The Lognormal Distribution and Cumulus Cloud Populations

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(Manuscript received 14 December 1976, in final form 24 March 1977)

ABSTRACT

It is shown that the lognormal distribution describes the frequency distributions of height, horizontal size, and duration of cloud and radar echo populations in many different regions and convective situations. Two hypotheses are suggested to explain this phenomenon. The first postulates a *growth* process of cloud parcels, in which growth by entrainment of environmental air occurs by a random process that obeys the law of proportionate effects. The second postulates a *formation* process for clouds, in which the clouds are formed by the merger of random boundary-layer convective elements.

The information presented in this paper should be useful for the parameterization of cumulus convection in larger scale models, and for the understanding and modeling of cloud formation and development.

1. Introduction

In a recent paper, López (1976) found that the frequency distributions of the height and maximum horizontal area attained by radar echoes of tropical disturbances in the northwest Atlantic, follow lognormal[2] distributions. At the same time, Biondini (1976) found that the duration and rainfall volume of Florida radar echoes are also lognormally distributed. A question immediately arises: Is this a characteristic particular to cumulus cloud populations of tropical maritime systems, or is this a general property of cumulus cloud populations everywhere? If lognormality is indeed a general characteristic of cumulus convection, the efforts to parameterize the effects of clouds in large-scale numerical models would be greatly benefited. In addition, by extending the theory of the genesis of lognormal distributions to cumulus convection, our knowledge of cloud-formation processes would be enhanced.

Accordingly, the objective of this paper is to determine if lognormality is a general characteristic of cumulus convection. For this purpose, many cloud and radar echo populations (for different regions and varying large-scale conditions) are examined to see if their frequency distributions of height, horizontal size and duration are lognormal. As will be seen, the vast majority of cases do indeed indicate lognormality. The few cases that depart from this tendency are shown to respond to physical bounds to growth which produce *truncated* lognormal distributions.

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[2] A variable is said to be lognormally distributed when its logarithm follows the normal probability law.

2. Characteristics of cumulus cloud populations

a. Height

Figs. 1 and 2 present the cloud and echo height distributions of six different cumulus cloud populations. For convenience they are separated into tropical and extratropical situations. The frequency distributions are plotted on logprobability paper: the ordinate is plotted in a logarithmic scale, while the abscissa is plotted in a normal probability scale. A lognormal distribution would thus describe a straight line in this coordinate system (Aitchison and Brown, 1957, pp. 31–35). The straight lines drawn through the different sets of points correspond to the lognormal distributions that best fit the data and give a smaller χ^2 value for each population. Table 1 shows the best-fit parameters of the different distributions.

A chi-square test for goodness-of-fit has been applied to these data sets. This test has been amply described in the literature (see, e.g., Meyer, 1975). For the purposes of this paper it suffices to say that if a population is distributed lognormally (with geometric mean and standard deviation μ' and σ', and with interval frequencies f_i') and if random samples are drawn from that population (with N interval frequencies f_i each), then the sample chi-squared statistic

$$\chi^2=\sum_{i=1}^{N}\frac{(f_i-f_i')^2}{f_i'}$$

would be distributed according to the χ^2 function with $N-3$ degrees of freedom. Moreover, this function has the property that large values for χ^2 have a small probability of occurring under random sampling. Thus, if a random sample from a supposedly lognormal popu-

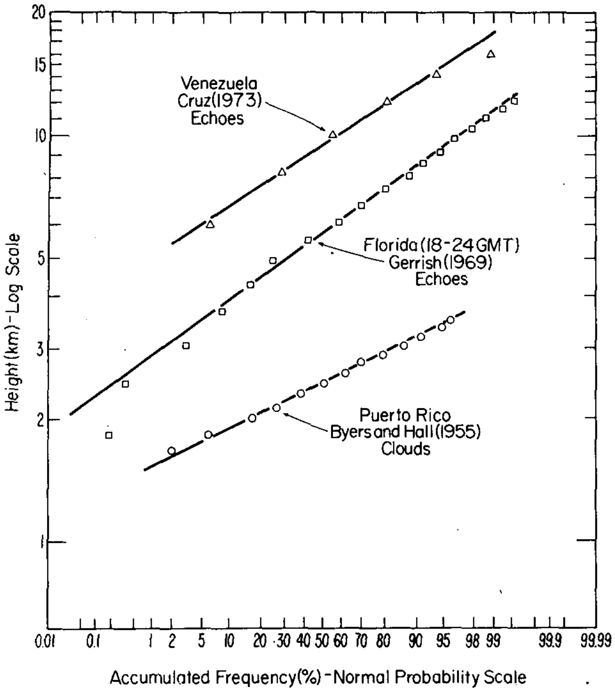

FIG. 1. Accumulated frequency distributions of cloud and radar echo height for tropical data sets. The straight lines correspond to the lognormal distributions that best fit the different data sets.

