Comments on "U. S. Navy Fleet Numerical Weather Central Operational Five-Level Fourth-Order Primitive-Equation Model"

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1 February 1977 and 31 May 1977

ABSTRACT

Although there is some ambiguity in the description of the U.S. Navy Fleet fourth-order primitive-equation model developed by Mihok and Kaitala (1976), the finite differences used for the continuity equation and pressure gradient term appear to contain second-order errors comparable to those of the original second-order model, and larger fourth-order errors. In the thermodynamics, moisture and momentum equations, there is partial cancellation of second-order errors, leading to a better approximation of the phase speed. However, in regions with strong horizontal variations of wind, the second-order errors in these equations are serious. These errors are due to the neglect of the truncation errors introduced by horizontal averaging in the staggered grid.

1. Introduction

We have read with interest the paper by Mihok and Kaitala (1976, hereafter MK). One of the most important points of the paper is the transformation of a previous second-order model by Kessel and Winninghoff (1972) into a fourth-order model.

However, there are certain ambiguities in the description of the model that need to be clarified. Mihok and Kaitala apply the classic "$\frac{4}{3}$ minus $\frac{1}{3}$" procedure (Kreiss and Oliger, 1972) to pairs of centered-difference approximations, but apparently they use horizontal averaging operators which prevent their result from having fourth-order accuracy. Their scheme appears to be only second-order, and the continuity equation as well as the pressure gradient terms of the momentum equation appear to contain truncation errors comparable to those of the original second-order model.

2. The numerical scheme in the Mihok and Kaitala model

It is unfortunate that the scheme in MK is incompletely described. In particular, several averaging operators are not explicitly defined, as for example $\overline{u\bar{\pi}}^*_{i+1,j}$ in Eqs. (64) and (65) and $\overline{u\bar{\pi}}_{i+1,j}$ in Eq. (63). However, from their statements, "all the averaged quantities are determined as before", and again "the averaged values are also determined as indicated in Eqs. (43) and (47)" (p. 1542 of MK), we can only assume that they have used a simple space average of the four closest values of the quantities being averaged.

In the continuity equation (65) of MK, the quantity $\overline{u\bar{\pi}}^*_{i+1,j}$ has apparently been defined as

$$\overline{u\bar{\pi}}^*_{i+1,j}=\tfrac{1}{4}(u\bar{\pi}_{i+1,j}+u\bar{\pi}_{i+2,j}+u\bar{\pi}_{i+1,j+1}+u\bar{\pi}_{1+1,j-1}). \quad (1)$$

This type of average in the "$\frac{1}{3}$" term *introduces second-order errors which are not eliminated by the "$\frac{4}{3}$ minus $\frac{1}{3}$" procedure*. In effect, if for simplicity we assume Cartesian geometry, a truncation error analysis of the second-order scheme (46) in MK yields

$$(u\bar{\pi}_{i+1,j}-u\bar{\pi}_{i,j})/d=(u\pi)_x+\tfrac{1}{4}(d^2/6)[(u\pi)_{xxx}] +\tfrac{1}{16}(d^4/120)[(u\pi)_{xxxxx}]+\ldots, \quad (2)$$

whereas

$$(\overline{u\bar{\pi}}^*_{i+1,j}-\overline{u\bar{\pi}}^*_{i-1,j})/2d =(u\pi)_x+\tfrac{1}{4}(d^2/6)[5.5(u\pi)_{xxx}+1.5(u\pi)_{xyy}] +\tfrac{1}{16}(d^4/120)[38.5(u\pi)_{xxxxx}+20(u\pi)_{xxxyy} +2.5(u\pi)_{xyyyy}]+\ldots. \quad (3)$$

0027-0644/78/1021-1023$05.00

Here $d=\Delta x=\Delta y$ and the right-hand side of Eqs. (2) and (3) is evaluated at the mass point i, j. From (2) and (3) we obtain the truncation error analysis of the "$\frac{4}{3}$ minus $\frac{1}{3}$" scheme in Eq. (65) of MK:

$$\begin{aligned}\tfrac{4}{3}[(u\pi_{i+1,j}-u\pi_{i,j})/d]-\tfrac{1}{3}[(\overline{u\pi}^*_{i+1,j}-\overline{u\pi}^*_{i-1,j})/2d]\\ =(u\pi)_x+\tfrac{1}{4}(d^2/6)[-0.5(u\pi)_{xxx}-0.5(u\pi)_{xyy}]\\ +\tfrac{1}{16}(d^4/120)[-11.5(u\pi)_{xxxxx}-6.66(u\pi)_{xxxyy}\\ -0.833(u\pi)_{xyyyy}]+\ldots\quad (4)\end{aligned}$$

