The Choice of Spectral Functions on a Sphere for Boundary and Eigenvalue Problems: A Comparison of Chebyshev, Fourier and Associated Legendre Expansions

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ABSTRACT

Modified Fourier series, as judged by criteria of accuracy, numerical efficiency and ease of programming, are the best choice of latitudinal expansion functions for general problems on the sphere. The pseudospectral and spectral methods, however, can be easily and successfully applied with all three types of orthogonal series. For special situations, such as when the latitudinal variable is stretched, Chebyshev polynomials are the only practical choice, but for orthodox problems on the globe, they are less efficient than the other two sets of functions. Although spherical harmonics have been universally employed in the past, Fourier series give comparable accuracy and are significantly easier to program and manipulate. Thus, in the absence of a special reason to the contrary, the simplest and most effective way to handle the north-south dependence of the solution to a boundary or eigenvalue problem on the sphere is to use a Fourier series in colatitude.

1. Introduction

The goal of the present work is to discuss the choice of spectral functions on the sphere for differential equations which are *independent* of time. This choice has already been analyzed for time-dependent marching problems by Orszag (1974). For boundary and eigenvalue equations, however, the factors influencing this choice are very different. The reason is that for nonlinear marching problems, the important issues are the advantages of the fast Fourier transform for evaluating nonlinear terms versus the disadvantages of severe, explicit time-stepping restrictions (the "pole" problem) for various expansions. For the time-independent, linear boundary and eigenvalue equations that will be discussed here, however, the "pole" problem is irrelevant and transforms are only a small part of the overall computation. Instead, accuracy, numerical efficiency and ease of programming are the decisive criterion in choosing a set of functions to solve these so-called "jury" problems. The eigenvalue calculations of Chapman and Lindzen (1970), Hollingsworth (1975), Moura and Stone (1976), Moura (1976), Boyd (1976, 1978b), Warn (1976) and Kasahara (1976), and the boundary value solutions of Matsuno (1970), Hong and Lindzen (1974) and Schoeberl and Geller (1976), to give but a partial list, all testify to the geophysical importance of solving this class of equations in spherical geometry.

The discussion will center on comparing Fourier series with associated Legendre functions (spherical harmonics). The special situations in which Chebyshev polynomials are the only effective choice will be described briefly in Section 4. For most problems on the sphere, all three expansions are accurate and easily programmed. Unlike the ambiguous results of Orszag (1974), however, it is possible to state the advantages and disadvantages of each type of expansion with some precision so that one can make the best choice for the problem at hand.

The next section is largely a review of the spectral and pseudospectral methods for various basis sets, but it also introduces the three types of functions that will be examined here and shows how to exploit the relationship between associated Legendre functions and the Gegenbauer polynomials to simplify the algorithms for spherical harmonics. Section 3 shows that the pseudospectral Fourier method is free of the need for additional polar constraints that complicates Fourier algorithms for marching problems and briefly reviews Orszag's analysis (with additional supporting material) of why Fourier series give roughly the same accuracy as spherical harmonics for a given number of retained terms. The final section sums up the advantages and disadvantages of each type of series.

2. The spectral and pseudospectral methods

Most geophysical boundary and eigenvalue problems arise from the assumption that the solution is sinusoidal in time and longitude, which usually allows separation of variables with respect to these two dimensions. The resulting equations can then be solved

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zonal wavenumber by zonal wavenumber with the wavenumber appearing only as a parameter. Even in more complicated problems, a complex Fourier series in longitude has all the properties one would want: infinite order convergence (defined later) for smooth functions, ease of programming and manipulation, and summability via the fast Fourier transform. Consequently, to my knowledge, no global basis functions other than Fourier series have been used to represent the longitudinal dependence of functions on the sphere. Thus, Fourier series in longitude will be used here, and the discussion will focus on the choice of expansion functions in latitude.