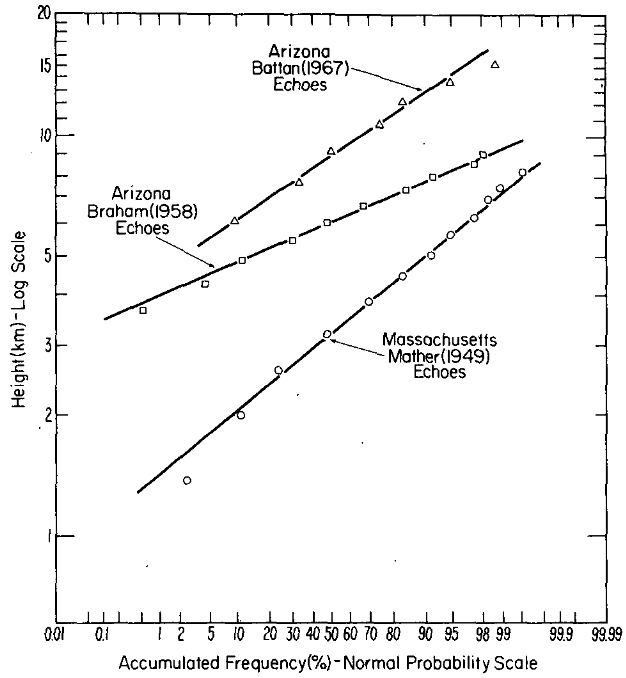

FIG. 2. As in Fig. 1 except for extratropical data sets.

lation (null hypothesis) produces a large χ^2 value which has a small probability of occurrence according to the corresponding χ^2 distribution, then the null hypothesis is rejected. The "small" probability level has been selected at 5% for the present application, which is a value commonly used. The next to the last column of Table 1 gives the corresponding χ^2 value for this probability for each of the distributions.

As can be seen from the table the χ^2 values obtained from the samples are in all cases smaller than the value at a level of significance of 5%. Thus the lognormal hypothesis is consistent with the data at this level of significance. In fact the probability of obtaining a value of χ^2 as large as or larger than the value measured is generally high (last column of Table 1). The case giving the worst fit is Battan (1967). It is shown below that data by Battan (1953) also produce the worst lognormal fit in respect to echo horizontal dimension. In both cases, however, the lognormal hypothesis cannot be rejected at a level of significance of 5%.

Notice how the lognormal probability law describes the height distribution of echoes and clouds in a wide variety of situations: from the trade winds to maritime and continental tropical conditions, and from moist to dry mid-latitude environments. Although the shape of the distributions is the same for all cases, the means and standard deviations vary considerably from case to case. In general, it can be observed from Figs. 1 and 2 that the echoes tend to increase (the geometric mean corresponds to the 50% intercept of the lines) as one moves from oceanic to continental conditions in the tropics, and from moist to dry conditions in the middle latitudes.

TABLE 1. Cloud and radar echo height distribution.

			Best-fit parameters (lognormal)					
Author	Location	Feature	Geometric mean	Geometric sigma	Degrees of freedom	χ^2	Value of χ^2 at a level of significance of 5%	Probability of χ^2 value as large as measured or larger
Gerrish (1969)	Miami over water	Echo height (km)	5.71	1.97	8	3.74	15.50	88%
Braham (1958)	Arizona	First echo height (km)	6.08	1.23	5	3.02	11.07	70%
Battan (1967)	Arizona	Echo height (km)	8.87	2.98	4	7.79	9.49	10%
Cruz (1973)	Venezuela	Max echo height (km)	9.38	2.99	4	2.93	9.49	55%
Byers and Hall (1955)	Puerto Rico	Cloud height (km)	2.44	0.54	8	2.79	15.50	95%
López (1976)	NW Atlantic	Echo height (km)	4.47	2.58	12	3.61	21.03	99%
Mather (1949)	Massachusetts	Echo height (km)	3.17	1.31	5	3.79	11.07	58%

