NOTES AND CORRESPONDENCE

Implicit Differencing of Predictive Equations of the Boundary Layer

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ABSTRACT

The Crank-Nicholson method may not give useful results in detailed prediction of the thermal planetary boundary layer unless time steps on the order of 10 s are used. In similar problems, lower order time differencing methods give reasonable results with time steps as large as 300 s. The reason for the superior behavior of the lower order schemes relative to straightforward application of the Crank-Nicholson technique is due to a better treatment of short waves which appear to be critically important in nonlinear terms.

1. Truncation and stability

The following equation is often solved in the prediction of a planetary boundary layer variable θ:

$$\frac{\partial\theta}{\partial t}=X1\frac{\partial\theta}{\partial z}+X2\frac{\partial^2\theta}{\partial z^2}, \tag{1}$$

where t is time, z height, and $X1$ and $X2$ are related to turbulent diffusivity or conductivity.

Many numerical approximations to (1) are of the form

$$\frac{\theta_j^{n+1}-\theta_j^n}{\Delta t}=\alpha\left[\frac{\theta_{j+1}^{n+1}-\theta_{j-1}^{n+1}}{2\Delta z}X1_j^n+\frac{\theta_{j+1}^{n+1}+\theta_{j-1}^{n+1}-2\theta_j^{n+1}}{(\Delta z)^2}X2_j^n\right]+\beta\left[\frac{\theta_{j+1}^{n}-\theta_{j-1}^{n}}{2\Delta z}X1_j^n+\frac{\theta_{j+1}^{n}+\theta_{j-1}^{n}-2\theta_j^{n}}{(\Delta z)^2}X2_j^n\right], \tag{2}$$

where superscript n refers to time step, subscript j to grid point, Δt is the time interval and Δz the grid size. Eq. (2) is a consistent approximation to (1) if $\alpha+\beta=1$ and unconditionally stable for constant $X1$, $X2$ if $\beta\leqslant\alpha$.

Richtmyer and Morton (1967, p. 189) describe particular variations of (2). The most commonly used form has $\beta=\alpha=\frac{1}{2}$ (the Crank-Nicholson scheme), and this is the only version of (2) having second-order accuracy in both time and space. For the constant coefficient case, the truncation error (TE) of (2) is given by

$$\mathrm{TE}=(\tfrac{1}{2}-\alpha)O(\Delta t)+O(\Delta t^2)+O(\Delta z^2). \tag{3}$$

A drawback of the Crank-Nicholson scheme is apparent in the solution of the constant coefficient case given by sums of Fourier components of the form

$$\theta_j^n=\lambda^n e^{ikj\Delta z}, \tag{4}$$

where λ is a constant to be determined, $i=\sqrt{-1}$, and k is a wavenumber. Substitution of (4) into (2) gives

$$\lambda=\frac{\dfrac{1}{\Delta t}+\beta\left[i\dfrac{\sin(k\Delta z)}{\Delta z}X1+\dfrac{2\cos(k\Delta z)-2}{(\Delta z)^2}X2\right]}{\dfrac{1}{\Delta t}-\alpha\left[i\dfrac{\sin(k\Delta z)}{\Delta z}X1+\dfrac{2\cos(k\Delta z)-2}{(\Delta z)^2}X2\right]}. \tag{5}$$

The absolute value of λ is less than 1 whenever $\beta\leqslant\alpha$. Therefore, the Crank-Nicholson scheme is unconditionally stable. However, this feature has relatively limited utility, since in the limit of large damping rate ($X2\rightarrow\infty$) or short waves ($k\Delta z\rightarrow\pi$), λ tends to the value $-\beta/\alpha$ for fixed Δt, while the actual solution tends toward an infinite damping rate in this limit. This means that the Crank-Nicholson scheme would not damp very short modes adequately unless sufficiently small time steps are used. It is clear, however, that a lower accuracy scheme with $\beta<\alpha>\frac{1}{2}$ would also be unconditionally stable and damp short waves more effectively, although the truncation error analysis [Eq. (3)] suggests that α should not be much greater than $\frac{1}{2}$.

This simple analysis is not directly applicable to nonlinear or variable coefficient cases of (2), but it does help justify using lower order time differencing in such experiments. In the next section, the nonlinear case of the thermal boundary layer is considered.