Representing the solution in the form

$$u=\sum_{-\infty}^{\infty} F_m(\phi)e^{im\lambda}, \tag{2.1}$$

where u and F_m may depend on height, λ is longitude and m the zonal wavenumber (always an integer), three different basis sets will be analyzed:

$$F_m(\phi)=\sum_{n=0}^{\infty}\begin{cases} a_n P^m_{n+|m|}(\phi) \\ b_n \sin^s\phi \cos(n\phi) \\ c_n T_n(\varphi)\end{cases} \tag{2.2}$$

which are the associated Legendre functions, modified Fourier series and Chebyshev polynomials, respectively. Note that ϕ is colatitude and φ is latitude, and also that

$$s=\begin{cases} 0, & m \text{ even} \\ 1, & m \text{ odd.}\end{cases} \tag{2.3}$$

The reason for extracting this "parity" factor is discussed below. Strictly speaking, a spherical harmonic is the product of an associated Legendre function with $e^{im\lambda}$ and a normalization constant, but since (2.1) is always assumed, I will loosely use "associated Legendre" and "spherical harmonic" interchangeably. All three sets of functions are complete on the sphere and have the property of "infinite order convergence" for smooth, infinitely differentiable functions on the sphere, which means that the error in truncating the expansion after N terms decreases faster than any finite inverse power of N as N is increased. (Accuracy and convergence are discussed in detail in Section 4.) Since every associated Legendre function can be written

$$P^m_n(\phi)=\sin^s\phi\, R_{mn}(\cos\phi), \tag{2.4}$$

where R_{mn} is a polynomial of degree at most $(n+|m|)$, it follows that every associated Legendre function can be written exactly as a finite modified Fourier series of at most $(n+|m|)$ terms. The Chebyshev polynomials are defined by the identity

$$T_n(\cos\phi)=\cos n\phi. \tag{2.5}$$

Because of (2.5), the spectral and pseudospectral algorithms for the Chebyshev polynomials follow almost trivially from those for modified Fourier series, so I will only describe the latter in this section. There are three important exceptions, however, to the general equivalence between Chebyshev and Fourier expansions, and these, which will be discussed in the next two sections, are the reasons why it is necessary to regard Chebyshev expansions in *latitude* and modified Fourier series in *colatitude* as independent choices of expansion functions. Eq. (2.5) also shows that modified Fourier series in colatitude are modified Chebyshev series in the variable $x=\cos\phi$. The importance of this relationship will be emphasized in Section 4. For m odd, the solution has branch points of the form $(1-x^2)^{\frac{1}{2}}$ $(=\sin\phi)$ at the poles, so the reason for extracting the "parity" factor $(1-x^2)^{s/2}=\sin^s\phi$ is to prevent the pole singularities from spoiling the rapid convergence of the expansion. Orszag (1974) discussed this need from a different but equivalent perspective.

The pseudospectral and spectral methods are both special cases of the so called "mean weighted residual" method (Finlayson, 1972). Given an equation

$$Lu=f, \tag{2.6}$$

where L is a linear operator, one chooses a set of basis functions $\{\theta_n(\phi)\}$ and weighting functions $\{w_i(\phi)\}$, and then determines the coefficients $\{a_n\}$ of the assumed expansion

$$u=\sum_{n=0}^{N-1} a_n\theta_n(\phi) \tag{2.7}$$

by demanding

$$\sum_{j=0}^{N-1}\langle w_i|L\theta_j\rangle a_j=\langle w_i|f\rangle,\quad i=0,1,\ldots,N-1, \tag{2.8}$$

where the inner product in the present case is defined by

$$\langle a|b\rangle=\int_0^\pi a(\phi)b(\phi)\sin\phi d\phi. \tag{2.9}$$

For an ordinary, second-order differential equation eigenvalue problem, i.e.,

$$\left.\begin{aligned} L&=a_2(\phi)\frac{d^2}{d\phi^2}+a_1(\phi)\frac{d}{d\phi}+a_0(\phi)-\lambda \\ f(\phi)&=0\end{aligned}\right\}, \tag{2.10}$$

where λ is the eigenvalue, Eq. (2.8) is then equivalent to the algebraic problem

$$MA=\lambda NA, \tag{2.11}$$

where the elements of A are the expansion coefficients of (2.7) and the elements of M and N are given by

$$m_{ij}=\langle w_i|a_2\theta''_j+a_1\theta'_j+a_0\theta_j\rangle \quad (2.12)$$

$$n_{ij}=\langle w_i|\theta_j\rangle, \quad (2.13)$$

where the primes denote differentiation with respect to ϕ. Eqs. (2.1)–(2.3) may also represent a system of coupled ordinary differential equations in ϕ, such as would result from applying finite differences in z to a two-dimensional boundary value problem in ϕ and z, if L is interpreted as a *matrix* operator and $u(\phi)$ and the $a_n(\phi)$ as column *vectors*.

