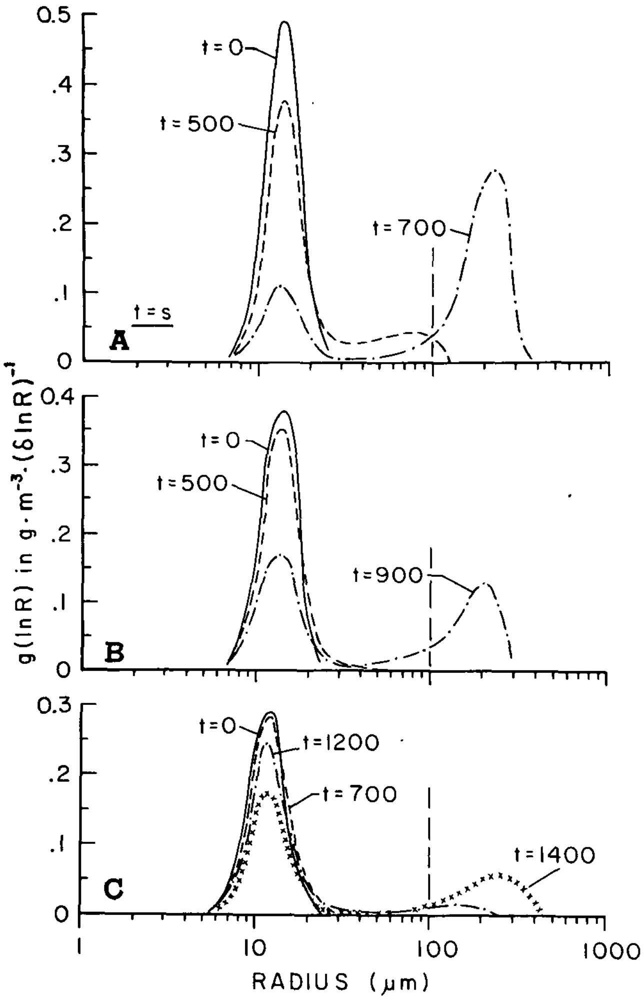

FIGURE 3.—Computed variation in the distribution of water mass density for $N_o=300$ cm^{-3}, $\nu_r=0.25$, and LWCs of (A) 2.5, (B) 2.0, and (C) 1.0 g·m^{-3}.

illustrate many common features; most notable is the lack of the pronounced water-density wall moving through log-radius space. In other words, as a distribution becomes colloidally more stable, the nature of the precipitation development process is transformed from one dominated by the rapid motion of a relative peak in water density (in the size range 20 μm$<r<$100 μm) to the slow motion of a thin band or wave front of LWC that eventually gives rise to a secondary mode of water density. This distinction is particularly important with regard to the parameterization of the collection process.

Figures 2 and 4 suggest that the computed autoconversion rates can be approximated by a family of parabolas on semilog plot. The family of parabolas can be described with an equation of the form

$$\frac{dM}{dt_{\text{Auto}}}=\exp\left[k'-\frac{1}{4a'}(t-h')^2\right] \tag{17}$$

where t represents the age of a parcel of droplets and k', a', and h' are coefficients, all of which are functions of the water content for a given initial concentration and *dispersion. Each curve in the family of parabolas is fitted* to eq (17), using the least-squares nonlinear regression model developed by Marquardt (1963). The source data for the regression model was computed with eq (15) following each 10-s integration time step in the numerical calculations. The results are summarized as follows:

When $N_o=100$ cm^{-3} and $\nu_r=0.25$,

$$h'=e^{7.13}m^{-1.44},$$

$$k'=-e^{2.001}m^{-0.478},$$

and

$$a'=e^{9.63}m^{-2.59}. \tag{18}$$

When $N_o=300$ cm^{-3} and $\nu_r=0.25$,

$$h'=e^{6.548}m^{-1.75},$$

$$k'=-e^{2.46}m^{-0.779},$$

and

$$a'=15.6\times10^4-4.8\times10^4m. \tag{19}$$

Here, m is the LWC in g·m^{-3}.

The autoconversion rates predicted with eq (17) are shown in figures 2 and 4 (dashed lines). The coefficients h' and k', which represent the vertex coordinates of the parabola, vary exponentially with the water content, m, for initial concentrations of 100 and 300 cm^{-3}. However, the coefficient a' changes from an exponential dependence on m when N_o is 100 cm^{-3} to a linear dependence on m when N_o is 300 cm^{-3}. In fact, the change in the character of the collection process from an unstable one to a stable one is further emphasized by the change in the behavior of the a' coefficient.

Additional numerical experiments with colloidally stable distributions all illustrate a change in the character of the collection process. By reducing the LWC to 1.0 g·m^{-3} for N_o equal to 300 cm^{-3} and to 0.75 g·m^{-3} for N_o equal to 100 cm^{-3}, one can see a reversal in trend in the variation of a'. That is, a' begins increasing with decreasing m. A similar behavior could be demonstrated by increasing the concentration about a given LWC. These results imply that, when a cloud distribution is stabilized to the degree that precipitation can best be described as drizzle, the formulation of the autoconversion rate by eq (17) is no longer justified. It is apparent that only the slightest perturbation (in the form of autoconversion) is then necessary for the precipitation process to be subsequently dominated by accretion. Fortunately, the conditions under which the above parameterization technique is least reliable are rarely encountered in cumuli.