Eq. (4) indicates that the "fourth-order" continuity equation (65) in MK contains both second-order errors similar to those of the simple centered scheme (2) and Eq. (46) in MK, and much larger fourth-order errors. Similar errors are present in the pressure gradient term of the momentum equations, but they can be simply corrected if Eq. (15) in MK is replaced by

$$\frac{\partial u_{j+\frac{1}{2}}}{\partial t}=-g\left[\frac{9}{8}\frac{h_{j+1}-h_j}{d}-\frac{1}{8}\frac{h_{j+2}-h_{j-1}}{2d}\right].\qquad (5)$$

Because of the horizontal averaging, the term $[(\frac{4}{3})\times\sin kd/2-(\frac{1}{6})\sin kd]$ appearing in Eqs. (19)–(29) in MK should be replaced by $[(31/24)\sin kd/2-(1/12)\times\sin kd-(1/24)\sin 3kd/2]$.

A truncation error analysis of the thermodynamic equation (64) in MK indicates that *there is partial cancellation of second-order errors* so that in a region of uniform flow $[(u\pi)_x=(u\pi)_y=0]$ the scheme contains no second-order errors. Therefore, it may be expected that phase errors in the advection of temperature will be considerably reduced. However, in regions of strong horizontal variations of wind the "fourth-order" scheme will contain second-order errors similar to those of the second-order scheme (55), and will be less accurate than a truly fourth-order scheme. It is not clear to us why MK chose to define $\tilde{T}$ in Eq. (55) with a four-point average [Eq. (52)] instead of using a two-point average which would make the scheme quadratically conservative (Bryan, 1967).

In the momentum equation (63) in MK there is some ambiguity in the definition of the averaged quantities appearing in the "$\frac{1}{3}$" terms. If the mass flux at a velocity point $\overline{u\pi}_{i+1,j}$ has been defined by

$$\overline{u\pi}_{i+1,j}=\tfrac{1}{4}(\overline{u\pi}^*_{i,j}+\overline{u\pi}^*_{i,j+1}+\overline{u\pi}^*_{i+1,j}+\overline{u\pi}^*_{i,j-1}),\quad (6)$$

then the scheme is quadratically conservative, but there is only a partial cancellation of second-order errors in a manner similar to the thermodynamic equation. An alternative definition is

$$\overline{u\pi}_{i+1,j}=\tfrac{1}{4}(u\pi_{i,j}+u\pi_{i+1,j+2}+u\pi_{i+2,j}+u\pi_{i+1,j-2})\quad (7)$$

in which case the flux terms in Eq. (63) of MK are truly fourth-order, but the scheme ceases to be quadratically conservative because the average (7) is inconsistent with the continuity equation (65) of MK. In this analysis we have not taken into account the second-order errors introduced by the average of the continuity equation which is used to forecast the quantity $\bar{\pi}u$ at a velocity point, where $\bar{\pi}$ is defined in Eqs. (49) and (53) of MK.

3. Conclusion

There is some ambiguity in the definition of certain horizontal averages in the "fourth-order" U. S. Navy Fleet model as described by Mihok and Kaitala (1976). If definition (1) has been used, as appears to be the case, then the horizontal averages introduce second-order truncation errors which are not cancelled in the "$\frac{4}{3}$ minus $\frac{1}{3}$" procedure used in MK. The "fourth-order" fluxes in the continuity equation contain second-order truncation errors which are similar to those of a simple second-order scheme, and much larger fourth-order errors. In the pressure gradient term, the "$\frac{4}{3}$ minus $\frac{1}{3}$" procedure apparently used [Eq. (15) in MK] will be no more accurate than the original second-order scheme, but the second-order errors can easily be corrected.

In the thermodynamic, moisture and momentum equation, the "$\frac{4}{3}$ minus $\frac{1}{3}$" procedure introduces a partial cancellation of second-order errors even if the simple space average (1) is used. This cancellation will decrease the phase speed errors of the numerical forecasts. However, in regions with strong horizontal variations of wind, the "fourth-order" scheme in MK will still contain large second-order truncation errors.

The difficulties pointed out here arise from the use of fourth-order differences on a staggered grid. The main advantage of staggered grids is that they resolve linear gravity waves better than a non-staggered grid, but in this case the advantage is offset by the fact that the errors in the continuity equation and pressure term are not better than those of a second-order scheme. We have recently presented several schemes which are truly fourth-order and conservative on a staggered grid (Kálnay-Rivas, 1976). We indicated, however, that they are unpractical, and that in our opinion, except for simple advective schemes, fourth-order differences should not be used with staggered grids. Therefore, a nonstaggered grid has been chosen for the fourth-order conservative GISS general circulation model of the global atmosphere (Kálnay-Rivas *et al.*, 1977).

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