The spectral method (also known as Galerkin's) and the pseudospectral method (also known as "orthogonal collocation") differ only in the choice of weighting functions, using

$$w_i=\theta_i(\phi) \quad \text{(spectral)}, \quad (2.14)$$

$$w_i=\delta(\phi_i) \quad \text{(pseudospectral)}, \quad (2.15)$$

respectively, where the collocation points ϕ_i for the pseudospectral method are the N interior roots of $\theta_N(\phi)$ and δ is the Dirac delta function. With this choice of ϕ_i, it can be shown [by multiplying each equation of (2.8) by $r_i\theta_k(\phi_i)$, $k=0, 1, \ldots, N-1$, and then adding the equations by summing over i] that the pseudospectral method is identical with the spectral if the integrations of the latter are evaluated by N-point Gaussian quadrature. (The r_i are the quadrature weights.) Fox and Parker (1968) and Orszag and Israeli (1974) show that there is very little difference between the spectral and pseudospectral methods for Fourier and Chebyshev series, even for equations with rapidly varying coefficients, as far as accuracy is concerned; no such comparison has been performed for associated Legendre functions. The important point is that for either method, the matrix elements of (2.8) can be numerically computed almost trivially if one can evaluate the basis functions and their derivatives at the Gaussian quadrature points.

For Fourier series, this is easy because the derivatives of trigonometric functions are known in closed form and the roots of $\cos(N\phi)$ are simply $\phi_i=[(2i-1)\pi/2N]$, and the quadrature weights $r_i=2/N$ for $i>0$ with $r_0=1/N$. The corresponding expressions for the Chebyshev polynomials follow trivially from those for Fourier series because of (2.5), but are given explicitly in Boyd (1978a). Both types of expansions can be summed up by using the fast Fourier transform, but because transforms constitute only a small part of the overall computation, it may be preferable (in the interests of saving programming and compilation time) to use the simpler recurrence relation

$$\theta_n(\phi)=2\cos\phi\theta_{n-1}(\phi)-\theta_{n-2}(\phi) \quad n=2, 3, \ldots. \quad (2.16)$$

Thus, for Fourier and Chebyshev series, computing the matrix elements is trivial.

For spherical harmonics, it is not. Calculating the derivatives is straightforward because of the trick I introduce here. The associated Legendre functions can be written in terms of the Gegenbauer polynomials[1]

$$P_n^m(\cos\phi)=\sin^m\phi C_{n-m}^m(\cos\phi) \quad (2.17)$$

which satisfy the extremely useful identity

$$\frac{d}{dx}C_{n-m}^m(x)=C_{n-m-1}^{m+1}(x) \quad (2.18)$$

from which it is trivial to derive

$$\frac{d}{d\phi}P_n^m=\cot\phi mP_n^m-P_{n-1}^{m+1} \quad (2.19)$$

and similarly for higher derivatives.[2] Thus, if one can evaluate the associated Legendre functions themselves for varying degree and order, it is easy to compute the derivatives.

Every standard texbook gives a simple three-term recursion formula for computing associated Legendre functions, but to quote Merilees (1973a): "the evaluation of P_n^m directly by recursion is difficult for higher orders and requires considerable care. This has been studied by Belousov (1962), who has developed an accurate algorithm which is, however, rather complicated." The problem is that the basic recursion relation for the Gegenbauer polynomials is weakly unstable for values of $\cos\phi$ corresponding to points near the poles, and this instability increases as m increases. For $m=0$ (Legendre polynomials), the instability is so weak that it can be safely ignored (Abramowitz and Stegun, 1965, p. 339), but for large m, one is doomed. The software for Belousov's algorithm and for computing the collocation points θ_i and the quadrature weights r_i, which are different for each value of m and are *not* known in closed form, are readily available (for example, from NCAR). Nonetheless, the need for these additional algorithms is an annoying complication in comparison to the simplicity of Fourier and Chebyshev expansions, for which the basis functions, collocation points and quadrature weights are all known in explicit, closed form: spherical harmonic mean weighted residual methods are harder to program than those for the other two sets of functions.

3. The absence of pole problems and polar constraints for the pseudospectral Fourier method

The great weakness of spectral and pseudospectral methods, other than those using spherical harmonic

[1] I use the Morse and Feshbach (1953) definition of Gegenbauer polynomials, but substitute C_n^m for their T_n^m. Abramowitz and Stegun (1965) use a *different* definition for the order of the Gegenbauer polynomials.