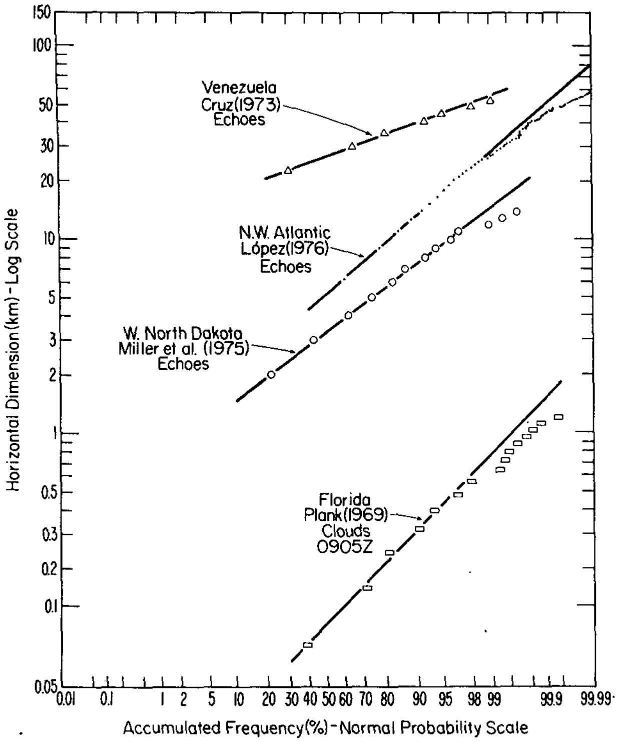

FIG. 3. Accumulated frequency distributions of cloud and echo horizontal dimension. The straight lines correspond to the lognormal distributions that best fit the different data sets.

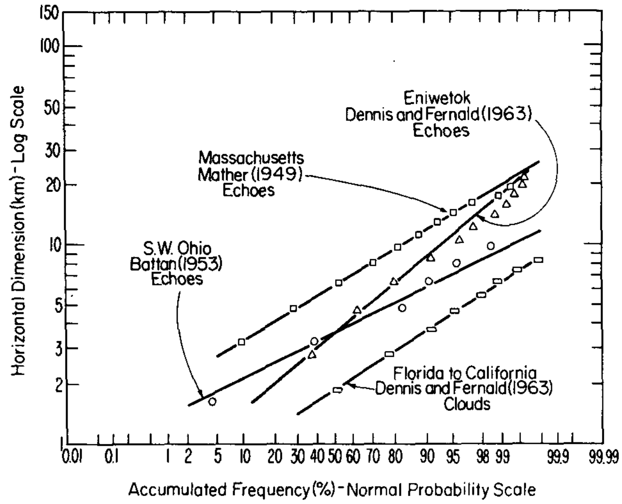

FIG. 4. Continuation of Fig. 3.

b. *Horizontal dimension*

Figs. 3 and 4 show the distribution of horizontal size of eight cumulus cloud populations. Originally the data were expressed variously as diameter, radius, length and area distributions. For purposes of comparison, all the data points have been reduced to an equivalent diameter or characteristic horizontal dimension. Again, the distributions have been plotted on logprobability paper. Similarly, straight best-fit lines have been drawn through the data points. Table 2 presents the corresponding parameters.

As in the case of height, the hypothesis that the frequency distributions of echo and cloud horizontal dimension are lognormal cannot be rejected at a level of significance of 5% or better. The fits are very good for the smaller 98% of the elements of the populations. The distribution of the larger 2% deviate somewhat from a lognormal law. This deviation, however, is not significant enough to warrant the rejection of the lognormal hypothesis. It will be shown that these small deviations correspond to a physical limit to growth. The resultant distribution can be better described by a truncated lognormal law. As in the case of the height distributions, the worst fit corresponds to the data obtained by Battan (1953). However, the data scatter is not large enough to justify rejecting the lognormality of this distribution at 5% level of significance. In general it can be said that the lognormal probability law

TABLE 2. Cloud and radar echo horizontal size distribution.

			Best-fit parameters (lognormal)					
Author	Location	Feature	Geometric mean	Geometric sigma	Degrees of freedom	χ^2	Value of χ^2 at a level of significance of 5%	Probability of χ^2 value as large as measured or larger
Hudlow (1971)	Barbados	Echo length (km)	2.30	8.16	1	0.10	3.84	75%
Cruz (1973)	Venezuela	Echo max. area (100 km²)	5.64	4.90	3	1.96	7.81	59%
Mather (1949)	Massachusetts	Mean echo diameter (km)	6.22	4.03	5	0.82	11.07	98%
Battan (1953)	SW Ohio	Max. horiz. echo dimension (km)	3.59	1.84	2	4.68	5.99	10%
Plank (1969)	Florida, 0905 GMT	Equivalent cloud diameter (km)	0.11	0.15	3	2.68	7.81	44%
Miller *et al.* (1975)	NW Dakota	Echo diameter (km)	3.31	2.99	5	0.56	11.07	99%
Dennis and Fernald (1963)	Florida to California	Cloud length (km)	1.85	1.32	2	0.32	5.99	85%
Dennis and Fernald (1963)	Eniwetok	Shower radius (km)	1.75	1.71	3	2.19	7.81	53%
López (1976)	NW Atlantic	Max. echo area (km²)	20.96	71.20	3	0.03	7.81	99%

