

NOTES AND CORRESPONDENCE

High-Latitude Truncation Errors of Box-Type Primitive Equation Models

EUGENIA KÁLNAY-RIVAS

Department of Meteorology, Massachusetts Institute of Technology, Cambridge 02139
and Goddard Institute for Space Studies NASA, New York, N. Y. 10025

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ABSTRACT

The "box-type" finite-difference method includes a weighted average of the pressure gradient with weights proportional to the surface of the grid walls. It is shown that this averaging introduces first-order truncation errors near the poles. An example is shown in which the relative error is of zero order and the scheme produces large distortions in the solution at high latitudes.

1. Introduction

The "box-method" designed by Kurihara and Holloway (1967) is a system of finite-difference equations which can handle irregular grids. It conserves kinetic energy in an autobarotropic atmosphere, and mean square potential temperature and total energy in an adiabatic, frictionless atmosphere. Only the first two conservation properties are necessary in order to avoid nonlinear instability.

The box-type finite differences have second-order truncation errors only if the grid is uniform or if it is defined through the use of a stretched coordinate (Kálnay-Rivas, 1972). In the Kurihara grid, to which the method was first applied, grid size varies discontinuously, especially near the poles. This variation is at least partly responsible for the small-scale errors observed at high latitudes. Another cause of these errors is the large size of the longitude steps near the poles. Both of these problems are corrected when a uniform latitude-longitude grid is used (e.g., Holloway *et al.* 1973).

We want to draw attention to another source of truncation errors characteristic of the box-type finite differences which is still present in a latitude-longitude grid and which produces strong distortions of the fields at high latitudes.

2. Truncation error analyses

Consider the meridional pressure gradient and the meridional mass divergence terms in the shallow water equations written in spherical coordinates, i.e.,

$$\frac{\partial hv}{\partial t} = -\frac{g}{a} \frac{\partial h}{\partial \phi} + \dots, \tag{1a}$$

$$\frac{\partial h}{\partial t} = -\frac{1}{a \cos \phi} \frac{\partial (hv \cos \phi)}{\partial \phi} + \dots. \tag{1b}$$

The box-type finite differences (Holloway *et al.*, 1973) corresponding to the same terms are

$$\frac{\partial hv}{\partial t} = -\frac{g \bar{h} \bar{h}_\phi \cos \phi}{a \cos \phi} + \dots, \tag{2a}$$

$$\frac{\partial h}{\partial t} = -\frac{(\bar{h} \bar{v} \cos \phi)_\phi}{a \cos \phi} + \dots. \tag{2b}$$

We are using the finite-difference notation

$$\bar{f}_j = (f_{j+1/2} + f_{j-1/2})/2, \quad f_{\phi_j} = (f_{j+1} - f_{j-1})/\Delta \phi,$$

and the variables have their standard meteorological usage.

These finite-difference equations [(2)] have a property analogous to the property of the continuous equations (1) in that

$$v \frac{\partial hv}{\partial t} + gh \frac{\partial h}{\partial t} = -\frac{g}{a \cos \phi} \frac{\partial (hv \cos \phi h)}{\partial \phi} + \dots, \tag{3}$$

which ensures the conservation of total energy. In Eq. (2a) it should be noted that the meridional pressure gradient is computed as the weighted average of the pressure gradient at the north and south walls of a box, with weights proportional to the wall's surface, i.e., to the cosine of latitude.