[2] Note that the collocation points never include the poles for any value of N or m, so (2.19) is never singular.

expansions, for solving time-dependent *marching* problems is that explicit time-stepping restrictions are very severe—the "pole" problem discussed in Orszag (1974). For the time-independent *jury* problems discussed here, however, there is obviously no "pole" problem because there is no time-stepping. Orszag (1974), however, found that it was necessary to impose additional constraints on Fourier series (besides merely extracting what he called the "parity" factor) to successfully solve marching equations on the sphere. This section will show that it is *not* necessary to impose such additional constraints in solving boundary and eigenvalue problems on the sphere via the Fourier pseudospectral method.

To borrow terminology from variational calculations (Strang and Fix, 1973), boundary conditions for differential equations can be classified as "essential" and "natural" where the classification depends on both the order of the governing equation and upon the method used to solve it as well as upon the boundary conditions themselves. "Essential" boundary conditions are those that must be satisfied by the expansion functions if the numerical method is to converge to the correct solution as the number of retained terms increases. "Natural" boundary conditions are those which need not be explicitly imposed upon the basis functions but which will be approximately satisfied to greater and greater accuracy by the approximate solution of the differential equation as the number of degrees of freedom increases. It can be shown (Orszag, 1974) that

$$\frac{d^kF_m}{d\phi^k}(\phi)=0 \quad \text{at} \quad \phi=0,\,\pi, \qquad \text{for } k=0,\,1,\,\ldots,\,|m|-1. \tag{3.1}$$

Associated Legendre functions satisfy these conditions automatically, but for modified Fourier series and for Chebyshev expansions, these conditions are *natural* boundary conditions. If one blindly applies the pseudospectral method using the basis functions as given by (2.2), one will obtain accurate solutions to the differential equation if the number of retained terms in the expansion is sufficiently large.

Orszag (1974), however, found a very different situation for his class of problems. The two-dimensional Laplacian operator on the sphere, given explicitly by (after separation of variables)

$$\Delta[F_m(\phi)e^{im\lambda}]=\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d}{d\phi}F_m\right)-\frac{m^2}{\sin^2\phi}F_m, \tag{3.2}$$

is a component of many differential equations on the sphere, time-dependent or otherwise. Orszag found that to avoid disaster from the second-order poles in (3.2) in applying the spectral method to (3.2), it was necessary to demand

$$f_m^{(N)}(\phi)=0 \ \text{at}\ \phi=0,\,\pi,\ \text{for all } N \quad \text{and for all } |m|\geqslant 2, \tag{3.3}$$

where $\sin\phi f_m^{(N)}(\phi)$ is the N-term approximation to $F_m(\phi)$. Although this is not explicit in Orszag's paper, for higher order equations it would be necessary to impose additional constraints. For one involving the biharmonic operator $\Delta^2(=\Delta\Delta)$, for example, it would be necessary to impose

$$\frac{d^2}{d\phi^2}f_m^{(N)}(\phi)=0 \quad \text{at} \quad \phi=0,\,\pi, \qquad \text{for all } N \text{ and all } |m|\geqslant 4. \tag{3.4}$$

Eq. (3.4) is a natural boundary condition for the Laplacian but an essential boundary condition for the biharmonic. Orszag solved a time-dependent problem involving only the Laplacian and found that if (3.3) was imposed, the numerical solution was very accurate, but if this constraint was not imposed, the numerical solution still converged as the time step was shortened—to a wildly wrong answer.

With the pseudospectral method, however, the situation is quite different because the set of discrete points at which the approximate solution is required to exactly satisfy the differential equation does not involve the poles. The integrations that define the spectral matrix elements, however, are from pole to pole, which makes the singularities lethal in the absence of (3.3). Thus Merilees (1973b) successfully solved time-dependent problems—without imposing (3.3)—using a Fourier pseudospectral approach. He noted, however, that his technique can be susceptible to instabilities that arise in the polar regions; Orszag (1974) suggests that the cure would be to impose the same constraints on the pseudospectral method as on the spectral, even though this is not nearly as crucial for the former as for the latter. For boundary and eigenvalue problems, however, such instabilities are irrelevant and the constraints (3.3) and (3.4) are unnecessary; explicit calculation shows that no ill-conditioning difficulties arise. For the eigenvalues of the Laplacian, for example, I found that with $N=60$ on a CDC 7600 computer (13+ decimal digits of precision), the Fourier pseudospectral method—without constraints except for extraction of the parity factor—computed the first 58 eigenvalues for $m=3$ and the first fifty for $m=10$ to within a maximum relative error of 2×10^{-12}.

