On the Use of Autoregressive-Moving Average Processes to Model Meteorological Time Series

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ABSTRACT

Statistical problems that may be encountered in fitting autoregressive-moving average (ARMA) processes to meteorological time series are described. Techniques that lead to an increased likelihood of choosing the most appropriate ARMA process to model the data at hand are emphasized. One specific meteorological application of ARMA processes, the modeling of Palmer Drought Index time series for climatic divisions of the United States is considered in detail. It is shown that low-order purely autoregressive processes adequately fit these data.

1. Introduction

Autoregressive-moving average (ARMA) models of meteorological time series have become popular. Examples include monthly precipitation (Delleur and Kavvas, 1978), annual streamflow (Carlson *et al.*, 1970), and a monthly drought index (Davis and Rapporport, 1974). ARMA models have become popular since they include a very wide range of stochastic processes that, because of their mathematical structure, are quite useful in forecasting and since they provide a convenient statistical representation of the data in terms of a small number of parameters.

Models of the ARMA type also are useful because of our concept of climate and the ways in which meteorological observations are manipulated in studies of climate. At time scales, ranging from months and seasons to ice ages, we think of climate as a persistent phenomenon. We may think of dry periods and wet periods, of cold periods and warm periods, of periods of zonal flow and periods of meridional flow. The short-period fluctuations are smoothed conceptually from our consideration. When dealing with meteorological observations, for purposes of climatological analysis, the data routinely are smoothed or averaged in time or space to remove the noise and, often, in the process are made relatively more persistent. Further, some meteorological data have an inherent persistence which may affect climatological averages on at least a seasonal basis (Madden, 1977). ARMA models are well suited to the analysis and forecasting of time series that are by nature or by manipulation persistent and, thus, are especially useful in climatological analysis.

Box and Jenkins (1976) are primarily responsible for making readily accessible the necessary statistical methodology for applying ARMA models to real data and for espousing the use of these models in forecasting. While ARMA processes have many advantages over other somewhat similar processes, their application to model meteorological data may require an increased degree of mathematical sophistication on the part of the researcher. Several potential pitfalls can arise when the uninitiated user attempts to apply these processes. Consequently, we believe that a real need exists to outline the statistical problems that may be encountered in fitting ARMA processes to meteorological time series. The main purpose of this paper is to describe these difficulties, along with indications of how they may either be avoided or taken into account through additional analysis.

The discussion will concentrate on one specific meteorological application of ARMA processes, the modeling of Palmer Drought Index for climatic divisions of the United States. However, the statistical problems that will be described are very fundamental and could occur in a wide variety of applications. We note, in particular, that Chander *et al.* (1979) have raised some of these issues in their comments on the Delleur and Kavvas paper.

In Section 2, ARMA processes are defined and their general properties are outlined, with particular emphasis on procedures for choosing the appropriate

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form of ARMA process for a given set of data. The modeling of the Palmer Drought Index climatic division time series is dealt with in Section 3. Finally, Section 4 consists of some concluding remarks.

2. ARMA processes

In this section we review some basic properties of ARMA processes required as background information for any applications. As much as possible, the approach of Box and Jenkins (1976) will be followed. It is assumed, for now, that we are dealing with a stationary stochastic process. The issue of how nonstationarity should be handled will be addressed later in this section (see Section 2d).

a. Definition

It is assumed that we are given a stationary stochastic process $\{Y_t: t = 1, 2, \cdots\}$ with mean μ. This Y_t process constitutes a mixed autoregressive (order p) moving average (order q) process, denoted by ARMA(p,q), if it can be expressed in the form:

$$Y_t - \mu = \sum_{i=1}^{p} \phi_i(Y_{t-i} - \mu) + a_t - \sum_{i=1}^{q} \theta_i a_{t-i}, \quad p, q \geq 0. \quad (1)$$

Here it is assumed that the sequence of a_t's is a white noise process, that is, consisting of uncorrelated random variables each with mean zero and variance $\sigma_a{}^2$. Usually, the additional assumption that the a_t's have a Gaussian (or normal) distribution (see Section 2d) is also necessary. Constraints on the parameters specified in (1) are required in order to ensure that certain properties, including stationarity, hold for the Y_t process.

