On the Thermodynamic Equation for Deep Convection

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7 September 1976 and 21 December 1976

ABSTRACT

A review is given of six equations that are used to approximate reversible saturated parcel ascent. These approximations are compared with the aid of several examples and their appropriateness for use in modeling deep clouds discussed.

The thermodynamic equation in cloud models must embrace both saturated and unsaturated motion. In unsaturated motion apart from sources and sinks of heat or moisture and apart from entrainment, the potential temperature of dry air and the vapor mixing ratio are conserved. (The thermodynamics can also be stated in terms of variables such as the potential temperature of moist air and specific humidity according to one's preference.) In saturated motion the potential temperature and mixing ratio are related through pressure. This relationship is complicated by the fact that air can be supersaturated and can contain both water and ice. Further, the water and ice can move relative to the air and may fall into unsaturated air.

A classical representation of saturated motion is the reversible saturation adiabat which can be written as (see Appendix for list of symbols)

\[ c_p d(ln\theta_d) + c(q_v + q_i) d(lnT) + d(L_v q_v/T) = 0, \]

where \( q_v + q_i \) is constant. The ice adiabatic can be written by replacing \( c \) by \( c_L q_v \) by \( q_v \), \( q_i \) by \( q_i \) and \( L_v \) by \( L_i \). This equation can be derived assuming adiabatic expansion of air that is saturated with respect to a plane surface of water or ice. Also the volume of air occupied by the water or ice is assumed to be small so that the perfect gas law can be applied. Eq. (1) is the starting point for the saturated thermodynamic approximations used in many cloud models where a term representing the fallout of water or ice may be added (e.g., Das, 1969). This note discusses some of the simplifications that have been made and and their appropriateness for representing basic deep convective thermodynamic.

In early cloud model development a convenient approximation to Eq. (1), valid for shallow convection, was initiated by Ogura (1963). This approximation can be written as

\[ dq_v = c_p d\theta_d + (L_v/\pi_{0v}) dq_i = 0, \]

where a constant nondimensional pressure \( \pi_{0v} \) needs to be specified and where \( \phi \) is conserved for both unsaturated and saturated motion. This approximation has also been used by Orville (1965, 1968) and Steiner (1973) for shallow convection and by Orville and Sloan (1970). Another approximation to Eq. (1) that is not as restrictive as Eq. (2) is

\[ c_p d(ln\theta_d) - c_p d\theta_d + d(L_v q_v/T) = 0, \]

which defines the equivalent potential temperature, i.e., the approximate potential temperature of a saturated parcel that is lifted reversibly and adiabatically until \( q_i = 0 \). Here \( \theta_d \) is conserved for saturated motion but not for unsaturated motion since \( L_v q_v (1/T) \neq 0 \). Thus \( \theta_d \) is not a convenient variable for use in representing resolvable scale thermodynamics in cloud models. However, the conservation of \( \theta_d \) has been used to derive the thermodynamic change of parcels that are known to be saturated (e.g., Murray, 1970).

An alternative variable, \( \theta_L \), has been proposed and used by Betts (1973). It approximates the potential temperature attained by evaporating all the liquid water in an air parcel through reversible adiabatic descent until \( q_i = 0 \). The corresponding differential approximation to Eq. (1) is

\[ c_p d(ln\theta_L) = c_p d\theta_d - d(L_v q_i/T) = 0. \]

If the variation in \( L_v/T \) is small, this approximation is similar to Eq. (3) provided \( dq_v = -dq_i \). It is also valid (i.e., \( \theta_L \) is conserved) for saturated and unsaturated motion provided \( q_i = 0 \) in unsaturated regions. Deardorff (1976) has summarized other advantages of using \( \theta_L \), particularly for representing subgrid-scale fluxes in saturated regions of cloud models.

Eq. (4) can be simplified using the power series expansion for an exponent by

\[ \theta_L = \theta_d - \frac{L_v \theta_d}{c_p T} q_i, \]

provided \( L_v/(c_p T) \ll 1 \). This form has been used for shallow convection by Betts (1973), Deardorff (1976) and Sommeria and Deardorff (1977). In the latter paper \( \theta_d \) was determined with the help of the following approximation:

\[ q_v(T, p) = q_v(T_L, p) [1 + B_v] /[1 + B_v q_v(T_L, p)], \]
where

\[ T_L = -\frac{T}{\theta_d} = T - \frac{(L_s/c_p)q_I}{\theta_d} \]

\[ B_1 = 0.622 \frac{L_s T_L^2}{(c_p R_d T_L^2)}. \]

This same technique has been used here for Eq. (5).

Another form of the thermodynamic equation has been investigated by Das (1969). His initial equation is the same as (1) with the addition of a term representing the change in entropy due to liquid entering or leaving a saturated parcel. Das argued that to within 10% the thermodynamic equation can be written as

\[ c_p d(\ln \theta_d) + (L_s/T) dq_s = 0. \quad (6) \]

This equation has been used to determine \( \theta_d \) and \( q_s \) in saturated or supersaturated regions by Soong and Ogura (1973), Soong (1974) and Williamson (1974).

