On the Nesting of Grids in Nonhydrostatic Computations

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ABSTRACT

The use of a nested grid system in the numerical integration of a nonhydrostatic system of hydrodynamic equations is investigated. Computational problems introduced by the abandonment of the hydrostatic approximation are noted.

Tests in which a solitary inertial gravity wave is propagated through a periodic domain indicate that the computational noise due to the grid interaction is negligible when an upstream or a two-step Lax-Wendroff differencing scheme is used. A precipitating cumulus cloud is then simulated with several arrangements of grid points. The "nested" results differ very little from those of the corresponding "fine grid" simulation. Computational noise cannot be detected in the grid interaction zone even when this zone is in the active cloud area.

1. Introduction

The numerical simulation of mesoscale and microscale atmospheric circulations generally requires fine grid-point resolution. Yet the total domain must be considerably larger than the immediate region of interest because the circulations are not isolated from their surroundings. An enlarged grid domain is also necessary in order to minimize the contaminating effects of artificial lateral boundary conditions. Since a uniform fine grid often requires excessive amounts of computer time and storage, it becomes necessary to restrict the fine mesh to a limited area of the grid domain.

Variable resolution can be achieved either by coordinate stretching or by the overlapping of fine and coarse grids. An advantage of the latter approach is that the coarse timestep can easily be made an integer multiple of the fine timestep, thereby permitting a further reduction in the computing time. Fig. 1 illustrates a nested grid arrangement which has been used to model a hydrostatic sea breeze circulation in two dimensions (Walsh, 1974). When this grid-point arrangement is used in place of a single uniform grid (Δx=0.5 km) covering the same domain, the number of grid points is reduced by a factor of 3.2 and the average timestep can be increased by a factor of 2.1. The number of required computations is therefore reduced by a factor of 3.2×2.1=6.7. Since computing costs typically vary as the product of the CPU time and the core region size, the cost of a simulation is reduced by a factor of 3.2×6.7, or about 21.4.

Fig. 2 illustrates the nesting of grids in two dimensions. Such an arrangement is advantageous in simulating a circulation such as the cumulus cloud which possesses three-dimensional characteristics. As discussed in Section 2, however, it is convenient to have coarse grid points in the fine mesh region when a nonhydrostatic system is simulated. The potential savings are therefore smaller than in the hydrostatic case. Nevertheless, the following cost-reduction factors would be

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Fig. 1. The nesting of grids in the two-dimensional sea breeze model of Walsh (1974). The vertical space increment Δz is constant through all (20 or 40) levels; Δx increases from 0.5 km on the innermost grid to 4.0 km on the outermost grids.

Fig. 2. The nesting of grids in the two horizontal dimensions, x and y. The symbols "x" and "o" represent points on the coarse and fine grids, respectively.
realized with the grid arrangement of Fig. 2 if the fine mesh covers 20% of the total domain:

\[ p = \frac{(\Delta x)_e}{(\Delta x)_f} \quad \text{Cost-reduction factor} \]

\[
\begin{array}{c|c}
2 & 6.0 \\
3 & 12.6 \\
4 & 16.4 \\
\end{array}
\]

Note that \( p \) is the ratio of the coarse and fine space increments. For example, if \( p = 3 \) and \( (\Delta x)_f = 0.5 \) km, then \( (\Delta x)_e = 1.5 \) km.

Nested grids have been used successfully in primitive equation integrations by Birchfield (1960), Hill (1968), Wang and Halpern (1970), Harrison and Elsberry (1972), and Mathur (1974), as well as by the groups at NCAR (Williamson and Browning, 1974) and NMC (Brown and Fawcett, 1972). The technique has generally not been used, however, to simulate smaller scale circulations for which the hydrostatic assumption is invalid. Cumulus cloud modelers (Hane, 1973; Wilhelnson and Ogura, 1972; Murray, 1970), for example, have abandoned the hydrostatic approximation, but have also reverted to uniform grids. This work investigates the feasibility of nesting grids in nonhydrostatic numerical models.