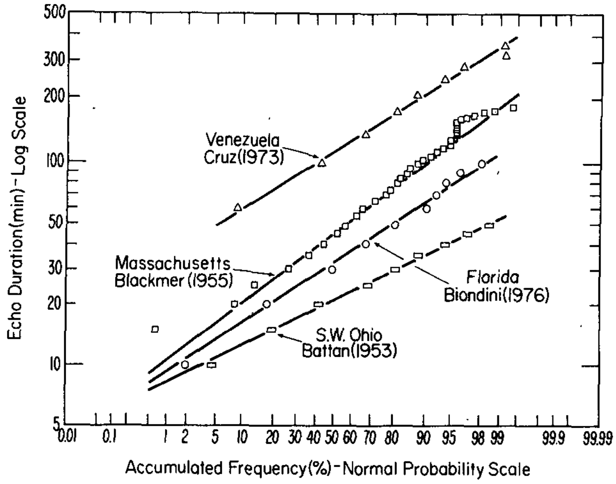

FIG. 5. Accumulated frequency distributions of radar echo duration. The straight lines correspond to the lognormal distributions that best fit the different data sets.

can be fitted very well to echo and cloud horizontal dimension distributions.

As in the case of height, it is worthwhile noting the wide range of climatic situations in which the lognormal has been found applicable: maritime and continental tropical regions; dry and moist, coastal and continental mid-latitudes. In general, continental tropical echoes tend to be larger than echoes in Atlantic tropical cloud clusters, and they are larger than the clouds induced by the Florida sea breeze.

c. Duration

Fig. 5 presents the lognormal plots of the frequency distributions of radar echo duration for four cumulus cloud populations. Again, best-fit straight lines have been drawn. Table 3 shows the pertinent statistical parameters. Once more, the hypothesis that the frequency distributions of echo duration are lognormal cannot be rejected at a level of significance of 5% or better. The lognormal law can be fitted very well to all of the echo duration distributions. Again, tropical and extratropical, maritime and continental populations all are lognormally distributed.

3. The lognormal distribution and cumulus cloud growth processes

a. The genesis of lognormal distributions

From the evidence presented in the last section it can be said that echo and cloud height, horizontal size and duration are distributed lognormally in a wide variety of convective situations. The question immediately arises: What is it in the formation and growth process of cumulus clouds that produces cloud populations which are lognormally distributed? It is reasonable to assume that the formation and growth processes in cumulus clouds follow the same stochastic processes that determine the genesis of the lognormal probability law. The lognormal distribution can be considered (Kapteyn, 1903; Aitchison and Brown, 1957, pp. 20–27) the frequency distribution of a variate that is subject to the law of proportionate effects, i.e., a variate whose change in value at any step of a process is a random proportion of the previous value of the variate. Thus, consider a variate whose value is x_0 at the start of the process and x_i at the ith step, reaching a value x_n at the end of n steps. At the ith step the change in value can be expressed as

$$x_i - x_{i-1} = \epsilon_i x_{i-1}, \qquad (1)$$

where ϵ_i is a random variable, independent of x. Considering n steps,

$$\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{x_{i-1}} = \sum_{i=1}^{n} \epsilon_i. \qquad (2)$$

If the change at each step is small,

$$\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{x_{i-1}} \sim \int_0^{x_n} \frac{dx}{x} = \ln x_n - \ln x_0, \qquad (3)$$

so that

$$\ln x_n = \ln x_0 + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n. \qquad (4)$$

By the additive form of the central-limit theorem, $\ln x_n$ is in the limit normally distributed and therefore x_n is lognormally distributed. So, something that is formed, grows or changes according to the law of proportional effects will yield a lognormal size distribution.

TABLE 3. Radar echo duration distribution.

			Best-fit parameters (lognormal)					
Author	Location	Feature	Geometric *mean*	Geometric *sigma*	Degrees of *freedom*	χ^2	Value of χ^2 at a level of significance of 5%	Probability of χ^2 value as large as measured of larger
Cruz (1973)	Venezulea	Echo duration (min)	113.98	75.93	5	3.30	11.07	65%
Blackmer (1955)	Massachusetts	Echo duration (min)	44.72	38.58	11	5.57	19.68	90%
López (1976)	NW Atlantic	Echo duration (min)	2.29	9.89	5	2.27	11.07	81%
Battan (1953)	SW Ohio	Echo duration (min)	21.01	10.41	4	2.53	9.49	64%
Biondini (1976)	Florida	Echo duration (min)	32.38	22.84	4	3.87	9.49	42%

Both Biondini (1976) and López (1976), working simultaneously and independently, have hypothesized that cumulus clouds could grow and develop by a process which follows the law of proportionate effects, thus producing the observed lognormal distributions of echo characteristics. Although Biondini speculates about a "multiplicative diffusion process" to explain cloud formation, neither he nor López discussed the particular physical mechanism that could be responsible for the characteristic growth patterns. In the next two sections two hypotheses are advanced to explain how clouds could develop according to the law of proportionate effects. The first has to do with growth by turbulent diffusion and the second with growth by the merger of smaller cloud elements.

b. *Stochastic growth process*

Applying the stochastic process described above to the *growth* of clouds, a process like this can be hypothesized:

1) The large-scale features of the flow (convergence field, thermal stability, etc.) produce an initial population of small convective elements, the average size of this initial population probably being related to the depth of the local boundary layer.