Given the orders p and q, the autoregressive parameters $\phi_1, \ldots, \phi_p$ and the moving average parameters $\theta_1, \ldots, \theta_q$ (along with μ and $\sigma_a{}^2$) must be estimated from a sequence of observations of the Y_t process. This estimation problem is nonlinear (when both p and q are greater than zero) and several approaches exist for which computer software has been implemented (e.g., Box and Jenkins, 1976; Jones, 1980). But an even more fundamental statistical problem for applications is the determination of the appropriate orders p and q.

One special case of the general ARMA(p, q) process is the autoregressive process of order p, obtained by taking q to be zero in (1) and denoted more simply as AR(p). For such processes, parameter estimation is a linear problem and, consequently, much more straightforward than with general ARMA processes. Purely autoregressive processes have found numerous meteorological applications, especially in the case where $p = 1$ (e.g., Leith, 1973). The AR(1) process is commonly referred to as a Markov or red noise process.

b. Properties

In this section we assume that the orders p and q are given, presumably having been already selected on the basis of some statistical criterion (Section 2c will deal with the problem of order determination) and describe some characteristics of ARMA(p, q) processes. Constraints must be placed on the possible values of the parameters specified in (1) for the Y_t process to be stationary as assumed. This stationarity assumption affects only the autoregressive parameters $\phi_1, \ldots, \phi_p$; specifically, the roots (as a function of x) of

$$1 - \sum_{i=1}^{p} \phi_k x^k = 0 \quad (2)$$

must lie outside the unit circle. For example, when $p = 1$, stationary is ensured by requiring $|\phi_1| < 1$.

Besides stationarity, the ARMA process should be invertible. Many users of ARMA processes are apparently unaware of the need for this requirement or its implications. An ARMA process is said to be *invertible* if it can be represented as an infinite order AR process, expressing the current observation of the process as an infinite weighted sum of previous observations, i.e.,

$$Y_t - \mu = \sum_{i=1}^{\infty} \pi_i(Y_{t-i} - \mu) + a_t. \quad (3)$$

Here the π_i's are functions of the original parameters of the ARMA model. Invertibility constrains only the moving average parameters $\theta_1, \ldots, \theta_q$; specifically, the roots of

$$1 - \sum_{i=1}^{q} \theta_k x^k = 0 \quad (4)$$

must lie outside the unit circle. For example, when $q = 1$, invertibility is ensured by requiring $|\theta_1| < 1$. The Appendix gives the formulas required for the inversion of an ARMA(2, 2) process, a model which is examined in Section 3.

Restricting consideration to only invertible ARMA processes ensures that the resultant model for the given meteorological time series, at least potentially, will be both physically meaningful and useful for forecasting. Otherwise, the current observation of the process may depend, not only on the previous observations as expressed by (3), but on future observations as well.

If any ARMA process can be represented as an infinite-order AR process and since, in practice, the π_i weights in (3) are very near zero after the first few terms, the question of why consider moving average terms at all must be answered. The most

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compelling reason is parsimony in model building, i.e., to model the data with a process which requires the estimation of as few parameters as possible, while still adequately fitting the data. Parsimony is advantageous because simpler models often have more straightforward physical interpretation and because the consideration of more parameters than necessary may lead to unstable parameter estimates and less accurate forecasts. For instance, an ARMA(1, 1) process requires an estimate of only two parameters (other than the mean and variance), while its equivalent representation in terms of an AR process might require considerably more than two parameters.

If ARMA processes involving moving average terms are to be accepted as plausible models for meteorological data, mechanisms by which these terms might arise are of interest. Several such mechanisms have been proposed. In particular, moving average terms arise when an AR(p) process has observational error. The resulting process is ARMA(p, p) (e.g., Box and Jenkins, 1976, p. 121). Boes and Salas (1978) proposed a process with shifting mean levels for modeling hydrologic data, and with an autocorrelation function that is identical to that for an ARMA(1, 1) process.

Finally, we remark that the estimation procedures commonly employed by computer software in fitting ARMA processes do not guarantee that the parameter estimates satisfy the stationarity and invertibility conditions (2) and (4). These conditions, nevertheless, can be guaranteed through the use of straightforward reparameterizations (Jones, 1980).

c. *Order determination*

A major problem in ARMA modeling is determining the order of the process. Box and Jenkins (1976) do not discuss objective procedures for order determination, and most computer software does not provide for automatic order selection. Consequently, researchers fitting ARMA processes to meteorological time series may encounter difficulty in choosing the appropriate values of p and q. Recently, several automatic procedures have appeared in the statistics literature, and we discuss in detail one of these procedures.