2. The meshing procedure

When using more than one finite difference mesh, one must allow the grids to interact in a physically reasonable manner without contaminating the solution with computational "noise" near the grid boundaries. This study will employ the two-way interaction scheme of Phillips and Shukla (1973) in which each grid receives information or "feedback" from the other grid(s). The feature of interest (sea breeze, cumulus cloud, etc.) can therefore influence and be influenced by its environment. This strategy has been used successfully in hydrostatic integrations (e.g., Walsh, 1974). The outermost column or row of fine mesh points is not computed when the fine grid is stepped ahead; the missing outermost values are obtained from the overlapping coarse grid values after the coarse timestep, when the two grids are at the same point in time.

In integrations of nonhydrostatic systems, the most common approach is to solve a Poisson equation (for pressure or a streamfunction) at each timestep. However, the boundary values for the Poisson solution on the fine grid are not available, since the overlapping coarse grid values exist only at times \( p \Delta t \). [Recall that \( p = (\Delta x)_e/(\Delta x)_f \).] The strategy here is to step the coarse mesh ahead in time, and then obtain the fine mesh boundary values from the overlapping coarse mesh values by a linear time-interpolation. This strategy requires that the Poisson variables be computed at the coarse grid points that lie within the fine mesh region (see Fig. 2). By contrast, the coarse mesh points within the fine mesh area are "dummy" points in the Phillips and Shukla (1973) scheme where the Poisson problem is not present. In the present strategy, only the Poisson variables (vorticity and streamfunction) need be computed at the coarse grid points that lie within the fine mesh area; even these values are subsequently replaced by the Poisson variables from the fine mesh (Appendix). Since the savings estimates in Section 1 allowed for the computation of all variables at all coarse grid points, the potential savings are somewhat greater than implied by these earlier estimates.

The fact that the Poisson variables (vorticity and streamfunction) differ by several orders of magnitude suggests the solution should be examined carefully in the region of the grid interaction. Therefore, the meshing technique is tested in the experiments of Sections 3 and 4 with particular emphasis on the solutions in the vicinity of the fine-grid boundaries. So that attention may be focused on the effects of the grid nesting, two-dimensional \((x-z)\) motions are simulated on a pair of grids which are nested only in the horizontal \((x)\) direction. In Section 3, a solitary internal gravity wave is propagated through a periodic domain containing the two grids. The effect of the differencing scheme (upstream vs Lax-Wendroff) is noted. The nesting technique is then used in Section 4 to simulate a cumulus cloud that is forced by an initial impulse of excess buoyancy and moisture.

3. Gravity wave tests

The prognostic equations for a two-dimensional Boussinesq fluid in the presence of a constant basic current \( U \) are

\[
\frac{\partial \xi}{\partial t} = -U \frac{\partial \xi}{\partial x} - f \frac{\partial \zeta}{\partial z} + \nu \nabla^2 \xi, \quad (1)
\]

\[
\frac{\partial \psi}{\partial t} = -U \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial z}, \quad (2)
\]

\[
\frac{\partial b}{\partial t} = -U \frac{\partial b}{\partial x} - N^2 w + \kappa \nabla^2 b, \quad (3)
\]