2) These elements then develop by a process whereby growth is a random proportion of the size of the element, i.e., by the process of proportionate effects.

3) In this way a cloud population develops whose eventual size distribution is lognormal.

In terms of the mass m of a cloud parcel, step 2 above can be represented as

$$\frac{dm}{dt}=m\epsilon, \tag{5}$$

where ϵ is a random number. This can be rewritten as

$$\frac{1}{m}\frac{dm}{dt}=\epsilon. \tag{6}$$

It can now be seen that ϵ is just the fractional entrainment rate for the cloud parcel. This quantity figures prominently in parametric models of cumulus clouds. It has been variously parameterized in terms of the mean vertical velocity of the cloud parcel and its radius (e.g., Simpson and Wiggert, 1969) and in terms of the parcel turbulence intensity and its radius (López, 1973).

In order to understand physically how the entrainment rate could be a random quantity, consider the growth of the parcel by entrainment as a flux of mass through the surface of the parcel. Thus

$$\frac{dm}{dt}=\rho_e\bar{v}_eA, \tag{7}$$

where ρ_e is the density of the entrained air and $\bar{v}_e$ the mean velocity of the entrained mass averaged over the surface area A of the parcel. Let m be the mass of the parcel, where

$$m=\rho_pV, \tag{8}$$

V being the valume and ρ_p the density of the parcel. Dividing Eq. (7) by (8) and setting $\rho_p\approx\rho_e$, one obtains

$$\frac{1}{m}\frac{dm}{dt}=\frac{A}{V}\bar{v}_e. \tag{9}$$

Let the ratio of the surface to the volume of the parcel be

$$\frac{A}{V}=\frac{K}{R}, \tag{10}$$

where R is the parcel's radius and K a geometric constant depending on the configuration of the parcel. Eq. (9) now becomes

$$\frac{1}{m}\frac{dm}{dt}=K\frac{\bar{v}_e}{R}. \tag{11}$$

By comparing Eqs. (6) and (11) it can be seen that the entrainment rate would be random (and the cloud parcel would grow according to the law of proportionate effects) if the velocity of entrainment averaged over the surface of the parcel were a random variable.

Now, the entrainment of outside air into the cloud parcel is controlled by the turbulence near the surface of the cloud. The convolutions and invaginations of the cloud surface trap the exterior air and draw it inside (Turner, 1962, 1963). Thus, the mean velocity of entrainment at a given moment during the growth of a parcel depends on the particular turbulent configuration of the parcel's surface. This is, of course, a random variable. Thus, the mean velocity of entrainment would be a random variable, also. As a consequence, the fractional entrainment rate would be random. The law of proportionate effects would apply, and the resulting distribution of cloud size would be lognormal.

From strictly dimensional arguments it can be reasoned (Telford, 1966; Morton, 1968) that the velocity of entrainment fluctuates randomly around a value that is determined by the root-mean-square of the velocity fluctuations inside the parcel (turbulence intensity). Thus, although the value of the velocity of entrainment for the particular times during the growth of a parcel is going to vary randomly, its time average is going to be dependent upon the average turbulent intensity of the cloud parcel. Upon integration of Eq. (11), one obtains

$$\ln m=\ln m_0+K\int_0^t\frac{\bar{v}_edt}{R}, \tag{12}$$

where m is the final mass at time t, and m_0 the initial mass of the cloud parcel. Notice that, in addition to the initial mass, the final size depends on the mean ratio between the velocity of entrainment (fluctuating randomly around a value determined by the degree of turbulence) and the radius of the parcel. Thus, although this hypothesis would explain the observed lognormal size distribution of fields of clouds, it also retains the R^{-1} and turbulence intensity dependency in the entrainment formulation that have been well established in the cumulus dynamics literature.

