Merging Finite-Difference Schemes Having Dissimilar Time-Differencing Operators

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ABSTRACT

A procedure is given here that allows two finite-difference schemes having dissimilar time-differencing operators (say, a horizontal advection-diffusion scheme and a vertical diffusion scheme) to be merged into a single equation at the cost of increasing storage requirements through the introduction of an additional time level. The accuracy and stability of the combined scheme are investigated.

1. Introduction

In the numerical modeling of the planetary boundary layer and in many other meteorological applications, the horizontal scale of interest is of much greater extent than that of the vertical. Suitable resolution of scales is achieved by selecting a corresponding difference in the size of the horizontal and vertical mesh. Explicit schemes are well-suited for treating horizontal variation since stability and accuracy can be achieved using time steps of reasonably large size. Such time steps are generally too large, however, if explicit differencing is used to incorporate vertical diffusion. Accordingly, implicit procedures are often employed to represent derivatives in the vertical direction. There are available a number of schemes well-suited for treating horizontal advection and diffusion and others for vertical diffusion. The time-differencing operators may differ, however, so that the methods cannot be combined directly. A procedure is presented here that allows schemes having different time-differencing operators to be merged at the cost of carrying an extra time level in storage. The method has applicability to a wide variety of problems.

2. The method

Consider the equations

$$\frac{\partial A}{\partial t}=\sigma\frac{\partial^2 A}{\partial z^2}, \tag{1}$$

$$\frac{\partial A}{\partial t}=-u\frac{\partial A}{\partial x}+K\frac{\partial^2 A}{\partial x^2}, \tag{2}$$

where A is the dependent variable, t denotes time and x, z are the horizontal and vertical coordinates, respectively. Here σ is the vertical diffusion coefficient, u the horizontal velocity and K the horizontal diffusion coefficient. Suppose that (1) and (2) are represented by the analogous finite-difference relations

$$L_t^1(A^n_{j,k})=L_z(A^n_{j,k}), \tag{3}$$

$$L_t^2(A^n_{j,k})=L_x(A^n_{j,k}), \tag{4}$$

each formula having the desirable properties of stability and accuracy. Here $A^n_{j,k}$ denotes the finite-difference approximation to $A(t,x,z)$ for $t=n\Delta t$, $x=j\Delta x$ and $z=k\Delta z$. Suppose that we wish to derive from these formulas a finite-difference analog to the equation

$$\frac{\partial A}{\partial t}=-u\frac{\partial A}{\partial x}+\sigma\frac{\partial^2 A}{\partial z^2}+K\frac{\partial^2 A}{\partial x^2} \tag{5}$$

having the same desirable properties as do (3) and (4); but suppose that $L_t^1\neq L_t^2$.

We consider here only three-level time-difference operators of the form

$$L_t^1(A^n)=(\Delta t)^{-1}(a_0A^{n+1}+a_1A^n+a_2A^{n-1}), \tag{6}$$

$$L_t^2(A^n)=(\Delta t)^{-1}(b_0A^{n+1}+b_1A^n+b_2A^{n-1}), \tag{7}$$

where the (suppressed) spatial grid indices are fixed for some (j,k). We attempt to find a linear combination of L_t^1 at time levels n and $n+1$, and a (possibly different) linear combination of L_t^2 at n and $n+1$ to arrive at the same four-level operator L_t; i.e., we seek constants α, β, γ and δ such that

$$\begin{aligned}L_t(A^{n+1})&=\alpha L_t^1(A^n)+\beta L_t^1(A^{n+1})\\&=\gamma L_t^2(A^n)+\delta L_t^2(A^{n+1}).\end{aligned} \tag{8}$$

We can then combine schemes to form

$$\begin{aligned}L_t(A^{n+1})=\alpha L_z(A^n)+\beta L_z(A^{n+1})\\+\gamma L_x(A^n)+\delta L_x(A^{n+1}),\end{aligned} \tag{9}$$

as an approximation to (5). Substituting (6) and (7) in (8) and equating coefficients of A at the same time level leads to the result

$$(\alpha,\beta,\gamma,\delta)=\alpha(1,\,-b_0b_2^{-1},\,a_2b_2^{-1},\,-a_0b_2^{-1}), \tag{10}$$

where we have used the fact that $\sum a_i=\sum b_i=0$ since L_t^1 and L_t^2 are derivative approximations. These constants are then used to form the finite-difference equation (9) through which we can divide by α to arrive at, in effect, a unique formulation.