In summary, the extra constraints needed for time-dependent equations as explained in Orszag (1974) for both the spectral and pseudospectral methods are unnecessary for boundary and eigenvalue problems when the pseudospectral method is used. For the Fourier spectral method, the absence of pole constraints is still risky, so for Fourier series on the sphere, the pseudospectral approach is preferable to the spectral.

It is all well and good to show that it is mathematically *unnecessary* to impose the full constraints (3.1) or even the more limited constraint (3.3) if one uses the pseudospectral method with Fourier series, but two questions still remain: first, is it *possible*, and second is it *desirable* from the standpoint of numerical efficiency to go ahead and impose the constraints (3.1)? The answers to both questions have been given by previous authors and do not require modification because of the different class of problems under consideration here, so I will merely summarize their conclusions.

The first question was answered by Merilees (1973a). It is always possible to impose (3.3) or (3.4) if this is convenient, but basis functions of the form

$$\theta_n(\phi)=\sin^{|m|}\phi \cos n\phi, \tag{3.5}$$

called the "modified Robert" functions by Orszag (1974), are horribly ill-conditioned for large zonal wavenumber. Thus, one can constrain basis functions to have low order zeros at the poles, but constraining them to have high order zeros for large values of m as the spherical harmonics automatically have is a road to numerical disaster.

The answer to the second question was given by Orszag (1974): it is not particularly desirable, from the standpoint of numerical efficiency, to impose (3.1) on Fourier basis functions. He argues "The detailed behavior expressed by the $\sin^{|m|}\phi$ factor [as in (3.5)] is usually of little direct interest in a numerical simulation . . . The above argument suggests that surface-harmonic series contain much information on the behavior of functions near the poles that is not of primary interest." Table 1, which lists exact and approximate eigenvalues for the spherical harmonics equation for $m=49$ as computed through both the modified Fourier and Chebyshev pseudospectral methods dramatically illustrates his point. (Note that this is a much larger value of m than is usually needed for meteorological applications, which rarely involve $m\geqslant 20$.) Counting multiple zeros in colatitude according to their multiplicity, the exact eigenfunction of degree n has 98 plus $(n-49)$ zeros, whereas the sum of the 60 expansion functions, whether Fourier or Chebyshev,[3] can have at most 61. Nevertheless, the Chebyshev expansion gives the lowest 14 eigenvalues with less than 1% relative error and the modified Fourier series approach gives the lowest 30 eigenvalues to within 1 part in 90 and the lowest 20 to within one part in 70 million.

TABLE 1. Chebyshev and Fourier pseudospectral approximations using 60 basis functions to the eigenvalues of the $m=49$ spherical harmonics.

Mode no.	Exact value	Chebyshev Approximate value	Chebyshev Relative error	Fourier Approximate value	Fourier Relative error
49	2450.	2449.9999956	2.0E-9	2450.000000004	1.6E-12
50	2550.	2549.9999989	4.0E-10	2550.000000003	1.2E-12
51	2652.	2652.0000004	1.5E-10	2652.000000009	3.6E-12
52	2756	2755.999995	2.0E-9	2756.000000001	0.4E-12
53	2862	2862.000062	2.0E-8	2862.000000007	2.8E-12
54	2970	2969.99921	2.6E-7	2970.000000010	3.4E-12
55	3080	3080.0072	2.3E-6	3080.000000020	6.9E-12
56	3192	3191.94	1.9E-5	3191.999999997	0.9E-12
57	3306	3306.30	9.1E-5	3305.999999994	1.8E-12
58	3422	3420.4	4.7E-4	3422.000000005	1.5E-12
59	3540	3544.8	1.4E-3	3539.999999997	0.9E-12
60	3660	3644.0	4.3E-3	3659.999999995	1.5E-12
61	3782	3815.0	8.7E-3	3781.999999998	0.6E-12
62	3906	3867.4	9.9E-3	3905.999999995	1.3E-12
63	4032	4151.4	3.0E-1	4031.999999998	0.5E-12
64	4160	4173.	3.1E-3	4159.999999994	1.5E-12
65	4290	—	—	4290.000000011	2.6E-12
66	4422			4421.99999973	6.1E-11
67	4556			4556.0000053	1.1E-9
68	4692			4691.999936	1.4E-8
69	4830			4830.00074	1.5E-7
70	4970			4969.9939	1.2E-6
71	5112			5112.044	8.6E-6
72	5256			5255.74	5.0E-5
73	5402			5403.15	2.1E-4
74	5550			5545.2	8.6E-4
75	5700			5712.3	2.2E-3
76	5852			5821.4	5.2E-3
77	6006			6072.6	1.1E-2
78	6162			6123.5	6.2E-3

Because of the importance of this second question—if satisfying (3.1) is really all that important for numerical efficiency, then one might as well use spherical harmonics and consign Fourier series to the trash heap for jury problems—it is instructive to examine this point from a slightly different angle which was not explicitly considered in Orszag (1974).