where \( u = -\partial \psi/\partial z \), \( v \), and \( w = \partial \psi/\partial x \) are the velocity components in the \( x \), \( y \), and \( z \) directions, \( \zeta = \nabla^2 \psi = \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial z^2 \) is the vorticity, \( b = g(T - T_o)/T_o \) is the buoyancy \((a = \text{adiabatic} \), \( o = \text{surface})\), \( N^2 = db/dz \) is the static stability, \( f \) is the Coriolis parameter, \( \nu \) and \( \kappa \) are the eddy diffusion coefficients for momentum and heat. The nonlinear advection terms are expressed in Jacobian form where \( F(\psi, \chi) = (\psi/\partial z)(\partial F/\partial z) - (\psi/\partial x)(\partial F/\partial x) \). Constant values are assigned to \( f \), \( N^2 \), \( \nu \), and \( \kappa \). The system (1)–(3) can be made hydrostatic simply by omitting \( \partial^2 \psi/\partial z^2 \) in the Laplacian operator (and by redefining \( \zeta \) as \( \partial \psi/\partial z \)). The Appendix describes the numerical techniques used to solve (1)–(3).
The initial conditions (4) for the gravity wave tests are shown in Fig. 4. An integration of the linear system should merely translate the initial patterns of Fig. 4 horizontally. The variable fields obtained by numerically integrating the linear version of (1)–(3) are indeed essentially undistorted, although slight irregularities develop near the fine grid boundary, as might be expected when the diffusion terms are omitted. Fig. 5 illustrates this behavior for the streamline field. Significantly, the irregularities are not noticeable at a slight distance downstream from the grid interaction region.

When the nonlinear advection terms are included, the numerical solution depends on the finite-difference representation of these terms. Both the upstream and Lax-Wendroff schemes were tested in this study. In each case, the advection terms distort the sinusoidal patterns given by (4). The distortion is apparent in Fig. 6, which shows the variable fields after 480 fine-grid timesteps (about 3.5 h) of an upstream test. The damping introduced by the finite-difference representations of the advection terms reduces the amplitude of

The wave corresponding to the plus sign in (6) will be nearly stationary for the same set of parameters. It should be excited only by numerical distortion of the traveling wave near the fine grid boundary.
provided that the value of $\Delta x$ is weighted by the fractions of the total domain that are coarse and fine. This fact, together with the agreement between the nested (upstream) and uniform coarse (upstream) results of Table 1, indicates that the grid nesting does not significantly alter the amplification or damping rates for disturbances having wavelengths of several kilometers. Finally, it should be noted that the lifetimes of systems such as cumulus clouds are considerably shorter than 3–4 h, so the amplitude changes of Table 1 are more extreme than those that will occur in the numerical integrations of Section 4.

4. Cumulus cloud simulations

As an application of the numerical strategy discussed in Section 2, a (nonhydrostatic) cumulus cloud will now be simulated on a system of two grids. The two dimensional slab-symmetric cloud model of Soong and Ogura (1973), which uses a modified upstream differencing scheme and parameterized microphysics, forms the basis for the tests described here. The model has been extended to three dimensions by Wilhelmson (1974). It is the desirability of reducing the computing time and storage required for three-dimensional cloud simulations that motivated this work.

Fig. 6. The fields of the vorticity $\zeta$, the north-south velocity $v$, and the buoyancy $b$ after 480 fine-grid timesteps (about 3.5 h) in Run 2a.
Table 1. Gravity wave phase speeds \(c_{ph}\) through 3.5 h and amplitudes after 3.5 h. See text for specified parameters.

<table>
<thead>
<tr>
<th>Run No.</th>
<th>Analytic (linear) (c_{ph}) (m s(^{-1}))</th>
<th>Analytic Amplitude</th>
<th>a. Upstream (c_{ph}) (m s(^{-1}))</th>
<th>a. Upstream Amplitude</th>
<th>b. Lax-Wendroff (c_{ph}) (m s(^{-1}))</th>
<th>b. Lax-Wendroff Amplitude</th>
<th>c. Upstream (single coarse grid) (c_{ph}) (m s(^{-1}))</th>
<th>c. Upstream Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Linear nonhydrostatic</td>
<td>-3.931</td>
<td>1000</td>
<td>-4.062</td>
<td>1188</td>
<td>-4.009</td>
<td>1023</td>
<td>-4.044</td>
<td>1300</td>
</tr>
</tbody>
</table>