It should be noted that in all of the entity type cloud models (e.g., Simpson and Wiggert, 1969; Weinstein and Davies, 1968; López, 1973) a different initial radius has to be specified to generate each cloud of different size under the same thermodynamic situation. In addition, nothing can be said about the different proportions of cloud types to be expected in a given case. The present hypothesis, however, by assuming a stochastic process and a characteristic initial convective element size, would explain the size distribution of the entire convective field under a given thermodynamic situation. The author is working on the theory of a stochastic cloud model based upon the present hypothesis to test the possibility of producing populations of clouds that are lognormally distributed in size and duration.

c. Stochastic formation process

In the previous section it was indicated that the lognormality of cloud and radar echo distributions can be explained by assuming that the clouds *grow* to their final size from an initially small volume by the process of random entrainment of environmental air. Lognormality can also be explained by postulating a process by which clouds of a particular size are *formed* by the merger of smaller elements. Thus, the following second hypothesis can also be made to explain the occurrence of lognormal cloud size distributions:

1) In a given region many small convective elements are randomly formed throughout the subcloud layer.

2) As they rise and expand they aglomerate into larger elements.

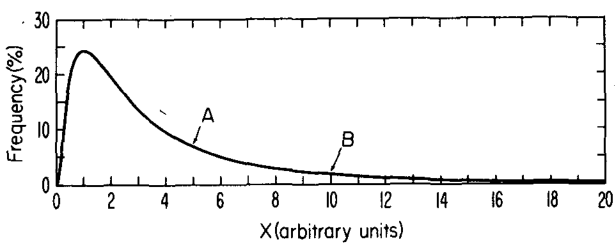

FIG. 6. A theoretical lognormal frequency distribution with geometric mean and standard deviation equal to 1 in arbitrary units of x. Points A and B are identified in Fig. 7.

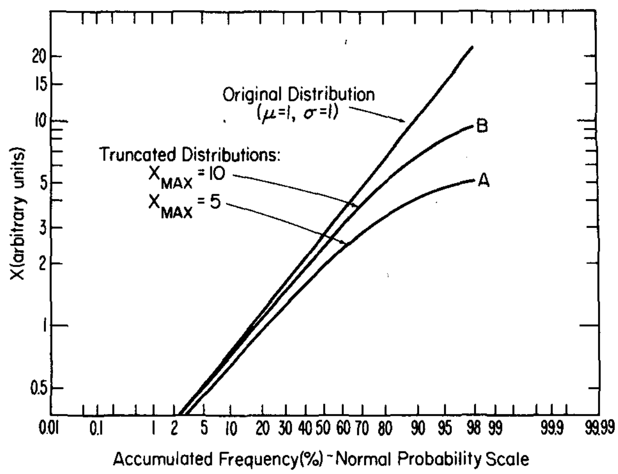

FIG. 7. The accumulated frequency distributions of a theoretical lognormal distribution (straight line) and the same for two truncated lognormals (curves A and B).

3) The larger elements, covering a greater area, intercept a larger number of other elements and thus grow more rapidly than the smaller ones, i.e., growth proceeds according to the law of proportional effects.

4) In this way a cloud population develops whose eventual size distribution is lognormal.

The author is also working on the development of a stochastic cloud formation model of this type.

This merging process probably continues during the latter stages of cloud life. Evidence for this mechanism can be seen in the studies of cloud mergers by Simpson and Woodley (1971) and López (1976). It is not the purpose of this paper to formulate the theoretical models that can be derived from the abovementioned hypotheses, but merely to advance these two hypotheses of growth and formation as possible explanations of the observed lognormal distributions of echo and cloud size and duration.

d. Truncated lognormals and limits to cloud growth

It was noted in Section 2 that some of the logprobability plots of size distributions departed to the right of a straight line (lognormal distribution) for the largest 2% of the echo and cloud sizes (see Fig. 3). This departure indicates that there are more echoes or clouds observed in that size range than are called for by the lognormal law, and at the same time that the lognormal predicts some elements with larger sizes that were not observed. This is exactly the situation to be expected if there is a physical limit to the growth of clouds.

In order to illustrate this limiting effect on a lognormal distribution some theoretical examples can be considered. Fig. 6 shows a theoretical lognormal frequency distribution with mean and standard deviation equal to 1 in arbitrary units of the variable x. The logprobability plot of this distribution is shown as the straight line of Fig. 7. The theoretical lognormal has