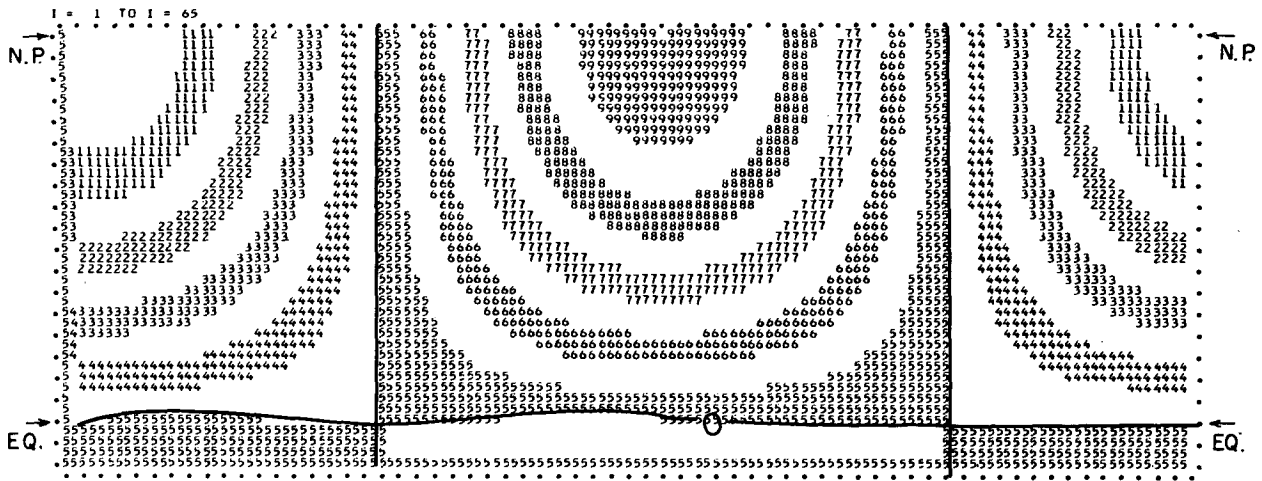


FIG. 1. Contours of the zonal (u) velocity field after a 6-day forecast using equations which do not contain a weighted average of the meridional pressure gradient. The position of the North Pole and the equator are indicated by arrows. The full contour line corresponds to $u=0$. The contour interval is $\sim 0.6 \text{ m s}^{-1}$. At the pole u is formally defined in terms of the stereographic velocities U, V and the longitude λ .

The truncation error analysis corresponding to the standard centered approximation of the pressure gradient is

$$\frac{\partial h_\phi}{\partial \phi} = \frac{\partial h}{\partial \phi} + \Delta\phi^2 \frac{1}{6} \frac{\partial^3 h}{\partial \phi^3} + O(\Delta\phi)^4, \quad (4a)$$

whereas that corresponding to the cosine-weighted approximation is

$$\frac{\overline{h_\phi \cos\phi}}{\cos\phi} = \frac{\partial h}{\partial \phi} + \Delta\phi^2 \left(\frac{1}{6} \frac{\partial^3 h}{\partial \phi^3} - \frac{1}{8} \frac{\partial h}{\partial \phi} \right) - \Delta\phi^2 \frac{\tan\phi}{4} \frac{\partial^2 h}{\partial \phi^2} + O(\Delta\phi)^4. \quad (4b)$$

Eq. (4b) suggests that near the poles, where $\tan\phi = O(\Delta\phi)^{-1}$, the cosine-weighted approximation

becomes first order in $\Delta\phi$. This is confirmed by the following analysis. Near the poles the free surface height can be approximated by

$$h = h_0 + Ax + Bx^2 + Cxy + Dy^2 + O(x^2 + y^2)^{\frac{3}{2}}, \quad (5)$$

where $x = a \cos\phi \cos\lambda$, $y = a \cos\phi \sin\lambda$ are the polar stereographic coordinates, and the x axis has been chosen parallel to the pressure gradient at the pole. Therefore

$$h = h_0 + \alpha \cos\phi + \beta \cos^2\phi + O(\cos^3\phi), \quad (6)$$

where $\alpha = aA \cos\lambda$, $\beta = a(B \cos^2\lambda + C \cos\lambda \sin\lambda + D \sin^2\lambda)$. The component of the truncation error proportional to $\tan\phi$ in (4b) can be evaluated, for example, at the point next to the poles, where $\sin\phi = 1 + O(\Delta\phi)^2$, $\cos\phi = \Delta\phi + O(\Delta\phi)^3$:

$$-\Delta\phi^2 \frac{\tan\phi}{4} \frac{\partial^2 h}{\partial \phi^2} = \Delta\phi^2 \frac{\alpha}{4} - \Delta\phi \frac{\beta}{2} + O(\Delta\phi)^3. \quad (7)$$