The governing equations are those for deep moist convection derived by Ogura and Phillips (1962) and modified by Wilhelmson and Ogura (1972) to include the buoyancy of water vapor and a non-isentropic base state. The system includes prognostic equations for the vorticity \(\zeta\), the potential temperature perturbation \(\theta'\), and the three classes of water substance: water vapor \(Q_v\), cloud water \(Q_c\), and rain water \(Q_r\). In a two-dimensional \((x,z)\) system, these equations are

\[
\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - w \frac{\partial \zeta}{\partial z} + \frac{\partial}{\partial x} \left[ \rho \frac{\partial \theta'}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \rho \frac{\partial \theta'}{\partial z} \right] + D_T, \tag{7}
\]

\[
\frac{\partial \theta'}{\partial t} = -u \frac{\partial \theta'}{\partial x} - w \frac{\partial \theta'}{\partial z} + \frac{\partial}{\partial x} \left[ \rho \frac{\partial \theta'}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \rho \frac{\partial \theta'}{\partial z} \right] + L, \tag{8}
\]

\[
\frac{\partial Q_v}{\partial t} = -u \frac{\partial Q_v}{\partial x} - w \frac{\partial Q_v}{\partial z} + P_3 - P_2 - P_5 + D_{Q_v}, \tag{9}
\]

\[
\frac{\partial Q_c}{\partial t} = -u \frac{\partial Q_c}{\partial x} - w \frac{\partial Q_c}{\partial z} + P_1 - P_2 - P_5 + D_{Q_c}, \tag{10}
\]

\[
\frac{\partial Q_r}{\partial t} = -u \frac{\partial Q_r}{\partial x} - w \frac{\partial Q_r}{\partial z} + P_3 - P_4 + P_5 - P_7, \tag{11}
\]

where the vorticity and streamfunction are related by \(\zeta = \nabla \psi\), \(u = \rho^{-1} \psi \partial \partial x\) and \(w = \rho^{-1} \psi \partial \partial z\) are the horizontal and vertical velocity components, \(\theta = \bar{\theta}(z) + \theta'\) is the potential temperature, \(\bar{\rho}(z)\) is the density, and \(D\) represents the sub-grid-scale diffusion of the various quantities. The parameterized microphysical processes include the condensation of water vapor \(P_2\), the evaporation of cloud droplets and raindrops \(P_3\) and \(P_4\), respectively) as formulated by Ogura and Takahasi (1971), the autoconversion \(P_3\) and collection \(P_4\) processes based on Kessler's (1969) formulation, and the redistribution of rainwater \(P_6\) as it falls. The reader is referred to Soong and Ogura (1973) for the details of the formulation.

Two different clouds (labeled "A" and "B") were simulated using both nested grids and single uniform grids. The grid spacing is summarized in Table 2. The nesting of the fine grid inside the coarse grid is the same as in Fig. 3 except that the coarse and fine grid points are coincident in the fine grid region. Computations are made for only half the domain because the absence of wind shear introduces symmetry about a central axis (\(x=0\)).

Cloud A is a cloud of moderate development in an environment characterized by

\[
\frac{\partial T}{\partial z} = -9.7^\circ C \text{ km}^{-1}, \quad z \leq 0.8 \text{ km},
\]

\[
\frac{\partial T}{\partial z} = -6.0^\circ C \text{ km}^{-1}, \quad 0.8 \text{ km} \leq z \leq 12.8 \text{ km},
\]

and a relative humidity profile that increases linearly from 0.75 at \(z=0\) to 0.88 at \(z=0.8\) km and then decreases linearly to 0.30 at \(z=8.4\) km. Above 8.4 km, the relative humidity is constant at 0.30. The surface (1000 mb) temperature is 25°C, while the cloud is initiated by a buoyant "bubble" in which \(\theta'_{max}=0.5^\circ C\) and which is saturated. The bubble occupies the region \(0 \leq x \leq 1.6\) km and \(1.2\) km \(\leq z \leq 2.0\) km.