The procedure of linearly combining a formula at successive time levels preserves the accuracy of the original formula as we now show by considering (1)

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and (3). Three-level approximations to the time derivative can generally be written in the form

$$L_t^1(A^n)=a_0\left(\frac{A^{n+1}-A^n}{\Delta t}\right)+(1-a_0)\left(\frac{A^n-A^{n-1}}{\Delta t}\right), \quad (11)$$

so that if A is the true solution to (1)

$$L_t^1(A_k^n)=a_0\left\{\left[\frac{\partial A}{\partial t}\right]_k^{n+\frac{1}{2}}+O[(\Delta t)^2]\right\}$$
$$+(1-a_0)\left\{\left[\frac{\partial A}{\partial t}\right]_k^{n-\frac{1}{2}}+O[(\Delta t)^2]\right\}. \quad (12)$$

Assume that (3) has error $O[(\Delta t)^2]+O[(\Delta z)^2]$. Then $L_z(A)$ must satisfy the relation

$$L_z(A_k^n)=a_0\left\{\left[\sigma\frac{\partial^2 A}{\partial z^2}\right]_k^{n+\frac{1}{2}}+O[(\Delta t)^2]+O[(\Delta z)^2]\right\}$$
$$+(1-a_0)\left\{\left[\sigma\frac{\partial^2 A}{\partial z^2}\right]_k^{n-\frac{1}{2}}+O[(\Delta t)^2]+O[(\Delta z)^2]\right\}. \quad (13)$$

Thus each operator L_t^1, L_z represents a second-order approximation to a particular $(a_0, 1-a_0)$ linear combination of the corresponding derivative at time levels $n+\frac{1}{2}$, $n-\frac{1}{2}$. It follows that an arbitrary linear combination of the difference equation (3) taken at successive time levels retains the accuracy of the original scheme. A similar argument holds for (4) and hence for the combined scheme (9).

Taking a linear combination of (3) or (4) at successive time steps preserves the stability of the original scheme if and only if the weighting coefficient applied at level n does not exceed that applied at level $n+1$. To see this, we again consider (3) and the linear combination

$$\alpha L_t^1(A_k^n)+\beta L_t^1(A_k^{n+1})=\alpha L_z(A_k^n)+\beta L_z(A_k^{n+1}). \quad (14)$$

If A_k^n is assumed to have the form

$$A_k^n=\xi^n e^{ikm\Delta z}, \quad (15)$$

substitution of (15) into (14) yields a characteristic equation of the form

$$\alpha(p_0\xi^2+p_1\xi^2+p_2)+\beta(p_0\xi^3+p_1\xi^2+p_2\xi)=0 \quad (16)$$

or

$$\left(\xi+\frac{\alpha}{\beta}\right)(p_0\xi^2+p_1\xi+p_2)=0. \quad (17)$$

Eq. (17) has three roots, two being the roots of the characteristic equation of the original scheme and the third being $\xi=-\alpha/\beta$, introduced as a result of the linear combination process. Stability is maintained if and only if all roots are $\leqslant 1$ in modulus. Hence stability of the combined scheme will be determined by the criterion for the original scheme if and only if the condition