Finally, the water-saturation pseudoadiabat described by

\[ c_p d(\ln \theta_d) + c_q d(\ln T) + d(L_s q_s/T) = 0 \quad (7) \]

is considered. It approximates (1) assuming that \( q_I \) is removed instantaneously from a parcel. Thus it represents the opposite extreme of (1) which assumes that no \( q_I \) is removed. The adiabat for precipitating rain generally would be somewhere between these two.

All the above approximations are known to represent shallow saturated thermodynamics reasonably well. What behavior do they exhibit, however, in relation to Eq. (1) for deep saturated thermodynamics? Of course, some of these approximations have already been investigated. However, the use of \( \theta_L \) for deep convection has not been reported to the author's knowledge, nor has a comparison of all the approximations previously mentioned. Therefore, several comparisons have been made for a saturated parcel rising from pressure level \( p_0 \) along a water or ice adiabat using Eqs. (1)–(7). The saturation vapor pressure used is based on Tetens' formula (Murray, 1967) and is

\[ q_{s0} = (3.8/p) e^{(T - 273)/(T - b)}, \]

where

\[ a = 17.27 \quad \text{for water}, \quad a = 21.87 \quad \text{for ice}. \]

\[ b = 36^\circ \text{C} \quad \text{for water}, \quad b = 7.66^\circ \text{C} \quad \text{for ice}. \]

Also,

\[ T = \pi \theta_d = \theta_d (p/1000)^\kappa, \quad \kappa = R_d/c_p. \quad (8) \]

Orville and Hubbard (1973) have indicated that for deep convection the use of a constant \( L_s \) or \( L_t \) can affect the thermodynamics. Here it has been held constant for simplicity since this will not significantly affect the comparisons to be presented.

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**Fig. 1.** Potential temperature differences between water saturated adiabats described by Eqs. (2)–(7) and the adiabat described by Eq. (1) for several initially saturated parcels that begin ascending at (a) \( p_0 = 900 \text{ mb} \), (b) \( p = 700 \text{ mb} \), (c) \( p_0 = 500 \text{ mb} \). The single asterisk indicates \( p_0 = 500 \text{ mb} \) and the double asterisk indicates \( p_0 = 700 \text{ mb} \).
Eqs. (1)–(5) can be used to calculate $\theta_d$ at lower pressure levels directly when $T$ is replaced by (8). Eqs. (6) and (7) must be numerically integrated upward since the coefficients vary with height. A pressure increment of 10 mb has been used for their integration. Potential temperature differences have been calculated for $i=2$–7 corresponding to Eqs. (2)–(7) using the reversible adiabatic as the reference, i.e.,

$$(d\theta)_i = (\theta_d)_i - (\theta_d)_1.$$ 

Potential temperature differences have been calculated since potential temperature appears in the buoyancy terms of cloud models such as Wilhelmsen’s (1974). Because $\tau<1$ they accentuate temperature differences. The results of calculations using water saturation for $p_0=900$, $\theta_d(p_0)=293$, for $p_0=700$, $\theta_d(p_0)=303$ and $p_0=500$, $\theta_d(p_0)=327$ are shown in Figs. 1a–1c.

Eq. (6) compares most favorably with the moist saturated adiabat. During the first half of the parcel ascent $(d\theta)_8$ is slightly positive indicating increased buoyancy associated with its use. Above 400 mb $(d\theta)_8$ is negative and decreases exponentially.

The $(d\theta)_1$ are always negative and decrease exponentially with $p$ in upper levels. This is the same result as Saunders (1957). At higher levels the $\theta$ values given by (6) lie between those of the saturated and pseudo adiabats. Overall Eq. (6) is a better approximation than (7) since the $c_d(\ln T)$ and $L_{d}(1/T)$ terms are comparable in magnitude but opposite in sign. Thus it is better to neglect both in (1).

The $(d\theta)_3$ are always negative and become noticeably worse than $(d\theta)_8$ after parcel ascent of several hundred millibars.

The $(d\theta)_2$ depend strongly on the choice of the constant $\pi_{80}$. For the $p_0=900$ mb case the absolute error is similar to $(d\theta)_8$ when $\pi_{80}$ is computed using $p_0=700$ mb and worse if $p_0=500$ mb. For $p_0=500$ mb the opposite is true. This suggests that unless the parcel origin can be assumed to occur within a limited vertical region, as in shallow convection, it is difficult to choose $\pi_{80}$ such that $(d\theta)_2$ is reasonably small. Thus, the use of (2) to represent the water saturation adiabat of deep convection is not suggested.