The development of the cloud is illustrated by the time-dependence of three quantities: the rainfall at the central axis, the maximum potential temperature perturbation \(\theta'_{max}\) in the grid domain, and the maximum vertical velocity in the domain. Results are presented for the three grid-point arrangements listed in Table 2: a uniform fine grid, a uniform coarse grid, and a fine grid nested inside a coarse grid, all covering the same space domain.

Fig. 7 shows the rainfall intensity (mm h\(^{-1}\)) and the accumulated rainfall (mm) at the central axis as functions of time. The three simulations are qualita-
The behavior of the maximum potential temperature perturbation $\theta_{\text{max}}$ is shown in Fig. 8. The three simulations again show good qualitative agreement, as $\theta_{\text{max}}$ in each case reaches approximately 2.5°C shortly after 20 min. The fact that the maximum potential temperature excess in the coarse grid cloud is smaller than in the other two by about 0.25°C is another indication that the nested grid arrangement allows for most of the development that is "missing" in the coarse grid cloud. It should also be noted that the oscillatory behavior in the $\theta_{\text{max}}$ curves after 40 min is due to the gravitational oscillations which have characterized the late decay stages of clouds produced by the University of Illinois cloud model (Soong and Ogura, 1973).

The maximum vertical velocities are compared in Fig. 9. The curves are very consistent with the rainfall and $\theta_{\text{max}}$ curves. The maximum vertical velocities of about 7 m s$^{-1}$ occur in the 20–25 min time period in each case, although the updraft in the decay stage of the fine grid cloud is somewhat stronger than the updraft in the same stage of the other two simulations. This result evidently accounts for the heavier rainfall from the fine grid cloud during the 25–30 min time period. Finally, it will be pointed out that the saturated region in each case extends from approximately 2.0 km to 4.5 km at the central axis and that the width of the cloud is about 3 km at the time of maximum rainfall intensity, or at about 30 min.

In order to test the nesting procedure more severely, a larger cloud ("B") is simulated by imposing a wider impulse in an environment that is characterized by stronger conditional instability at low and middle levels. As indicated in Table 2, three cases are again distinguished by the grid point arrangements. The same space domain is modeled with 1) a uniform fine grid, 2) a nested arrangement (50% coarse, 50% fine) in which the cloud is entirely within the fine grid at all times, and 3) a nested arrangement (75% coarse, 25%
Fig. 10. The rainfall intensity (mm h⁻¹) and the accumulated rainfall (mm) at the central axis for cloud simulations B1 (--), B2 (--), B3 (--).  

For comparison with cloud A, the rainfall intensity and accumulated rainfall at the central axis are shown in Fig. 10 for the three simulations of cloud B. The later but much stronger intensity maximum is apparent, and all three simulations produce approximately 2.5 cm of rain. The vertical velocities, which peak at approximately 12 m s⁻¹ at t = 32 min in all cases, are also considerably stronger than in cloud A.

The major difference in the simulations is that the total rainfall is reduced from the fine grid value by about 4% when the mesh is 50% coarse and 50% fine, while the rainfall is reduced from the fine grid value by about 11% when the mesh is 75% coarse and 25% fine. Fig. 11, which compares the cloud areas at t = 40 minutes, shows that a slightly smaller volume is saturated in the 75%/25% case. Fig. 11c shows that the cloud extends about 1 km beyond the fine grid boundary at this time. Significantly, the x-dependence of all the variables is quite smooth in the vicinity of the fine grid boundary despite the fact that the two grids are interacting in an active cloud region. As examples, the fields of the horizontal velocity component u and the rainwater mixing ratio Q_r are plotted in Fig. 12 as functions of x for several levels at t = 32 min. Neither the u values at k = 8 (an outflow level) nor the −u values at k = 3 (an inflow level) appear to be spuriously affected by the grid interaction. The same is true for the rainwater mixing ratios at levels 8 and 9, which are plotted in Fig. 12b. (The Q_r values at level 3 are not plotted since level 3 is not saturated at the fine grid boundary.)