In summary, the calculated autoconversion curves should well simulate the collection process in cumulus clouds when used in conjunction with the Kessler accretion model as described in eq (3). The calculated autoconversion curves have been fitted to eq (17). This equation with the coefficients defined by eq (18) and (19), which represent the coefficient behavior for two specific initial conditions, fitted the calculated curves well, particularly in the region of the maximum autoconversion. The greatest error in the fitted curves occurs at small LWCs. This maximum error appears, however, in the early stages of conversion when the conversion rate is at least an order of magnitude below its peak value. Equation (17) and the corresponding equations describing the be-

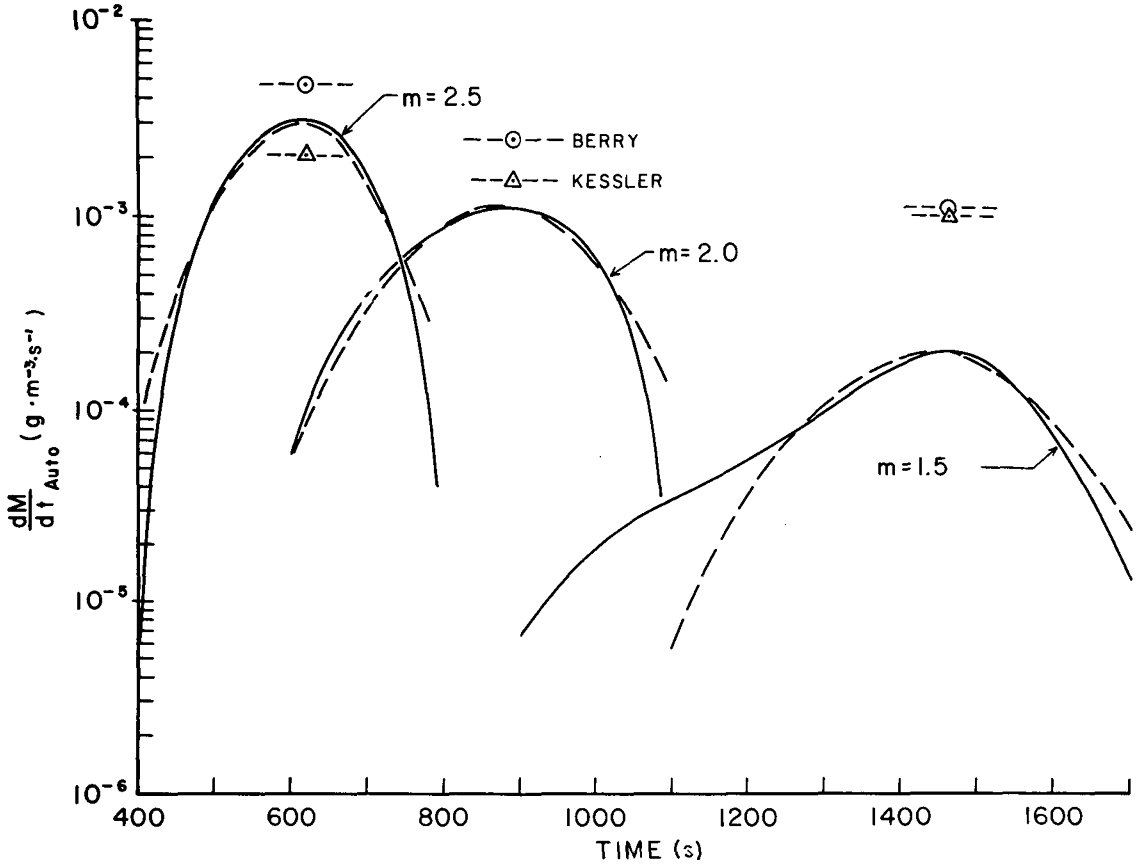

FIGURE 4.—Calculated autoconversion rates (solid lines) and regression formulation curves (dashed lines) for $N_o = 300$ cm^{-3}, $\nu_r = 0.25$, and LWCs of 2.5, 2.0, and 1.5 g·m^{-3}.

havior of its coefficients by no means represent a general formulation of the conversion process. These results imply the merit of continuation of this technique of defining autoconversion to determine a general formulation for cumulus clouds. Again, one must be cautious with regard to the general formulation, particularly in polluted air masses when the distribution is stabilized to a high degree.

7. COMPARISON OF NEW FORMULATION WITH PREVIOUS AUTOCONVERSION FORMULATIONS

In addition to the calculated autoconversion rates and the parabolas fitted by regression, figures 2 and 4 also contain the predicted autoconversion rates based on Kessler's and Berry's formulations. Because neither Kessler's nor Berry's formulation involves any time dependence in the autoconversion process, these are simply indicated above the vertex points of the parabolas. If one wishes to use a simple formulation independent of time to parameterize the conversion process, one would expect it to have a value that is nearly the average of the time-dependent conversion. Looking at figure 2 for $N_O = 100$ cm^{-3}, we find that Kessler's simpler formulation better approximates this condition. However, as evidenced in figure 2, Kessler's formulation produces the largest error at smaller LWCs. In spite of its simplicity both in formulation and development, Kessler's model of conversion is a better approximation to my time-dependent calculations based on the Berry (1965) model than is Berry's model. This is particularly true if the autoconversion is used in conjunction with Kessler's accretion model [eq (3)].

The fact that Berry's formulation tends to overpredict in spite of its lack of time dependence suggests that his formulation does, in fact, simulate the full mass transport (accretion plus conversion) rather than conversion only. Therefore, Berry's formulation should be used with considerable caution.

8. CONCLUSIONS

The new technique of calculating autoconversion outlined above has the distinct advantage that, in addition to being time dependent, it is directly compatible with Kessler's accretion model. The numerical experiment having N_O equal to 100 cm^{-3} and the radius dispersion equal to 0.25 indicated that Kessler's autoconversion is a reasonably good approximation for maritime clouds. However, figures 2 and 4 suggest that autoconversion is, in general, a nonlinear function of cloud water content. Thus, the extension of the Kessler autoconversion formulation to continental clouds may introduce considerable error.

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