$$\left|\frac{\alpha}{\beta}\right|\leqslant 1 \quad (18)$$

holds. An analogous result holds for the advection scheme (4); *viz.*,

$$\left|\frac{\gamma}{\delta}\right|\leqslant 1. \quad (19)$$

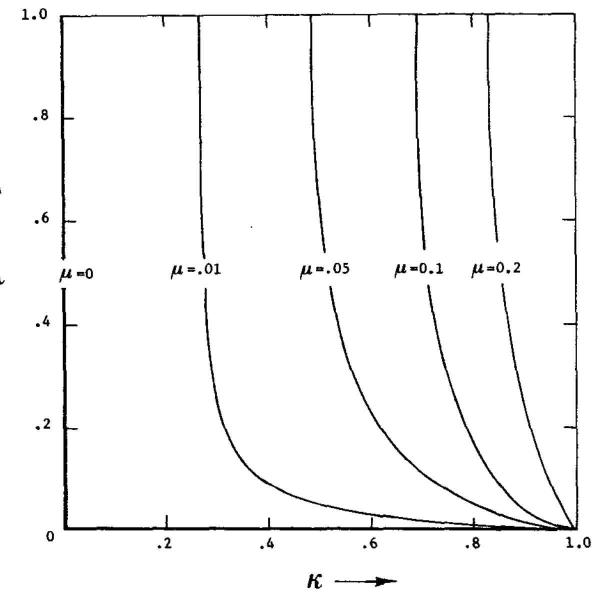

FIG. 1. Stability region for (25) in terms of the parameters κ, λ, μ. For constant μ, stability prevails in the region to the left of the line indicated by μ=constant.

Once schemes (3) and (4) are selected for the purpose of mergence, the ratios α/β and γ/δ will be fully determined by (10); thus we have no control over whether or not (18) and (19) are satisfied except through the choice of schemes. While we cannot give any general criterion that will be sufficient to insure the stability of the combined scheme (9), the criteria (18) and (19) will constitute necessary conditions.

3. Example

Let us represent (1) by the scheme

$$\tfrac{3}{2}A_{j,k}^{n+1}-2A_{j,k}^n+\tfrac{1}{2}A_{j,k}^{n-1}=L_z(A_{j,k}^n), \quad (20)$$

where

$$L_z(A_{j,k}^n)=\lambda(A_{j,k+1}^{n+1}-2A_{j,k}^{n+1}+A_{j,k-1}^{n+1}) \quad (21)$$

with $\lambda=\sigma\Delta t/(\Delta z)^2$ (Richtmyer and Morton, 1967, p. 190). Represent (2) by the leapfrog/DuFort-Frankel scheme

$$\tfrac{1}{2}A_{j,k}^{n+1}-\tfrac{1}{2}A_{j,k}^{n-1}=L_x(A_{j,k}^n), \quad (22)$$

where

$$L_x(A^n_{j,k}) = -\kappa(\tfrac{1}{2}A^n_{j+1,k} - \tfrac{1}{2}A^n_{j-1,k}) + \mu(A^n_{j+1,k} - A^{n+1}_{j,k} - A^{n-1}_{j,k} + A^n_{j-1,k}) \quad (23)$$

with $\kappa = u\Delta t/\Delta x$ and $\mu = K\Delta t/(\Delta x)^2$.

We choose, in accordance with (10),

$$(\alpha,\beta,\gamma,\delta) = \alpha(1,1,-1,3) \quad (24)$$

to arrive at a scheme corresponding to (9) which, when divided through by α yields the formula

$$\tfrac{3}{2}A^{n+2}_{j,k} - \tfrac{1}{2}A^{n+1}_{j,k} - \tfrac{3}{2}A^n_{j,k} + \tfrac{1}{2}A^{n-1}_{j,k} = L_z(A^n_{j,k}) + L_z(A^{n+1}_{j,k}) - L_x(A^n_{j,k}) + 3L_x(A^{n+1}_{j,k}). \quad (25)$$

This scheme has error $O[(\Delta t)^2] + O[(\Delta x)^2] + O[(\Delta z)^2]$ in keeping with the accuracy of (20) and (22). Since $|\alpha/\beta| = 1$ and $|\gamma/\delta| = \frac{1}{3}$, the necessary conditions for stability (18) and (19) are satisfied. The stability region for (25) has been determined empirically and is shown in terms of the parameters κ, λ and μ in Fig. 1. For constant μ, stability prevails in the region to the left of the line indicated by μ=constant.

4. Summary

A procedure has been presented that allows the merging of two finite-difference schemes (e.g., schemes governing horizontal advection and vertical diffusion) having dissimilar time-differencing operators. The combined equation retains the accuracy of the two constituent schemes. The conditions under which computational stability is preserved have been discussed. The advantages of the method are offset by the need to store the dependent variable at an additional time level.

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REFERENCE

Richtmyer, R. D., and K. W. Morton: *Difference Methods for Initial-Value Problems*, 2nd ed. Interscience, 405 pp.