The Stability of Some Planetary Boundary Layer Diffusion Equations

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ABSTRACT

A study is made of the stability properties of certain coupled and decoupled nonlinear diffusion equations. The equations are of the form currently used in many high-resolution models to parameterize the stably stratified boundary layer. It is shown that some parameterization schemes for a decoupled thermal diffusion equation possess an unrealistic instability feature. However, the stability of the schemes is demonstrated in the more natural setting of a coupled, momentum and heat, set of diffusion equations.

1. Introduction

Diffusive processes in the atmospheric boundary layer above the so-called surface layer are often parameterized in high-resolution numerical models with a nonlinear diffusion representation. Numerous formulations of this "local exchange coefficient" genre, wherein the diffusion coefficients are related explicitly to the local values of the shear and static stability, have been proposed and tested in recent years (see, e.g., Pandolfo, 1971; Louis, 1979; Blackadar, 1979; McNider and Pielke, 1981; Keyser and Anthes, 1982).

A conclusion reached by Brown and Pandolfo (1982, hereafter referred to as BP) casts doubt upon the utility of these "local exchange coefficient" schemes. They argue that one particular (but not uncommon) finite difference representation of the nonlinear diffusion equation can suffer from a numerical instability. This particular interpretation of their results is clearly of some interest and concern.

In this note an examination is undertaken of the stability of the differential form of both coupled and decoupled nonlinear diffusion equations. The study of the decoupled, single diffusion equation described in the next section facilitates comparison with BP, mitigates the forementioned cause for concern, and directs attention to the issue of the physical acceptability of instability features associated with differential diffusion equations. This issue is then pursued further with reference to a coupled set of diffusion equations for momentum and potential temperature.

2. A decoupled diffusion equation

Consider the conventional diffusion equation,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial \theta}{\partial z} \right), \quad (1)$$

with $K$ dependent on the spatial structure of the $\theta$ field. In BP the diffusion coefficient $K$ was given the form

$$K(\theta) = K_0 \left( 1 - \frac{\gamma}{\partial z} \right), \quad (2)$$

with $K_0$ and $\gamma$ assumed to take positive constant values. This latter form was adopted as a first approximation to a more physically based representation of $K$ in terms of a Richardson number $Ri$ e.g., Eq. (25) of BP,

$$K = K_0 (1 - |\alpha| |Ri|)^2, \quad Ri \geq 0. \quad (3)$$

In this formulation $\alpha$ is the Monin-Obukhov constant and it was noted in BP [Eqs. (27), (28)] that Eq. (2) is a linear form of Eq. (3) if $Ri$ is identified solely with $\partial \theta/\partial z$. The system is decoupled in the sense that there is not a linked momentum diffusion equation.

Here we first examine the stability properties of the system of Eqs. (1), (2). Steady-state solutions satisfy the relation

$$K_0 (1 - \gamma S) S = \text{constant},$$

where $S = \partial \theta/\partial z$ in the steady state. It follows that $S$ itself is a constant.

Now consider small perturbations of the potential temperature field ($\theta'$) away from such an equilibrium state. The linearized equation governing the behavior of the perturbation potential temperature ($\theta'$) is

$$\frac{\partial \theta'}{\partial t} = (1 - 2\gamma S) \frac{\partial^2 \theta'}{\partial z^2}. \quad (4)$$
Multiplying Eq. (4) by $\theta'$ and integrating by parts over a spatial segment $(0, d)$, we obtain
\[
\frac{\partial}{\partial t} \int_0^d \frac{1}{2} \theta'^2 \, dz = K_0(1 - 2\gamma S) \oint \frac{\partial^2 \theta'}{\partial z^2} \, dz \int_0^d \frac{\partial \theta'}{\partial z} \, dz. \tag{5}
\]

The boundary conditions at $z = (0, d)$ require consistency with the specification of the unperturbed field. Thus, if either $\theta$ or $\partial \theta / \partial z$ is specified at the boundary points then $\theta'$ or $\partial \theta' / \partial z$ will vanish at the end points. Then it follows from Eq. (5) that the amplitude of the potential temperature perturbations will grow (i.e., the basic state will be unstable) if
\[
K_0(1 - 2\gamma S) < 0. \tag{6}
\]

Apart from notational differences this instability criterion, derived here for the differential system (1), (2), is identical to that derived in BP for the related differential-difference system. This equivalence is a strong indicator that the source of the instability is essentially related to the continuous differential system rather than a numerical feature.