Table 2 lists the lowest 20 nonzero Fourier coefficients for the lowest $(n=49)$ associated Legendre function, normalized so that its maximum value over the interval is one. Since the Fourier basis functions satisfy the simple bound

$$|\theta_n(\phi)|\leqslant 1 \text{ for } \phi\in[0,\pi] \text{ for all } n \text{ and } m, \tag{3.6}$$

Table 2 lists the lowest 20 nonzero Fourier coeffi-

TABLE 2. The first 20 non-zero Fourier components for the lowest $m=49$ spherical harmonics.

n	u_n	n	u_n
0	0.115	20	0.00343
2	−0.220	22	−0.00137
4	0.195	24	0.000495
6	−0.159	26	−0.000161
8	0.119	28	0.0000464
10	−0.0820	30	−0.0000119
12	0.0519	32	0.00000268
14	−0.0302	34	−0.000000523
16	0.0160	36	0.0000000872
18	−0.00777	38	−0.0000000122

[3] For convenience, Chebyshev basis functions which have a first order zero at each pole were used. This is optional; it is never necessary to impose any constraints on Chebyshev polynomials for any value of m, not even a "parity" factor.

it follows that the maximum absolute value of the error over the interval (the error in the so-called "L_∞" norm) that results from truncating the expansion after a finite number of terms is bounded by the sum of the absolute values of all the neglected coefficients. Note how rapidly the coefficients decrease once the series begins to converge: this is the practical meaning of the "infinite order" convergence property discussed earlier. It is easy to see that truncating the expansion with just seven terms will give a maximum error of only 5%, despite the huge value of the zonal wavenumber. Clearly, matching the exact behavior of solutions on a sphere near the poles for large m is just not very important in obtaining accurate numerical results.

In summary, two difficulties that reduce the usefulness of modified Fourier series for time-dependent problems—severe time-stepping restrictions and the need to impose additional polar constraints such as (3.3)—do not arise when solving time-independent jury problems via the Fourier pseudospectral method. The fact that modified Fourier series only approximately satisfy the "natural" boundary conditions (3.1) which are automatically and exactly satisfied by associated Legendre functions does not significantly reduce the numerical effectiveness of the Fourier series versus the associated Legendre series, even for very large values of the zonal wavenumber.

4. Variable stretching, limited domains and Chebyshev polynomials

The great strength of Chebyshev polynomials is their ability to retain the property of infinite order convergence where Fourier or associated Legendre expansions are either inefficient or useless. The basic tool in determining the convergence order of an expansion is integration by parts. If

$$f(0)=f(\pi)=0, \tag{4.1}$$

then the conventional expression for the Fourier coefficients of the expansion of $f(x)$ on $x\in[0,\pi]$ can be integrated by parts to give

$$a_n=(2/\pi n^3)\left[\left. f^{(2)}(x)\cos(nx)\right|_0^\pi - \int_0^\pi f^{(3)}(x)\cos(nx)dx\right], \tag{4.2}$$

which implies, since the boundary term and integrand can both be bounded by constants which are independent of n, that

$$a_n \sim O(n^{-3}) \text{ as } n\rightarrow\infty. \tag{4.3}$$

If $f(x)$ is periodic so that $f^{(k)}(0)=f^{(k)}(\pi)$ for all k where the superscript denotes differentiation with respect to x, then one can integrate by parts an arbitrary number of times to show that the a_n decrease at least as fast as $O(n^{-j})$ where j is arbitrary. When the coefficients decrease faster than any finite power of $(1/n)$, the convergence is said to be "infinite order." Orszag (1974) shows that, even with the slight complication of the $\sin^s\theta$ factor, modified Fourier series on the sphere have this property, and he gives a closely related argument to prove the same for series of associated Legendre functions. Because of (2.5), the Chebyshev coefficients of $f(x)$ are identical to the Fourier cosine coefficients of $f(\cos\phi)$. If $f(x)$ is not periodic, however, the surface term in (4.2) will not vanish and (4.3), i.e., third-order convergence, is the best one can obtain from Fourier series. Since $f(\cos\phi)$ is *always* a periodic function even if $f(x)$ is not, Chebyshev—but *not* Fourier series—retain the property of infinite order convergence so long as $f(x)$ is analytic everywhere on the expansion interval.

