NOTES AND CORRESPONDENCE

On Fourier Filtering Near the Poles

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1. Introduction

In a recent paper by Holloway *et al.* (1973) the authors state "that it is desirable to transform vector components to a polar stereographic projection before Fourier filtering is performed." The purpose of this paper is to demonstrate that truncating the transformed variables beyond component N is equivalent to the complete filtering of the original variables beyond component $N+1$ and the partial filtering of components N, $N+1$. Further we shall show that this filtering process has an effect on the maximum time step permitted in a calculation in the sense that the limiting time step for component N is increased by a factor of $\sqrt{2}$.

2. The filtering process

If we define

$$W_S = U_S + iV_S \quad (1)$$

$$W = u + iv \quad (2)$$

where U_S, V_S are the stereographic components of the horizontal wind; u, v are the zonal and meridional components, and $i=\sqrt{-1}$, then the transformations relating W_S and W are

$$W_S = iWe^{i\lambda} \quad (3)$$

$$W = -iW_S e^{-i\lambda} \quad (4)$$

where λ is the longitude.

Then if $A(n)$ represents the Nth coefficient of the Fourier transform of A, namely

$$A(n) = \frac{1}{2\pi}\int_0^{2\pi} Ae^{-in\lambda} d\lambda, \quad (5)$$

it follows that the Fourier transforms of W_S and W are related by

$$W_S(n) = iW(n-1), \quad (6)$$

$$W(n) = -iW_S(n+1). \quad (7)$$

Now let $W_S|_N$ be the stereographic variable W_S whose Fourier series is truncated beyond wavenumber N. Then it follows that

$$W_S|_N = \sum_{n=-N}^{N} W_S(n)e^{in\lambda} \quad (8)$$

and in view of Eq. (4), and using (6), we obtain

$$W|_N = \sum_{m=-(N+1)}^{N-1} W(m)e^{im\lambda}, \quad (9)$$

which is the corresponding variable in spherical coordinates.

If, on the other hand we had truncated the original variable beyond wavenumber $N+1$, then

$$(W|_{N+1}) = \sum_{m=-(N+1)}^{N-1} W(m)e^{im\lambda} + W_N e^{iN\lambda} + W_{N+1}e^{i(N+1)\lambda}. \quad (10)$$

Comparing (9) and (10), we note that employing the process described by (9) rather than (10) produces a partial filtering of components N and $N+1$ as well as a phase shift of these components.

It is clear that the transformation to stereographic components could be accomplished by use of (9), but it it not obvious what additional benefits are derived from this process over a direct truncation as represented by (10). Nevertheless, Holloway (private communication) found that in one particular test run the time step had to be reduced from 600 to 400 sec when the stereographic transformations were removed. This appears to be somewhat paradoxical, since the variables which are stereographically transformed, truncated, and transformed back to spherical coordinates contain shorter wavelengths than those merely truncated at the given wavenumber. In what follows we shall show by a simple

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analysis that the stereographic transformations do indeed permit a longer time step, by effectively reducing the frequency of the components N and $N+1$.

3. The model

In order to proceed with the demonstration it is not necessary to consider a spherical coordinate system, nor is it necessary to work with a grid. In fact, the essence of the demonstration can be accomplished by considering the *small* oscillations of an incompressible fluid whose mean height is H. Further, we may consider the motion to take place in a plane region and to be periodic in both directions. We shall imagine that x corresponds to longitude, the direction in which we will perform filtering and y corresponds to latitude, with the corresponding wind components u and v. The deviations of the geopotential height from its mean ($\bar{\phi}=gH$) is denoted by ϕ, and the linearized equations governing the variations are then

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=-\frac{\partial \phi}{\partial x}\\ \frac{\partial v}{\partial t}&=-\frac{\partial \phi}{\partial y}\\ \frac{\partial \phi}{\partial t}&=-\bar{\phi}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\end{aligned}\right\}. \tag{11}$$

Now if our periodic domain is of length 2π in both directions and since the problem is linear, we may consider solutions of (11) having the following form,

$$(u,v,\phi)=[u(n),v(n),\phi(n)]e^{i(nx+ly)} \tag{12}$$

where l, n are integers. We retain the l as a parameter, but otherwise will not use it; as we will filter in the x-direction only.

Substituting (12) into (11), we obtain

$$\left.\begin{aligned}\frac{du(n)}{dt}&=-in\phi(n)\\ \frac{dv(n)}{dt}&=-il\phi(n)\\ \frac{d\phi(n)}{dt}&=-i\bar{\phi}[nu(n)+lv(n)]\end{aligned}\right\}. \tag{13}$$

Before proceeding it should be reemphasized that the complete solution of the problem is obtained by summing over those wavenumbers permitted on some grid that we may have defined. In a latitude-longitude grid, the spacing of grid points becomes very small near the poles, resulting in a restrictively small time step to ensure stability. In the above analogous problem we will filter out high wavenumber components in order to avoid small time steps.