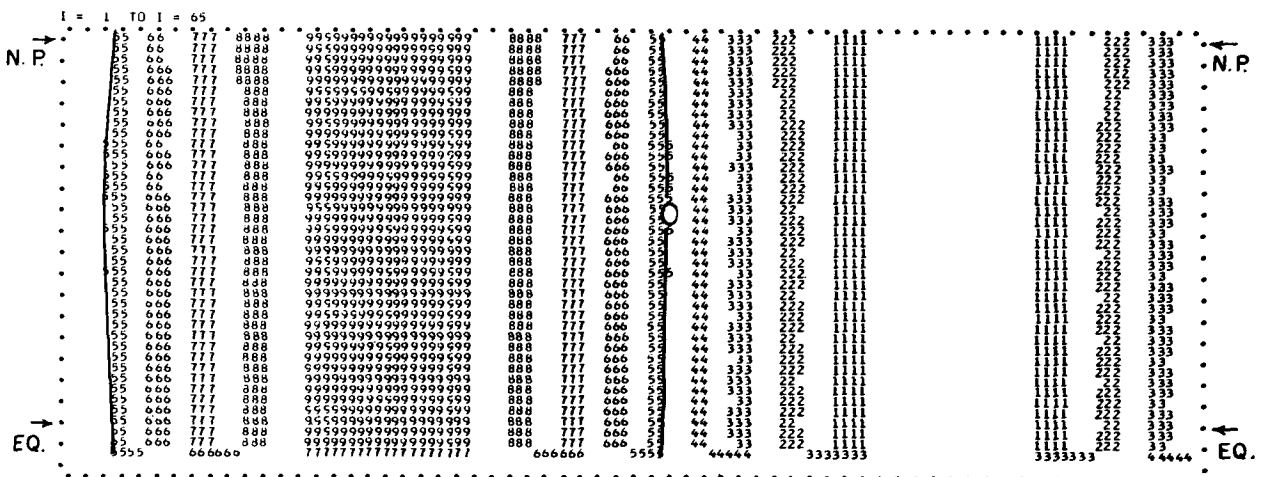


FIG. 2. As in Fig. 1 except for the meridional (v) velocity field.

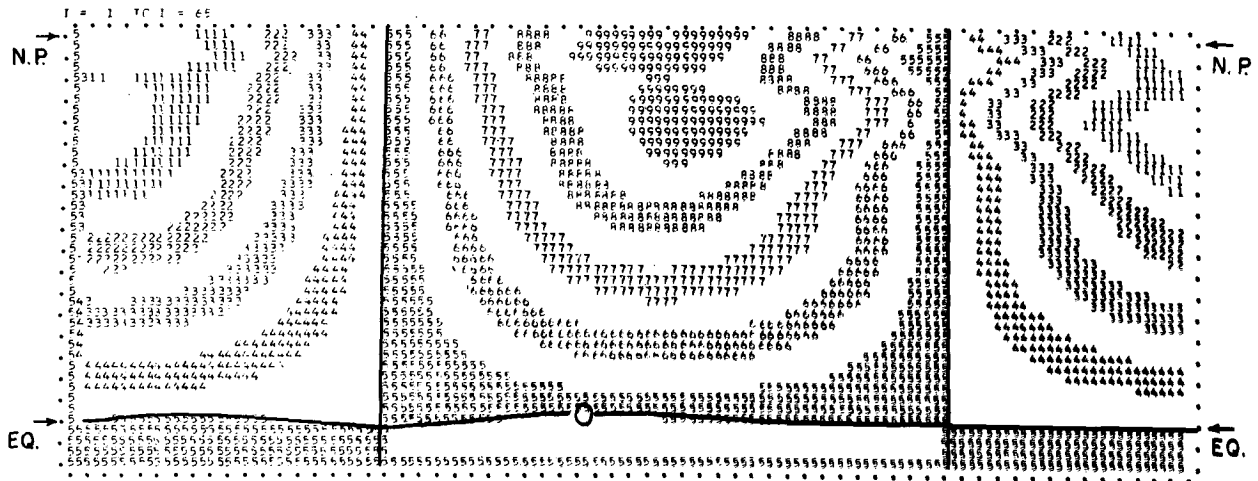


FIG. 3. As in Fig. 1 except the equations used do include a weighted average of the meridional pressure gradient.

As expected, unless $\beta \equiv 0$, the cosine-weighted approximation of the meridional pressure gradient introduces first-order truncation errors near the poles.