One procedure for choosing the orders of an ARMA process is called the Bayesian Information Criterion (BIC). Although based on a Bayesian statistical argument (Schwarz, 1978), it can still be employed in a non-Bayesian context as an automatic selection procedure. Suppose we are given n observations, $Y_1, \ldots, Y_n$, of the time series to be modeled. For trial values of p and q, an ARMA(p, q) process is fit to the data. One measure of the goodness of fit of the model is the mean squared error (or estimated variance of the residuals)

$$S^2_{p,q} = [n - (p + q + 1)]^{-1} \sum_{t=1}^{n} \hat{a}_t^2. \quad (5)$$

Here $\hat{a}_t = Y_t - \hat{Y}_t$, where $\hat{Y}_t$ is the estimated value of Y_t obtained, at least formally, by substituting the estimates of the $p + q + 1$ parameters (including the mean μ) into (1).

As mentioned in Section 2b, parsimonious models are desired. The BIC procedure achieves parsimony by invoking a penalty for the number of parameters ($p + q + 1$) required by the model. The orders p and q that minimize the quantity

$$\mathrm{BIC}(p, q) = n \log S^2_{p,q} + (p + q + 1) \log n \quad (6)$$

are selected. We note that the first term on the right-hand side of (6) is a measure of goodness of fit of the model, while the second term is a penalty function for the number of parameters required.

We mention some other procedures which have been proposed for ARMA process order selection. Akaike's Information Criterion (AIC) (e.g., Akaike, 1974) is closely related to the BIC procedure, involving only a change in the form of penalty function in (6). Ozaki (1977) has justified the application of the AIC procedure to ARMA processes, while Jones (1975) has suggested the use of this criterion in modeling meteorological time series with AR processes. The BIC technique, nevertheless, has been shown to perform well relative to the AIC technique for some meteorological applications (Katz, 1979).

Mean squared error alone (5) has been commonly employed in meteorological applications as a criterion for choosing among several potential models, for example, when performing regression analysis. Choosing the orders p and q of an ARMA process for which (5) is a minimum, however, generally leads to unnecessarily large values of p and q or overfitting. The mean squared error must be penalized [e.g., as in (6)] in order to obtain a parsimonious model.

Another procedure involves a diagnostic check of the residuals as recommended by Box and Jenkins. If the correct values of p and q are chosen, the residuals a_t should be uncorrelated. The "portmanteau lack of fit test" (Box and Jenkins, p. 290) is used to determine whether the estimated residuals $\hat{a}_t$ are significantly autocorrelated. This procedure is available with some computer software for fitting ARMA processes and has been employed in meteorological applications. Unfortunately, it is quite inadequate as an order selection criterion since it does not take into account directly how well the model fits the data.

The most conclusive approach to order selection would be to simply compare how well each of the potential models forecasts for an independent set of data (i.e., data not used in estimating the parameters of the model). Unfortunately, sufficient additional data generally are not available to provide

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an adequate forecasting test. While some data from the original set of observations intended for use in fitting the ARMA models could be saved for use in making forecasts, this reduction in sample size results in a loss in precision of the estimated parameters of the ARMA processes.

Finally, we mention a fundamental philosophical issue in model building, regarding all considerations of order determination, that should be kept in mind. Since there is no reason to expect, a priori, that meteorological observations should be generated by an ARMA process, such a model can only be viewed as an approximation. Thus, in some sense, no statistical model can be viewed as the true model or the correct choice.

d. Complications

For completeness, several other considerations in modeling meteorological time series using ARMA processes should be mentioned, even though we do not deal with these issues in this paper. Most of the statistical theory derived for ARMA processes (methods of parameter estimation, etc.) depend on the assumption that the error term a_t in (1), and hence Y_t, has a Gaussian distribution. On the other hand, it is well known that many meteorological variables have highly non-Gaussian distributions. For instance, the distribution of precipitation amounts is nearly always positively skewed.