It is natural to inquire whether this instability is a physically acceptable feature of the differential system. The energy source for the growth of a perturbation away from an equilibrium steady-state field is the basic state potential temperature field itself, and the factor that permits this growth is the particular formulation of the diffusion coefficient $K$. However, the specification of $K$ is, in essence, an attempt to parameterize the effects of turbulence in the boundary layer as a process that diffuses the mean temperature field toward an equilibrium state at a rate that is related to the value of the diffusion coefficient $K$. If the formulation of $K$ in Eq. (2) is either an attempt to specify a desired state $S = \gamma^{-1}$ or to model a decrease in the diffusivity as $S \rightarrow \gamma^{-1}$, then the existence of this instability feature runs counter to the underlying philosophy.

In the next section we reconsider this issue in the more natural setting of a coupled momentum and potential temperature diffusion system. Here it is helpful to simply note in passing that:

1) A physical interpretation of the instability in terms of a negative value of the "effective" diffusion coefficient, $K_0(1 - 2\gamma S)$, is evident on examination of Eqs. (4), (6). (This interpretation was considered and rejected in BP, but essentially on the basis of the behavior of perturbations about a state of zero, rather than nonzero, potential temperature gradient.)

2) It is perhaps worth emphasizing specifically here that, with this nonlinear specification of the diffusion, there is an instability feature that is related to negative values, not of $K$ itself, but of the "effective" diffusion coefficient. In particular, this instability can occur for values of $S$ for which $K$ itself is positive, i.e., the domain $[1 - 2\gamma S, 1 - S]$. To underline this point we observe that a diffusion coefficient resembling Eq. (3) with $K$ always positive definite, viz.,
\[
K = K_0 \left(1 - \gamma \frac{\partial \theta}{\partial z}\right)^2, \tag{7}
\]
also possesses an instability feature.

3) A diffusion coefficient that exhibits a similar functional form to Eqs. (2) and (7) in the limit of $|\gamma \partial \theta / \partial z| < 1$, viz.,
\[
K = K_0 \left(1 + \gamma \frac{\partial \theta}{\partial z}\right)^{-1}, \tag{8}
\]
is stable for all values of the equilibrium potential temperature gradient $S$.

4) Simple differential-difference systems based upon Eqs. (1), (2), and (3), (8) can be constructed whose instability features replicate those of the corresponding differential system. This latter result underlines the non-numerical source of the instability. It can be shown as follows. Consider the governing equations to be discretised in space to yield

**SYSTEM A:**
\[
\frac{\partial \theta_j}{\partial t} = K_0(1 - \gamma \theta_j) \theta_j, \tag{9a}
\]

**SYSTEM B:**
\[
\frac{\partial \theta_j}{\partial t} = K_0(1 + \gamma \theta_j^{-1}) \theta_j, \tag{9b}
\]

where
\[
\theta_j = (\theta_{j+1/2} - \theta_{j-1/2})/(\Delta z),
\]
the internal grid points are defined by $z = j(\Delta z)$, with $j = (0, 1)$ only, and the domain boundary is located at $z = -1/2(\Delta z), 1/2(\Delta z)$. On prescribing a constant flux $F(\Delta z)$ at the boundaries the resulting equations for $\theta_0$ and $\theta_1$ can be combined to yield the following sets:

**SYSTEM A:**
\[
\begin{align*}
\frac{\partial a}{\partial t} &= 2 F - 2K' a(1 - \gamma a) \\
\frac{\partial b}{\partial t} &= 0
\end{align*} \tag{10a}
\]

**SYSTEM B:**
\[
\begin{align*}
\frac{\partial a}{\partial t} &= 2 F - 2K' a(1 + \gamma a)^{-1} \\
\frac{\partial b}{\partial t} &= 0
\end{align*} \tag{10b}
\]

where
\[
a = \theta_1 - \theta_0, \quad b = \theta_1 + \theta_0, \quad K' = K_0(\Delta z)^2, \quad \gamma' = \gamma/(\Delta z).
Now examining the stability of small perturbations \((a')\) about a basic state \(\overline{a} (\neq 0)\), then the resulting linearized equations are