4. The time integration and filtering

Let us now integrate Eqs. (15)–(17) by use of a leapfrog scheme. Then if A^t denotes the value of A at time step t we have,

$$\left.\begin{aligned}\hat{u}(n)^{t+1}&=u(n)^{t-1}-in\phi(n)^t2\Delta t\\ \hat{v}(n)^{t+1}&=v(n)^{t-1}-il\phi(n)^t2\Delta t\\ \hat{\phi}(n)^{t+1}&=\phi(n)^{t-1}-i\bar{\phi}[nu(n)^t+lv(n)^t]2\Delta t\end{aligned}\right\}, \tag{14}$$

where the "hat" indicates that we will subsequently filter the forecast variables.

If we consider the direct filtering of the variables beyond wavenumber N, then we have as the filtering step

$$[u(n),v(n),\phi(n)]^{t+1}=R(n)[\hat{u}(n),\hat{v}(n),\hat{\phi}(n)]^{t+1} \tag{15}$$

where

$$\begin{aligned}R(n)&=1 \quad \text{if } |n|\le N\\ R(n)&=0 \quad \text{otherwise.}\end{aligned} \tag{16}$$

It is apparent then that our criterion for stability will be determined from consideration of solution of (14) when $n=N$. It can be shown that the solution will be stable if

$$(\Delta t)^2\le\frac{1}{(l^2+N^2)\bar{\phi}}. \tag{17}$$

If, on the other hand we first transform the variables according to transformation (3) and then truncate that series beyond wavenumber N, then we can show that

$$\left.\begin{aligned}u(n)^{t+1}&=\tfrac{1}{2}\hat{u}(n)^{t+1}[R(n+1)+R(1-n)]\\ &\quad+\frac{i}{2}\hat{v}(n)^{t+1}[R(n+1)-R(1-n)]\\ v(n)^{t+1}&=\tfrac{1}{2}\hat{v}(n)^{t+1}[R(n+1)+R(1-n)]\\ &\quad-\frac{i}{2}\hat{u}(n)^{t+1}[R(n+1)-R(1-n)]\\ \phi(n)^{t+1}&=R(n)\hat{\phi}(n)^{t+1}\end{aligned}\right\}, \tag{18}$$

where $R(n)$ is given by (16).

From (18) we note that if $|n|\le N-1$, then the system composed of (14) and (18) is identical to that previously considered. Further, the case $n=N+1$ has no effect on the stability. We are therefore concerned only with the case $n=N$.

If we substitute (18) into (14) we obtain the modified sequence,

$$\left.\begin{aligned}u(N)^{t+1}&=u(N)^{t-1}-(l+iN)\phi(N)^t\Delta t\\ v(N)^{t+1}&=iu(N)^{t+1}\\ \phi(N)^{t+1}&=\phi(N)^{t-1}+(l-iN)\bar{\phi}u(N)^t2\Delta t\end{aligned}\right\}. \tag{19}$$

It may then be shown that (19) will be a stable sequence if

$$(\Delta t)^2 \leq \frac{2}{(l^2+N^2)\bar{\phi}}. \tag{20}$$

Comparing (20) and (17) we note that for *this component*, the time step required for stability is larger by a factor of $\sqrt{2}$ when the stereographic transformation is employed. This does not mean that the computation employing the stereographic transform can use a time step larger by a factor of $\sqrt{2}$, since the time step may be limited by the frequency corresponding to component $N-1$. In fact we could only benefit from this procedure if N and $N-1$ were different by a factor greater than about $\sqrt{2}$. Only $N \leq 3$ satisfies this criterion, so that perhaps the stereographic transformation ought to be used only very near the poles.

It does not *appear* that the stereographic transformations would have a serious effect on the meteorological modes, since, when we add an advection term to the right-hand side of (11)–(13), the low frequency mode is the same in both cases. This implies that the eigenvalues associated with at least the very short wavelength meteorological modes is not affected by the stereographic transformation process. However, the filtering procedure as applied near the poles is operating on rather low wavenumber components, and we cannot make any further statements on this point using the above analysis.

5. Closing remarks

We have demonstrated by a simple model the effects on the stability criterion produced by the application of a stereographic transformation to the wind field before Fourier filtering. It is still not clear if there are any benefits to this procedure vis-a-vis directly truncating the variables beyond one less wavenumber. If, in fact, the stereographic transformation does not seriously alter the meteorological mode then there would seem to be some benefit. However, this will require further experimentation and analysis.

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REFERENCE

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