The point of reviewing this elementary and moderately well-known analysis is that it is sometimes necessary to modify boundary and eigenvalue problems on the sphere in such a way that Fourier and Legendre series lose their infinite-order convergence. Boyd (1976, 1978b, 1978c), for example, discusses the use of variable stretching to resolve critical latitudes for planetary waves. The solution is not periodic in the transformed variable, so the Fourier coefficients of the solution would satisfy (4.3) at best. Associated Legendre functions have the same drawback. Chebyshev polynomials, however, retain rapid convergence and were quite successful for this class of problem. Planetary waves on the midlatitude beta-plane, Dickinson's (1968) "polar cap" modes on a computational domain embracing only part of a hemisphere—Chebyshev polynomials possess the flexibility to handle almost any manipulation or transformation of the problem which is convenient. Modified Fourier series and associated Legendre functions, in contrast, are useful only on a global or hemispherical domain in colatitude.[4]

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[4] For Fourier series, there are occasional exceptions to this statement. If an eigenvalue equation is in so-called normal Liouville form $[u''+q(x)u=\lambda u(x)]$, then the boundary conditions (4.1) plus the differential equation itself imply the "natural" boundary conditions $f''(0)=f''(\pi)=0$. This causes the boundary term in (4.2) to vanish, and two integrations by parts demonstrate that $a_n\sim O(n^{-5})$. Birkhoff and Fix (1970) show that Fourier series give results comparable to or even a little better than orthogonal polynomials for such problems unless extremely high accuracy is needed: finite-order convergence is not always a great evil if the order of convergence is high enough. The planetary wave equation of Boyd (1978b), however, is not in Liouville normal form and cannot be transformed into it without introducing singularities, so one has no choice but Cheybyshev polynomials.

It was noted in Section 2, however, that a modified *Fourier* series in colatitude is in fact a modified *Chebyshev* series in $x=\cos\phi$. This fact now assumes paramount importance because it implies that if one shifts and stretches x—not ϕ—one can still use modified Fourier series on a non-global, non-hemispherical domain or with variable stretching to resolve a critical latitude *without* sacrificing the property of infinite order convergence. In such applications, however, it is best to think of these expansion func-

TABLE 3. A comparison of the spherical harmonic, Fourier and Chebyshev expansion coefficients for the lowest symmetric gravitational mode of the diurnal tide.*

n	Chebyshev	Fourier	Spherical harmonic
0	−0.0750	−0.00394	0.282
2	0.173	−0.203	−0.638
4	−0.223	0.382	0.621
6	0.204	0.273	−0.336
8	−0.147	0.110	0.117
10	0.0879	−0.0293	−0.0283
12	−0.0456	0.00588	0.00504
14	0.210	0.000786	0.000686
16	−0.00874	0.0000871	0.000074
18	0.00332	0.000008	0.000006
20	−0.00117		
22	0.000383		
24	−0.000118		
26	0.000034		
28	0.000009		
30	0.000002		

* The spherical harmonics are normalized with respect to integration over the surface of the sphere. The Chebyshev coefficients are for Chebyshev basis functions which vanish at the poles.

Unfortunately, Chebyshev polynomials in latitude are significantly less efficient than the other two expansions for ordinary problems on the sphere (as I shall show through an example), so their use is likely to be confined to beta-plane geometry or situations where ease of programming is paramount. The theory of the diurnal tide is given in Chapman and Lindzen (1970); here, it will to suffice to note that the sun-following tidal modes are the $m=1$ eigenfunctions of a second-order differential operator in ϕ. This example has been chosen because the modes fall into two classes: gravitational modes which have most of their amplitude equatorward of $\pm 30°$ latitude, and rotational modes which have most of their amplitude poleward of these latitudes. The Chebyshev polynomials have their collocation points concentrated near the poles, so one would expect that the Chebyshev polynomials would perform better on the rotational modes than on the gravitational. Tables 3 and 4, which compare the expansion coefficients for different series for a gravitational and a rotational mode, respectively, show that this is indeed the case: 11 polynomials give the same accuracy for the high-latitude mode as 16 polynomials for the tropical mode. Alas, nine and ten trigonometric functions, respectively, give the same accuracy. There is no significant difference between the Fourier series (modified Chebyshev polynomials in x) and associated Legendre expansions, but both are markedly better than Chebyshev polynomials in φ, even though the latter also share the property of infinite-order convergence.

tions as Chebyshev polynomials multiplied by $(1-x^2)^{s/2}$ rather than as trigonometric functions. It is in this sense that the remark "modified Fourier series are useful only on a global or hemispherical domain" is true.