The $(\theta_d)_1$ and $(\theta_d)_3$ are always positive and tend to deviate substantially from the saturated adiabatic values in upper levels. This implies a significant increase in buoyancy. Fig. 2 illustrates the difference in buoyancy between the saturated adiabatic values and buoyancy values determined for the case $p_0=900$ mb. The buoyancy difference $(dB)_i$ is computed as

$$(dB)_i = B_i - B_1,$$

$$B_i = \frac{(\theta_d)_i}{\theta_d(p)} + 0.61(q_e)_i - (q_l)_i, \quad i=3, 4, 6, 7,$$

$$\theta_d(p) = 298 + (900-p)(3/50)[^\circC].$$

where $B_i$ represents the buoyancy terms (e.g., Wilhelmsen, 1974) and where $\theta_d(p)$ represents an environmental potential temperature. Computations showed that the differences are due mainly to the $(d\theta)_i/\theta_d(p)$ term. Thus $(dB)_i$ changes like $(d\theta)_i$. Because of the large positive $(d\theta)_1$, $(d\theta)_3$, and $(dB)_i$ the liquid water potential temperature formulations do not seem appropriate for representing deep thermodynamics.

In a shallow but precipitating cloud model there is some difficulty in using the liquid water potential temperature. Below the cloud, for example, rain can exist and evaporate in an unsaturated environment. The change in rain content cools the air and so $\theta_d$ decreases. A formulation of the evaporation rate is then needed in terms of $\theta_L$ and $q_r$ (the precipitating rain). A unique formula, however, is not possible. This is because $q_r$ is not uniquely related to temperature $T$ unless $q_r=q_{es}$ (see Deardorff, 1976). Thus any precipitating water $q_r$ should be treated separately and the change in $\theta_L$ due to the evaporation of rain computed from changes in $d\theta_L$ and $dq_r$. Then the subgrid parameterization of $\theta_L$ and of the buoyancy term become more complicated than in Deardorff (1976) if subgrid evaporation in an unsaturated air is to be represented.

So far only examples of saturated thermodynamics for air that contains no ice has been considered. In deep convection rising parcels will eventually contain only ice although significant ice may not occur in strong cumulonimbus updrafts until homogeneous freezing around $-36^\circ$C or about 300 to 250 mb (Young, 1975). Figs. 3a and 3b show $(d\theta)_i$ for Eqs. (3), (4), (6) and (7)
and parcels rising along ice adiabats with $p_0$ and $\theta_0$ as before. The results indicate larger positive $(d\theta)_5$ than for the corresponding warm saturated rise. The negative $(d\theta)_3$ are again smaller than those from the pseudo adiabat $(d\theta)_7$. The $(d\theta)_4$ are again positive and notably worse than the others as $p$ decreases. As expected these results show the same tendencies as those in Figs. 1 and 2 for the water saturation cases.

The thermodynamics of parcels containing both water and ice is more complicated, particularly if water and ice coexist inside a cloud. Basic questions include how to determine the amount of moisture available for condensation or deposition (i.e., what reference saturation vapor pressure to use), how to represent the Bergeron process, and how to include the effects of freezing. The latter has been investigated by Orville and Hubbard (1973) and Chappel and Smith (1975). Answers to these questions will modify parcel adiabats presented here but will not remove any bad tendencies that these adiabats have. Entrainment and turbulence will also affect parcel adiabats. In light of these considerations and one’s objectives an appropriate thermodynamic simplification can be chosen.

A good approximation to Eq. (1) for representing deep, saturated, convective thermodynamics appears to be Eq. (6). The use of Eq. (4) defining the liquid water potential temperature does not appear to be appropriate. This is unfortunate because of the properties of $\theta_L$, particularly its appropriateness for use in specifying subgrid scale processes. However, it is possible to use $\theta_L$ to describe subgrid-scale processes and convert this information into subgrid-scale representation for $\theta_d$ and $q_e$. This is currently being done for a three-dimensional model by the author in cooperation with Dr. Joseph Klemp at NCAR.

**Acknowledgments.** Some of the integrations in this note were carried out by Carl Youngblut. This research has been sponsored by the National Science Foundation under Grant ATM 75-19336.

**APPENDIX**

**List of Symbols**

- $c$: specific heat of water ($4.2 \times 10^3$ $\text{J kg}^{-1} \text{K}^{-1}$)
- $c_i$: specific heat of ice ($2.1 \times 10^3$ $\text{J kg}^{-1} \text{K}^{-1}$)
- $c_p$: specific heat at constant pressure for dry air ($1 \times 10^3$ $\text{J kg}^{-1} \text{K}^{-1}$)
- $L_v$: latent heat of vaporization ($2.5 \times 10^6$ $\text{J kg}^{-1}$)
- $L_i$: latent heat of sublimation ($2.85 \times 10^6$ $\text{J kg}^{-1}$)
- $p$: atmospheric pressure (mb)
- $p_0$: atmospheric pressure at the initial level of rising parcels
- $R_d$: specific gas constant ($287$ $\text{J kg}^{-1} \text{K}^{-1}$)
- $q_i$: ice mixing ratio
- $q_v$: vapor mixing ratio
- $q_s$: saturation mixing ratio over water
- $q_{sw}$: saturation mixing ratio over ice
- $T$: temperature
- $\theta_d$: potential temperature of dry air
- $\theta_v$: equivalent potential temperature
- $\theta_L$: liquid water potential temperature
- $\pi$: nondimensional pressure

**REFERENCES**