A standard statistical technique for circumventing this difficulty is to transform the original observations, obtaining new observations having an approximate Gaussian distribution. Then ARMA processes are fit directly to the transformed data. Transformations such as the logarithm, square root and cube root commonly have been applied to meteorological variables.

Nonstationarity is another issue which is of particular interest when dealing with meteorological time series. To remove nonstationarities from the original data, Box and Jenkins (1976) suggest repeated differencing of the time series (i.e., forming the differenced series $W_t = Y_t - Y_{t-1}$) and then fitting ARMA processes to the differenced data. Computer software often includes an option for differencing the data a specified number of times.

Since many business and economic time series tend to exhibit gradually changing levels about which the observations fluctuate, differencing may be appropriate in this case. But meteorological variables do not typically exhibit such behavior and, although researchers have tried to model differenced meteorological time series, this operation is probably inappropriate for such data. Finally, we note that it may be necessary to remove regular seasonal or diurnal cycles that might be present with meteorological data.

3. Palmer Drought Index

In this section we present an illustration of the application of ARMA processes to the modeling of Palmer Drought Index (PDI) data. Individual ARMA models were fit to PDI time series for 344 climatic divisions of the conterminous United States. The discussion of these statistical results will rely on several concepts concerning ARMA processes introduced in Section 2. We will attempt to establish, in particular, that only low-order AR processes, not ARMA processes with both autoregressive and moving average terms, are required to adequately model these meteorological time series.

a. Description of data

First, we briefly review the method by which the PDI is determined. The reader is referred to Palmer (1965) for a more complete description. The index is meant to be a measure of relative drought severity. It is a relative measure in the sense that it contains adjustments for the normal climate of the region. The PDI is generally computed monthly and is based on a quantity known as the monthly moisture anomaly. Roughly speaking, this moisture anomaly value is determined by comparing the observed precipitation with a calculated precipitation value appropriate for the location and time of year. Droughts (or wet periods) are made persistent by making the index a function of the current month's moisture anomaly value and the previous month's PDI value, except at the beginning of a dry or wet period.

The specific data analyzed were obtained from the National Climatic Center (Asheville, North Carolina). They consist of monthly PDI values by climatic division for the time period, 1931–75 ($n = 540$ for each time series). The PDI values were calculated using meteorological data (i.e., monthly mean temperature and total precipitation) derived by averaging over the stations within the division.

b. Results

ARMA processes with seven different combinations of the orders p and q were fit to the 344 climatic division PDI time series, using the BIC procedure to determine the best-fitting models. The results are presented in Table 1. For nearly every division, a low-order AR process, usually AR(1) but occasionally AR(2), is selected. A process that involves a moving average parameter is chosen for only one division. If, instead of the BIC procedure, the AIC procedure is used, the orders selected are quite similar (detailed results not included in this paper). While the AIC procedure does have a tendency to choose higher order AR processes than the BIC procedure, a process that involves a moving average parameter is still rarely ever chosen.

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We note that Davis and Rappoport (1974) chose an ARMA(2, 2) process to fit the PDI time series for one particular climatic division, central Ohio. Their selection of an ARMA(2, 2) process was based largely on the outcomes of the portmanteau lack of fit test, whose inadequacy was discussed in Section 2c. Such a choice of model, involving moving average parameters, appears to conflict with our results. In particular, the BIC procedure selects an AR(1) process for this climatic division.

In an attempt to resolve this lack of agreement, the ARMA(2, 2) model obtained by Davis and Rappoport is more closely examined. The actual fitted model is

$$Y_t = (1.344)Y_{t-1} - (0.431)Y_{t-2} + a_t - (0.419)a_{t-1} + (0.034)a_{t-2}, \quad (7)$$

where $\hat{\mu} \approx 0$. Assuming the invertibility condition (4) is satisfied, (7) has an equivalent representation as an AR process (3). The ARMA process with the parameters specified in (7) is invertible and can be approximated adequately as an AR(2) process with parameters $\hat{\phi}_1 = 0.925$ and $\hat{\phi}_2 = -0.078$ [see (A5) in Appendix]. This result indicates that the central Ohio PDI data could be modeled more parsimoniously than (7) in terms of an AR process of at most order two. In particular, the moving average terms in (7) are apparently unnecessary.