As an alternative, one can use Chebyshev polynomials in latitude rather than x as done by Boyd (1976). The great advantage of this is that Chebyshev polynomials in latitude do not need parity factors or boundary constraints of any kind for any zonal wavenumber (including $m=0$). It is unnecessary, in contrast to Cartesian geometry (Boyd, 1978a), to combine the polynomials into basis functions that vanish at the sidewalls. When the variable is stretched, the absence of the "parity factor" greatly simplifies programming.

In summary, Chebyshev polynomials in latitude are not recommended for solving general problems on the sphere. When variable stretching is employed, when the computational domain is not global or hemispherical, or when one wishes to change from a spherical to a beta-plane geometry, modified Chebyshev polynomials in x or (less efficiently on the sphere) Chebyshev polynomials in φ are the only practical options. It is because of these special situations that these functions cannot be omitted from any thorough discussion of the choice of basis sets on the sphere.

5. Conclusion

As shown in Section 2, the spectral and pseudospectral methods are easy to apply for all three types of expansions, but in relative terms, associated Legendre functions are significantly harder to program and manipulate than the other two types of series. Nonetheless, it may be desirable in some special situations to use the spherical harmonics spectral method anyway. An example is Laplace's tidal equation for fixed frequency. The Fourier pseudospectral approach would generate a dense matrix which would require $O(N^3)$ operations for its solution, but because of the simplicity of Laplace's tidal equation and the fact that the spherical harmonics are the exact eigenfunctions of the two-dimensional Laplacian operator on the surface of a sphere, the associated Legendre spectral method yields a symmetric, tridiagonal ordinary algebraic eigenvalue problem which can be solved in only $O(N^2)$ operations. Unfortunately, among the

TABLE 4. A comparison of the spherical harmonic, Fourier and Chebyshev expansion coefficients for the second lowest symmetric rotational mode of the diurnal tide.

n	Chebyshev	Fourier	Spherical harmonic
0	0.347	1.135	−0.270
2	−0.369	2.308	0.470
4	−0.342	1.469	0.771
6	0.0867	0.483	0.326
8	0.0506	0.0917	0.069
10	−0.0158	0.0112	0.00884
12	−0.00230	0.000950	0.000773
14	0.00144	0.000059	0.000049
16	−0.000086	0.000003	0.000002
18	0.000058	0.000000	0.000000
20	0.000013		
22	0.000000		

examples quoted in the Introduction, this happy situation is exceptional. For the two-layer baroclinic instability study of Moura and Stone (1976), for example, one cannot exploit the sparsity of the matrices generated by using spherical harmonics because the resulting system of equations is a generalized rather than an ordinary algebraic eigenvalue problem which does not necessarily have only real eigenvalues. Since so much previous theoretical and observational work has been cast in terms of associated Legendre functions, it still may be useful to employ them anyway to facilitate comparison with past analyses.

Ultimately, the choice of algorithm can only be made in the context of a specific problem, but the analysis above gives some clear guidelines. Chebyshev polynomials are the jack-of-all-trades: useful when the latitudinal variable, the geometry or the domain are changed. Spherical harmonics, though almost universally employed in the past, are not one basis set but rather a different set for each zonal wavenumber. In consequence, the collocation points and the functions themselves are nontrivial to calculate (though software is readily available) and the computational bookkeeping is more complicated. Fourier series, however, have the algorithmic simplicity of Chebyshev polynomials and the efficiency of spherical harmonics where efficiency is judged by accuracy for a given number of retained terms. Therefore, in the absence of special situations which favor other types of series such as those discussed just above and in the previous section, the pseudospectral, modified Fourier series method is the best choice for general problems on the sphere. It is preferable to spherical harmonics because it is easier to program; it is preferable to Chebyshev polynomials in because it is more efficient.

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