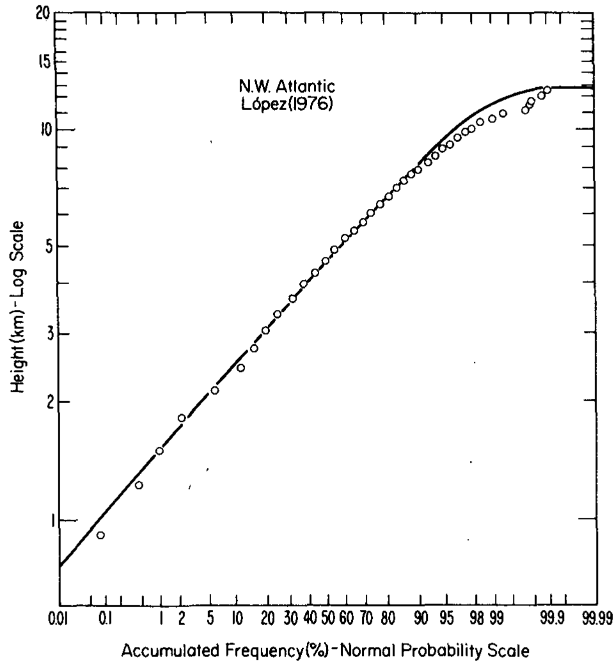

FIG. 8. Accumulated frequency distribution of radar echo height. The solid curve corresponds to the truncated lognormal distribution that best fits the data.

been arbitrarily truncated at $x=5$ (point A) and $x=10$ (point B) (Fig. 6). The logprobability plots of the resulting (renormalized) distributions are shown as curves A and B in Fig. 7. Notice that the curves depart to the right of the parent full lognormal in the same way as the size distributions of Fig. 5.

Accordingly, truncated lognormals were fitted to the three distributions that departed most from a full lognormal in the highest values. These three distributions were not pictured in Figs. 1–5, but their regular lognormal-fit parameters are shown in Tables 1–3. Figs. 8–10 portray the original distributions and the corresponding truncated lognormal fits. These were truncated at the maximum observed dimension. In general, the truncated lognormals fit the observed distributions much better than the corresponding regular lognormals do, especially in the range of largest sizes and durations. Still, however, some departure from the theoretical distributions toward higher frequencies is noticed in all three examples. It must be recalled that the theoretical distributions were arbitrarily truncated at the observed maximum dimension. This procedure tacitly assumes that only clouds that would grow to be smaller than the maximum size were initially generated. In reality, however, clouds might be generated that potentially could be larger than the maximum permitted by the environment. Those clouds would then be stunted in their development and would appear to be of the same size as clouds that just reach unhindered the maximum size. Thus the frequencies of the upper size range would appear inflated as observed.

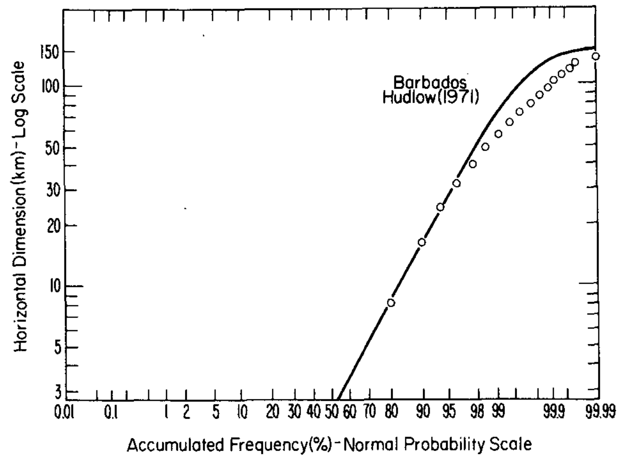

FIG. 9. Accumulated frequency distribution of radar echo horizontal dimension.

4. Summary and conclusions

This paper has shown that the lognormal distribution describes well the frequency distributions of height, horizontal size and duration observed in cloud and radar echo populations in many different regions and convective situations.

Two hypotheses have been suggested to explain this phenomenon. One has to do with the growth process of cloud parcels by mixing with environmental air. Here *growth* by entrainment occurs by a random process that obeys the law of proportionate effects. The second hypothesis postulates a *formation* process of clouds by mergers of smaller elements. Here the size of the cloud being formed by the conglomeration of random boundary layer convective elements depends on how many elements have joined already.

The importance of the ideas presented in this paper should again be emphasized. First, here is the possibility of an analytical expression for cloud populations.

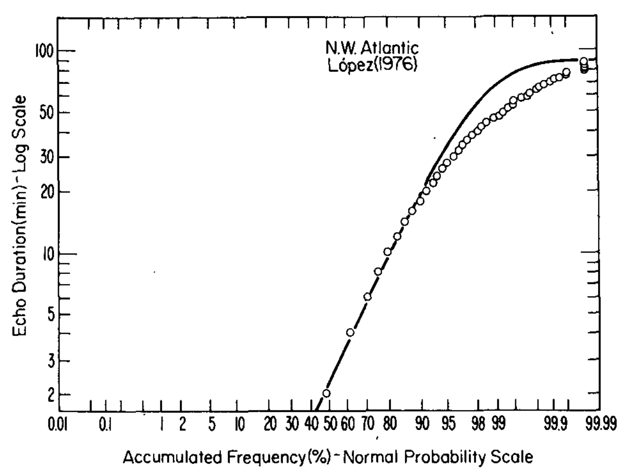

FIG. 10. Accumulated frequency distribution of radar echo durations.