We can conclude, then, that for most of the conterminous United States AR processes of low order, usually order one, are quite adequate for modeling PDI time series. This conclusion that only a smaller class of ARMA processes need be considered in modeling such data has some related implications. Besides not having to deal with the conceptual difficulties that arise in the physical interpretation of general ARMA processes, the computationally burdensome problem of the simultaneous estimation of both autoregressive and moving average parameters is avoided.

4. Concluding remarks

We have presented a critical examination of the use of ARMA processes to model meteorological time series. Several methodological techniques, that researchers who apply these processes may be unaware of, have been introduced. These techniques lead, in particular, to an increased likelihood of choosing the most appropriate ARMA process to model the data at hand, consequently resulting in a more parsimonious model-building approach.

As a meteorological application, ARMA processes were fit to drought index time series by climatic division. The implementation of a procedure that identifies the appropriate choice of ARMA process was demonstrated for these data. It was shown that low-order AR processes adequately fit such time series and that more general ARMA processes need not be considered.

TABLE 1. ARMA models for climatic division PDI time series.

Model	Frequency (relative frequency) of selection
AR(1)	308 (89.5%)
AR(2)	22 (6.4%)
ARMA(1, 1)	0 (0.0%)
AR(3)	8 (2.3%)
AR(2, 1)	1 (0.3%)
AR(1, 2)	0 (0.0%)
AR(4)	5 (1.5%)
Total	344

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APPENDIX

Inversion of ARMA(2, 2) Process

From (1), the ARMA(2, 2) process can be expressed as

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}. \quad (A1)$$

We assume that θ_1 and θ_2 satisfy the invertibility condition (4) which, for $q = 2$, reduces to

$$\theta_1 + \theta_2 < 1, \quad \theta_2 - \theta_1 < 1, \quad \text{and} \quad |\theta_2| < 1. \quad (A2)$$

Thus (A1) can be reexpressed in the form of an infinite order AR process (3). Let $\theta_1{}^*$ and $\theta_2{}^*$ be the solutions to

$$1 - \theta_1 x - \theta_2 x^2 = (1 - \theta_1{}^* x)(1 - \theta_2{}^* x). \quad (A3)$$

Since the π_i weights in (3) decrease (in absolute value) rapidly toward zero, only the first few weights need be computed. Straightforward algebra gives

$$\left.\begin{aligned}
\pi_1 &= \phi_1 - (\theta_1{}^* + \theta_2{}^*) \\
\pi_2 &= \phi_2 + \phi_1(\theta_1{}^* + \theta_2{}^*) - [(\theta_1{}^*)^2 + \theta_1{}^*\theta_2{}^* + (\theta_2{}^*)^2] \\
\pi_3 &= \phi_2(\theta_1{}^* + \theta_2{}^*) + \phi_1[(\theta_1{}^*)^2 + \theta_1{}^*\theta_2{}^* + (\theta_2{}^*)^2] \\
&\quad - [(\theta_1{}^*)^3 + \theta_1{}^*(\theta_2{}^*)^2 + (\theta_1{}^*)^2\theta_2{}^* + (\theta_2{}^*)^3]
\end{aligned}\right\} . \quad (A4)$$

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The Davis and Rappoport (1974) parameter estimates (7) are $\hat{\phi}_1 = 1.344$, $\hat{\phi}_2 = -0.431$, $\hat{\theta}_1 = 0.419$ and $\hat{\theta}_2 = -0.034$. The given values of $\hat{\theta}_1$ and $\hat{\theta}_2$ satisfy the invertibility requirements (A2), and (A3) yields $\hat{\theta}_1^* = 0.489$ and $\hat{\theta}_2^* = -0.070$. Substituting $\hat{\phi}_2$, $\hat{\phi}_2$, $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ in (A4), we obtain $\hat{\pi}_1 = 0.925$, $\hat{\pi}_2 = -0.078$, and $\hat{\pi}_3 = -0.0009$. Since $\hat{\pi}_3$ is negligible, this ARMA(2, 2) process can be adequately numerically approximated as an AR(2) process, i.e.,

$$Y_t - \mu \approx (0.925)(Y_{t-1} - \mu) - (0.078)(Y_{t-2} - \mu) + a_t. \quad \text{(A5)}$$

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