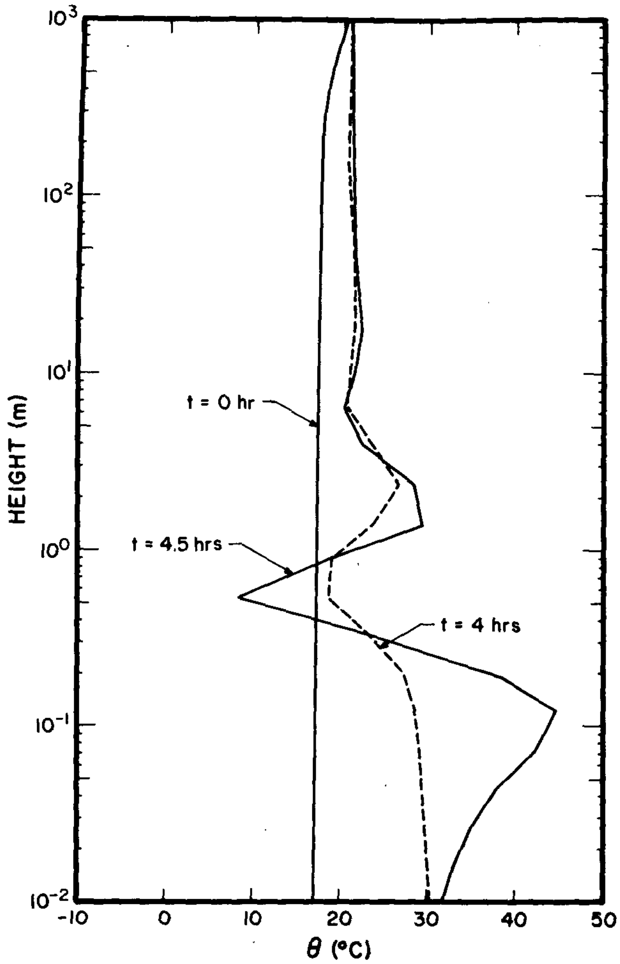

FIG. 1. Time evolution of potential temperature in the lowest 100 m of the atmosphere using Eq. (2) with $\alpha = 1$, $\beta = 1$, $\Delta t = 60$ s.

2. Nonlinear numerical experiments

The boundary layer potential temperature (θ) variation may be simulated by the following version of Eq. (1):

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial\theta}{\partial z}\right). \quad (6)$$

In this study, the thermal dependence of K is given by the KEYPS formula

$$K=\frac{kz}{2^{\frac{1}{2}}}\left\{-\gamma\frac{g}{T}\frac{\partial\theta}{\partial z}(kz)^2+\left[\left(\gamma\frac{g}{T}\frac{\partial\theta}{\partial z}(kz)^2\right)^2+4u_*^{\,2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \quad (7)$$

where k is the von Kármán constant, g the acceleration of gravity, u_* the friction velocity, T temperature and γ is an empirically determined constant. The present experiments were performed with $\gamma=14$, $k=0.4$, $u_*=0.3$ m s^{-1}. Eq. (6) is based upon the turbulent energy equation and has also been recently used by Zdunkowski *et al.* (1975).

The KEYPS formula is extremely sensitive to $\partial\theta/\partial z$. Very slight and otherwise unimportant computational fluctuations of $\partial\theta/\partial z$ can produce peculiar height variations of K which are suppressed by replacing K at each grid point with the smoothed average

$$K_j=(K_{j+1}+K_{j-1}+2K_j)/4.$$

In addition, if the maximum K occurs above 480 m, the value at 500 m is taken as the maximum and decreased above this according to

$$K(z)=K(500\text{ m})e^{-z/500\text{ m}},\quad z>500\text{ m}.$$

In order to improve vertical resolution at low levels, a logarithmic height transformation is used below about 200 m, i. e.,

$$\zeta=L\ln\left(\frac{z}{z_0}\right). \quad (8)$$

The transformed form of (8) is

$$\frac{\partial\theta}{\partial t}=\frac{\partial K}{\partial\zeta}\frac{\partial\theta}{\partial\zeta}\left(\frac{L}{z_0e^{\zeta/L}}\right)+K\left[\frac{\partial^2\theta}{\partial\zeta^2}\left(\frac{L}{z_0e^{\zeta/L}}\right)^2-\frac{\partial\theta}{\partial\zeta}\frac{L}{(z_0e^{\zeta/L})^2}\right]. \quad (9)$$

Grid spacing is defined by

$$\left.\begin{array}{l}L=1,\quad z_0=0.01\text{ m}\\ \Delta\zeta=\frac{1}{2},\quad \Delta z=86.7\text{ m above } z=200\text{ m}\end{array}\right\}.$$

A boundary condition of the form

$$\theta=16\cos\left(\frac{2\pi}{24h}t\right)+290\text{ K}$$

is applied at $z=z_0$, simulating a large diurnal temperature oscillation.

Fig. 1 illustrates forecasts using the Crank-Nicholson scheme with 60 s time step. Large oscillations develop within 2 h and appear to have a vertical scale of about six grid intervals. These oscillations grow in amplitude during the first 6 h of surface heating, and then gradually diminish after surface cooling commences. However, the results remain physically meaningless.