3. Numerical experiment

We performed a numerical experiment that confirms the seriousness of this error. A hemispheric shallow water equations model with a latitude-longitude non-staggered grid was used. At the pole a small polar cap was defined on which the height h and the stereographic velocities U and V were forecasted. Three sets of finite difference were used:

$$\frac{\partial h v}{\partial t} = -\frac{g}{a} \frac{-\phi}{h_{\phi}} + \dots \quad (8a)$$

$$\frac{\partial h}{\partial t} = -\frac{(\overline{h v \cos \phi})_{\phi}}{a \cos \phi} + \dots \quad (8b)$$

$$\frac{\partial h v}{\partial t} = -\frac{g}{a} \frac{\overline{h h_{\phi} \cos \phi}}{\cos \phi} + \dots \quad (9a)$$

$$\frac{\partial h}{\partial t} = -\frac{(\overline{h v}^{\phi} \cos \phi)_{\phi}}{a \cos \phi} + \dots \quad (9b)$$

$$\frac{\partial h v}{\partial t} = -\frac{g}{a} \frac{-\phi}{h h_{\phi}} + \dots \quad (10a)$$

$$\frac{\partial h}{\partial t} = -\frac{(\overline{h v}^{\phi} \cos \phi)_{\phi}}{a \cos \phi} + \dots \quad (10b)$$

Eqs. (8) do not contain the average of the meridional pressure gradient. Nevertheless, they have a property similar to (3) and therefore conserve total energy. Eqs. (9) also share this property and are energy-conserving, but they do include a weighted average of

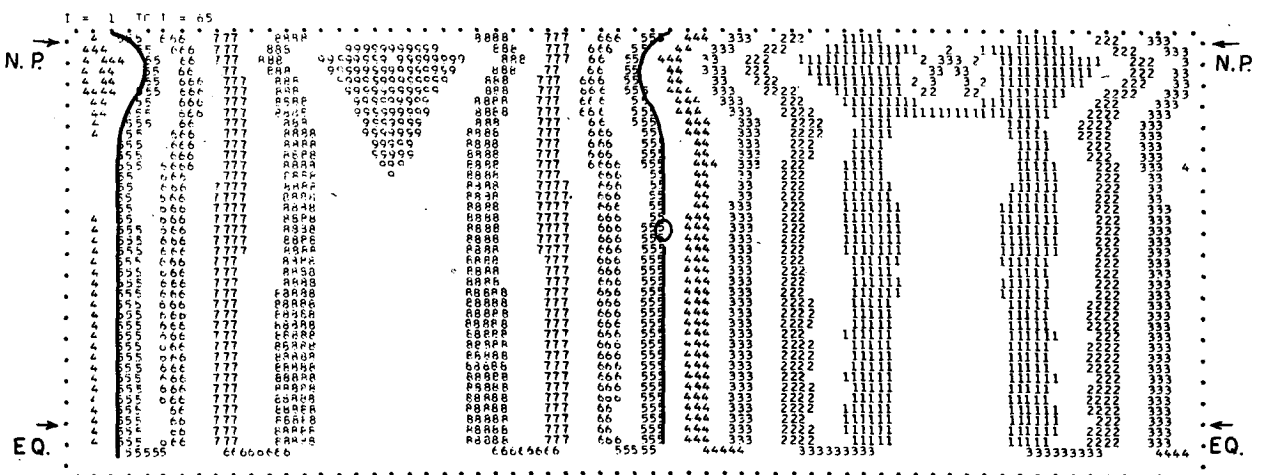


FIG. 4. As in Fig. 3 except for the meridional (v) velocity field.

the meridional pressure similar to the one made in (2). Eqs. (10) are the same as (9) but the averaging of the pressure gradient has been eliminated and therefore they do not conserve energy. (Nevertheless, the non-conservation of energy is due only to the exchange terms between kinetic and potential energy, and therefore the numerical model is still free from nonlinear instability.)

We performed a 6-day numerical integration starting from the non-trivial steady-state solution of the shallow water equations previously used by Umscheid and Rao (1971) and by Williamson and Browning (1973). For this set of initial conditions $\alpha=0$, so that the absolute truncation error in Eq. (9) is of first order in $\Delta\phi$ and the relative error is of order zero, since the pressure gradient itself is of order $\Delta\phi$ near the pole.

Figs. 1 and 2 show the contours of the zonal and meridional velocity fields obtained using Eqs. (10). The results coincide within four significant figures with those obtained using Eqs. (8) and have maximum errors of 0.02 and 0.09 m s^{-1} , respectively. Figs. 3 and 4 show the velocity fields obtained using Eqs. (9). The zonal and meridional velocity fields have maximum errors of 0.82 and 3.7 m s^{-1} , respectively. Clearly, the

truncation error discussed here produces large distortions of the velocity fields at high latitudes and is especially serious for long-term computations of the meridional wind and of meridional transports.

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