This is of crucial importance for the parameterization effort. Second, the evidence presented here indicates that cloud formation and/or growth may not be an entirely deterministic process, but may instead involve a significant stochastic mechanism that follows the law of proportionate effects.

Acknowledgments. Part of this work was done while the author was at the University of Puerto Rico under financing by National Science Foundation Grant OCD 73-00605 A03. The rest of the work was done at the Wave Propagation Laboratory, NOAA.

REFERENCES

Aitchison, J., and J. A. C. Brown, 1957: *The Lognormal Distribution*. Cambridge University Press, 176 pp.

Battan, L. J., 1953: Duration of convective radar cloud units. *Bull. Amer. Meteor. Soc.*, **34**, 227–228.

——, 1967: Silver iodide seeding and radar echoes from convective clouds. Sci. Rep. No. 23, Institute of Atmospheric Physics, University of Arizona, 11 pp.

Biondini, R., 1976: Cloud motion and rainfall statistics. *J. Appl. Meteor.*, **15**, 205–224.

Blackmer, R. H. Jr., 1955: The lifetime of small precipitation echoes. *Preprints Fifth Conf. Radar Meteorology*, Ft. Monmouth, N. J., Amer. Meteor. Soc., 103–108.

Braham, R. R., Jr., 1958: Cumulus cloud precipitation as revealed by radar—Arizona, 1955. *J. Meteor.*, **15**, 78–83.

Byers, H. R., and R. K. Hall, 1955: A census of cumulus-cloud height versus precipitation in the vicinity of Puerto Rico during the winter and spring of 1953–1954. *J. Meteor.*, **12**, 176–178.

Cruz, L. A., 1973: Venezuelan rainstorms as seen by radar. *J. Appl. Meteor.*, **12**, 119–126.

Dennis, A. S., and F. G. Fernald, 1963: Frequency distribution of shower sizes. *J. Appl. Meteor.*, **2**, 767–769.

Gerrish, H. P., 1969: Mesoscale studies of instability patterns and winds in the tropics. Radar Meteorological Laboratory, Institute of Marine and Atmospheric Sciences, University of Miami, 77 pp.

Hudlow, M. D., 1971: Three dimensional model of precipitation echoes for x-band radar data collected during BOMEX. Bomex Bulletin No. 10, BOMAP Office, NOAA, 51–63.

Kapteyn, J. C., 1903: *Skew Frequency Curves in Biology and Statistics*. Astronomical Laboratory, Groningen: Noordhoff (Quoted by Aitchison and Brown, 1957.)

López, R. E., 1973: A parametric model of cumulus convection. *J. Atmos. Sci.*, **30**, 1354–1373.

——, 1976: Radar characteristics of the cloud populations of tropical disturbances in the northwest Atlantic. *Mon. Wea. Rev.*, **104**, 269–283.

Mather, J. R., 1949: An investigation of the dimensions of precipitation echoes by radar. *Bull. Amer. Meteor. Soc.*, **30**, 271–277.

Meyer, S. L., 1975: *Data Analysis for Scientists and Engineers*. Wiley, 355–358.

Miller, J. R., A. S. Dennis, J. H. Hirsch and D. E. Cain, 1975: Statistics of shower echoes in western North Dakota. *Preprints 16th Conf. Radar Meteorology*, Houston, Tex., Amer. Meteor. Soc., 391–396.

Morton, B. R., 1968: Non-similar turbulent plumes. Dept. of Mathematics, GFDL Pap. No. 7, Geophysical Fluid Dynamic Lab., Monash University, Victoria, Australia, 50 pp.

Plank, V. E., 1969: The size distribution of cumulus clouds in representative Florida populations. *J. Appl. Meteor.*, **8**, 46–67.

Simpson, J., and V. Wiggert, 1969: Models of precipitating cumulus towers. *Mon. Wea. Rev.*, **97**, 471–489.

——, and W. L. Woodley, 1971: Seedling cumulus in Florida: New 1970 results. *Science*, **172**, 117–126.

Telford, J. W., 1966: The convective mechanism in clear air. *J. Atmos. Sci.*, **23**, 652–666.

Turner, J. G., 1962: The motion of buoyant elements in turbulent surroundings. *J. Fluid Mech.*, **16**, 1–16.

——, 1963: The flow into an expanding spherical vortex. *J. Fluid Mech.*, **18**, 195–208.

Weinstein, A. I., and L. G. Davies, 1968: A parameterized numerical model of cumulus convection. Dept. of Meteorology, Pennsylvania State University, NSF Rep. No. 11, GS-777, 44 pp.