Since this numerical instability cannot occur on a constant coefficient linear basis, it may be a nonlinear computational instability which could have its origin in aliasing error of short waves. If this were the case, it could be eliminated by adequately damping short scales by either reducing Δt or increasing α and decreasing β. The time step needs to be reduced to about 10 s before a stable result is produced by the Crank-Nicholson scheme (Fig. 2), and an experiment with $\frac{1}{2}$ s time steps agrees to within a few tenths of a degree with the 10 s time step solution.

Such time steps are rather inefficient for the present problem where the forcing has a 1-day periodicity. First-order time differencing would usually resolve a 1-day period oscillation sufficiently well with much larger time steps. First-order differencing with $\alpha>\beta$ would also more correctly damp the short waves, thereby inhibiting aliasing error and non-linear instability.

Experimentation with $\alpha=0.55$, $\beta=0.45$ and $\alpha=0.60$, $\beta=0.40$ produced unstable results, but selection of $\alpha=0.75$, $\beta=0.25$ gave the stable solution shown in Fig. 3 for a time step of 300 s. These results agree to within about 0.1°C with the Crank-Nicholson solution of Fig. 2 and are obtained in only a small fraction of the computer time.

3. Conclusions

All experiments that we have tried with $\alpha\geqslant 0.75$ and $\beta\leqslant 0.25$ have given useful results. These include cases with soil layers, heat balance conditions at the surface, radiative heating, and several different K profiles. In general, these cases exhibit behavior similar to that depicted in Fig. 1 as α, β tend toward 0.5 when time steps on the order of minutes are used. The problem arises only when upward turbulent heat flux exists, and only when the surface layer (below about 50 m) is treated as a predictive layer rather than a constant flux layer. The particular minimum value of α that gives best results depends upon the experimental conditions through the magnitude of the upward heat flux.

There are other solutions to the problem. The DuFort-Frankel scheme was used successfully by Zdunkowski *et al.* (1975). However, this scheme is inconsistent and requires careful testing in each application. Garder and Raymond (1974) used a finite element technique based on a Galerkin approximation in space. The time derivative was approximated with the Crank-Nicholson scheme using a predictor-corrector technique on the nonlinear term, and stable results were obtained with a 20 min time step. Application of this approach to our problem revealed a stabilizing effect in the predictor-corrector technique, but not in the finite element spatial approximation. The results are very similar to those of Figs. 2 and 3 for time steps of 5 min. However, the iterative nature of the predictor-corrector technique reduces its computational efficiency.

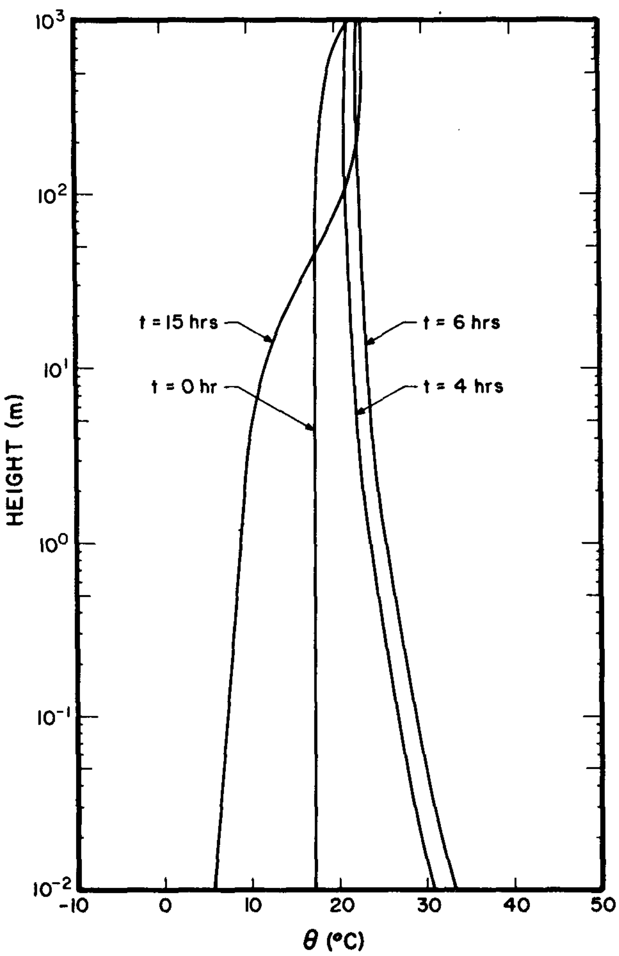

Fig. 2. As in Fig. 1 except for Δt = 10 s.