5. Summary and conclusions

The results of Sections 3 and 4 indicate that it is indeed feasible to use a nested grid network in nonhydostatic computations. The spurious “noise” in the grid interaction region is very minor in the gravity wave tests and is not detectable in the cumulus cloud simulations. The phase speed of the propagating gravity wave is nearly identical to the theoretical (linear) value, while the damping is essentially the same as that which would take place on a uniform fine grid.
with its environment in a more physically realistic manner without a priori interference from the artificial lateral boundary conditions that frequently contaminate limited area models.

The extension of grid nesting to both horizontal dimensions of a three-dimensional model will enable the technique's value to be more fully realized, as it is the three-dimensional models that have been most severely limited by cost considerations. The savings can be further increased by the use of more than two mesh sizes, although it is probably worthwhile to increase \((\Delta x)^2 / (\Delta s)^2\) to its maximum feasible value before introducing the complexity of additional grids.

An attractive feature of a nested grid arrangement is the ease with which one can allow for the movement of the fine mesh so as to "follow" a meteorologically active region. Present plans include the use of a moveable grid to retain fine mesh handling of a moving cumulus system. The same strategy can be of use in modeling other propagating mesoscale and microscale disturbances.

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APPENDIX

Numerical Techniques

The strategy for solving the model equations involves the choice of a differencing scheme and the choice of a method for solving the Poisson equation for \(\psi\). The Poisson equation is solved by performing a Fourier analysis in the \(x\)-direction, followed by the solution of the harmonic equations in the \(z\)-direction.

For the gravity wave tests, the variables are staggered in space as shown in Fig. A1. Two differencing schemes
are compared. The first, the one-sided time upstream space-differencing scheme, has the advantage of speed and small storage requirements. However, the strong implicit damping effect (Molenkamp, 1968) is a disadvantage in simulating small-scale circulations in which large horizontal gradients may actually exist.

The second differencing method is the two-step Lax-Wendroff scheme in which "primed" quantities are computed at the half-timestep \((n+\frac{1}{2})\Delta t\) by using a Lagrangian (i.e., uncentered) formulation of the advection terms. The prognostic equations are then written in flux form (centered differences) in terms of the "primed" quantities, and the extrapolation is made through the full timestep from \(n\Delta t\) to \((n+1)\Delta t\). While this scheme is more time consuming than the upstream scheme, the artificial damping is reduced. The Lax-Wendroff scheme has been used successfully in nested grid models by Phillips and Shukla (1973), who used the shallow water equations, and by Walsh (1974), who used the hydrostatic version of the Boussinesq equations.

The nesting of the grids is illustrated in Fig. 3 where \(\Delta x\) (and hence \(\Delta t\)) are twice as large on the coarse grid as on the fine grid. The grid interaction proceeds as follows for the upstream scheme:

A1) Step coarse grid from \(n\Delta t\) to \((n+1)\Delta t\).
A2) Step fine grid from \(n\Delta t\) to \((n+\frac{1}{2})\Delta t\).
A3) Obtain boundary values \((\psi, b, v)\) for fine grid from coarse grid values at \(n\Delta t\) and \((n+1)\Delta t\) by interpolating in \(x\) and \(t\).
A4) Step fine grid from \((n+\frac{1}{2})\Delta t\) to \((n+1)\Delta t\).
A5) Obtain boundary values \((\psi, b, v)\) for fine grid from coarse grid values at \((n+1)\Delta t\) by interpolating in \(x\).
A6) Replace coarse grid values which lie inside fine grid boundaries by interpolating in \(x\).
A7) Go to A1) with \(n \leftarrow n+1\).

All interpolations are linear. Interpolated values are indicated by parentheses in Fig. A1.

The Lax-Wendroff integration proceeds similarly, although interpolations must also be performed for the "primed" quantities.

REFERENCES