**SYSTEM A:**

\[
\frac{\partial a'}{\partial t} = -2K'(1 - 2\gamma\overline{a})a',
\]

(11a)

**SYSTEM B:**

\[
\frac{\partial a'}{\partial t} = -2K'(1 + \gamma\overline{a})^{-2}a'.
\]

(11b)

It follows directly from Eqs. (11a), (11b) that the differential-difference System A is unstable for \((1 - 2\gamma\overline{a}) < 0\), while System B is unconditionally stable. These results parallel those associated with the parent differential systems of Eqs. (1), (2) and (1), (8). In particular, the results for System B indicate that a formulation of the diffusion coefficient, that is not physically unreasonable, can be constructed that is stable for both differential and differential-difference systems.

3. Coupled diffusion equations

We now consider the system

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial z} \left\{ K_M \frac{\partial u}{\partial z} \right\},
\]

(12a)

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left\{ K_H \frac{\partial \theta}{\partial z} \right\},
\]

(12b)

where \(u\) and \(\theta\) denote respectively the horizontal velocity and potential temperature, and \((K_M, K_H)\) the associated diffusion coefficients that are dependent upon the spatial structure of the \(u\) and \(\theta\) fields. Two particular formulations of these coefficients will be studied. The first, due to Blackadar (1979) and utilized by McNider and Pielke (1981) and Keyser and Anthes (1982), is of the form

\[
K_{M,H} = \left\{ \begin{array}{ll}
\alpha \left( 1 - \frac{\text{Ri}}{\text{Ric}} \right) \frac{\partial u}{\partial z}, & 0 < \text{Ri} < \text{Ric} \\
0, & \text{Ri} > \text{Ric},
\end{array} \right.
\]

(13)

where \(\alpha = \alpha^*l^2\), and \(\alpha^*\) is a constant and \(l\) the turbulent mixing-length scale. Here \(\text{Ric}\) is the critical Richardson number and it is argued that \(l\) is effectively a constant for \(z \approx 200\) m. In our analysis we shall assume \(\alpha\) to be constant. The boundary layer is assumed to be stable and non-turbulent if \(\text{Ri} > \text{Ric}\); while the turbulence is assumed to induce a diffusivity effect proportional to the product of the shear and the depression of \(\text{Ri}\) below \(\text{Ric}\) for \(0 < \text{Ri} < \text{Ric}\). For this system there are two types of equilibrium state for \(\text{Ri} \leq \text{Ric}\). These are characterized either by constancy of shear and potential temperature gradient, or with \(\text{Ri} = \text{Ric}\) throughout the flow domain. Clearly the latter state is conceived to be a desirable, marginally non-turbulent equilibrium state, and this state should be stable to small perturbations. The appropriate criterion for the former state is not so clear-cut. It could be argued that instability of this state does not necessarily constitute an unacceptable feature since it would imply an adjustment toward the alternative type of equilibrium state. However, this suggestion is unsatisfactory since a special case of the first state is \(\text{Ri} = \text{Ric} = \text{constant}\). Again linear hydrodynamic stability theory suggests that an uniformly stably stratified, constant shear flow is stable in the inviscid limit. Thus we tentatively conclude that the former state should also be one of stability.

We now proceed with a stability analysis of system (12), (13) for the first type of equilibrium state. The linearized equations governing the behavior of small perturbations of the system away from a basic state of uniform shear, \(\Lambda\), and uniform stratification \(N^2 = (g/\theta)/(\partial \theta / \partial z)\), are

\[
\frac{\partial u'}{\partial t} = \frac{\partial}{\partial z} \left\{ \alpha |\Lambda| \left[ 2 \frac{\partial u'}{\partial z} - (\text{Ric}\Lambda)^{-1} \frac{\partial \theta^*}{\partial z} \right] \right\},
\]

(14a)

\[
\frac{\partial \theta^*}{\partial t} = \frac{\partial}{\partial z} \left\{ \alpha |\Lambda| N^2 \left[ \left( 1 + \frac{N^2}{\text{Ric}\Lambda^2} \right)^{-1} \frac{\partial u'}{\partial z} \right. \right. \\
\left. \left. + \left( 1 - \frac{2N^2}{\text{Ric}\Lambda^2} \right)^{-1} \frac{\partial \theta^*}{\partial z} \right] \right\},
\]