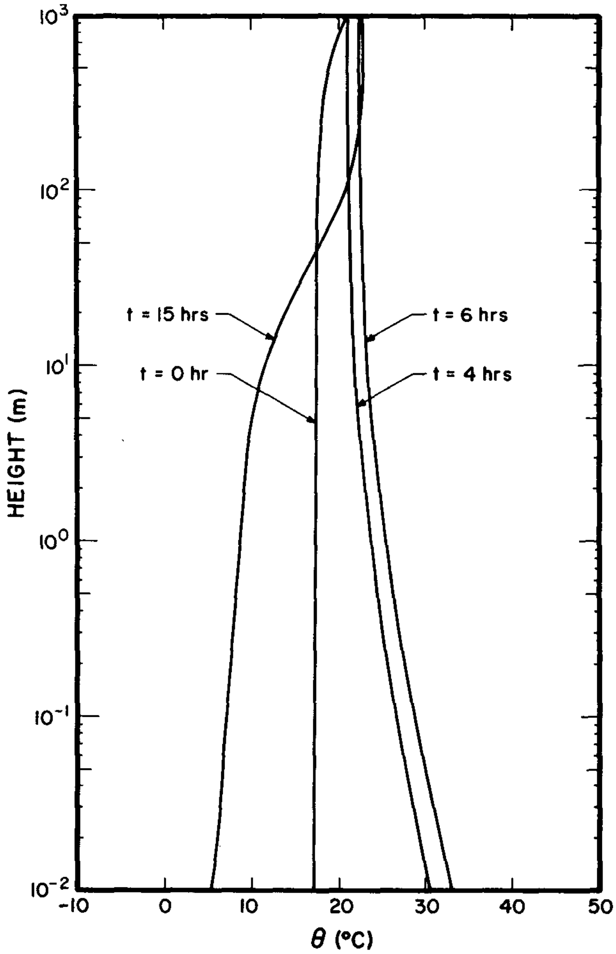

Fig. 3. As in Fig. 1 except for α = 1.5, β = 0.5, Δt = 300 s.

Yamada and Mellor (1975) chose $\alpha=1$, $\beta=0$ in a stable computation that supports our conclusion that stable solutions may be expected with $\alpha \geqslant 0.75$, $\beta \leqslant 0.25$. We have found that stable solutions with a diurnal periodicity and time steps on the order of 10 min depend very little upon the particular value of α if it is not greater than 1. However, computational damping of longer modes increases with α, and the truncation error analysis [Eq. (3)] suggests use of as small a value as is permitted by stability considerations.

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An Example of Polar Air Modification over the Gulf of Mexico

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ABSTRACT

A satellite picture shows many aspects of modification of continental air over warm water. The estimated average vertical flux of sensible and latent heat over 24 h was 10^3 W m^{-2} (1.44 cal cm^{-2} min^{-1}). Maximum rates were estimated to be an order of magnitude higher.

There is considerable interest in air-sea exchange and air mass modification at this time as a result of the Global Atmospheric Research Program (GARP). Several recent experiments such as the GARP Atlantic Tropical Experiment (GATE) and the Air Mass Transformation Experiment (AMTEX) have tended to focus attention on such processes. Many satellite photographs provide illustrations of some aspects of the modification processes and the resulting cloud distributions. An interesting example is discussed here.

In late November of 1970, a surface cold front and its accompanying continental air mass moved southward across the Gulf States and the Gulf of Mexico. The surface front arrived at the Texas coast about 00 GMT on the 23rd and crossed the Yucatan Peninsula late on the 24th. The temperatures in the cold air were well below normal for the season, being colder than 0°C at the surface at some stations along the U. S. Gulf Coast. Indicators of the flow characteristics and of the modification of the continental polar air are shown in a satellite picture of the cloud distribution over the area at 1659 GMT 24 November 1970 (Fig. 1). The land areas along the south and east coasts of the United States were free of clouds. The leading edge of the cold air, the cold front, from Central America east-northeast across central Cuba and then northeast parallel to the east coast of the United States, was indicated by the "rope cloud" and the segments of cumulus bands or streets along or a short distance ahead of the leading edge of the cold air. Similar examples have been published (Parmenter, 1967; Eichenlaub and Garrett, 1972). We have added some comments concerning the related energy exchange processes.

One indicator of the modification in the polar air is the layer of stratocumulus behind the surface front. The northern edge of this layer was located some 75–100 km offshore and closely paralleled the coastline, and indeed reflected some of the capes and bays remarkably well. This also indicates the uniformity of the moisture and heat fluxes. The cloud streets, oriented in the direction of the flow in the cold air, were at an appreciable angle to the coastline (except along the east and west sides of the Florida Peninsula).

A Lagrangian approach allows an estimation of the sensible and latent heat fluxes into the polar air flowing southward across the Gulf. Trajectory evaluations, based on a combination of surface streamlines