(14b)

where \(\theta^* = \gamma\theta^*/\theta\). Eliminating \(u'\) in favor of \(\theta^*\) and then seeking solutions of the form

\[
\theta^* \propto \exp(\sigma t + iz),
\]

the following equation can be derived for the growth rate \(\sigma\):

\[
\sigma = -\alpha |\Lambda| n^2 [1 \pm (1 - Q/p^2)^{1/2}],
\]

(15)

where

\[
\begin{align*}
p &= \frac{3}{2} (1 - \frac{2}{3} \text{Ric}/\text{Ric}) \\
Q &= (2 - \text{Ric}/\text{Ric})(1 - \text{Ric}/\text{Ric}) \\
\text{Ri} &= N^2/\Lambda^2
\end{align*}
\]

The growth rate \(\sigma\) will be positive, and the system unstable, if either \((p < 0)\) or \((p > 0)\) and \(Q < 0\). However, for \(0 < \text{Ri} < \text{Ric}\), these conditions are not met and the system is stable. The decay rate of the perturbations is proportional to the shear and the square of the wavenumber \((n)\) of the perturbation. A disconcerting feature is that the system exhibits a discontinuous response to perturbations in the neighborhood of \(\text{Ri} = \text{Ric}\). There is finite damping of perturbations for \(\text{Ri} = \text{Ric}\) and zero damping for \(\text{Ri} = \text{Ric}\). A practical (but perhaps minor) consequence of this feature, noted by one reviewer, is that this scheme might produce different results with different computing systems. This effect would arise because different word sizes, or slightly different mathematical
functions, would influence the space and time occurrence of $\text{Ri} = \text{Ri}_c$, and hence of the “on-off” diffusion mechanism.

The second formulation of the diffusion coefficients $K_{M,H}$ that we consider is of the form

$$K_{M,H} = \alpha(1 + b \text{Ri})^{-m} \left| \frac{\partial u}{\partial z} \right| .$$

For $m = 1$, this is an extension of the form considered in the previous section, while for $m = 2$ it is equivalent to that proposed by Louis (1979). In this case the growth rate of perturbations satisfies the equation

$$[\sigma + \alpha |\Delta| n^2 (1 + b \text{Ri})^{-m}]$$

$$\times [\sigma + \alpha |\Delta| n^2 (2 + mb \text{Ri})(1 + b \text{Ri})^{-m}] = 0.$$ 

It follows that in this case the system is stable for all values of the basic state $\text{Ri}$, and now the decay rate is proportional to the modulus of the shear, the square of the wavenumber of the perturbation, and the factor $(1 + b \text{Ri})^{-m}$.

The “stability” result obtained in this section for coupled diffusion equations is at variance with the mild instability observed by BP in numerical computations with a boundary layer flow model that included such a coupling. There are several possible explanations for this discrepancy. They include the round-off error effect at $\text{Ri} = \text{Ri}_c$ referred to earlier or the occurrence of some other form of numerical instability (associated possibly with the finite-difference form of the diffusion terms or with some other component of the model). It is not possible to pinpoint the reason for the discrepancy here, but it is perhaps significant to note that “round-off” variations constituted the trigger for the variations observed in the experiments of BP.

4. **Concluding remarks**

The analysis of the stability properties of the differential form of certain nonlinear boundary layer diffusion equations has formed the crux of this study. The desirability of examining the stability of such systems was advocated in BP. There is a postulate that is implicit in our discussion, viz., a requisite feature of diffusive parameterization schemes is that they should exhibit stability to small perturbations about an equilibrium state.

The study showed that for a decoupled thermal equation the stability is dependent upon the particular formulation of the coefficient of diffusion. This particular instability, contrary to the inference in BP, is essentially non-numerical in nature. The response of a coupled momentum and thermal diffusion systems is different. In particular, it is shown that the differential form of two recently proposed parameterization schemes (Blackadar, 1979; Louis, 1979) are both unconditionally stable to perturbations of a uniformly stratified and uniformly sheared flow.

**REFERENCES**


