NOTES AND CORRESPONDENCE

Spatial Smoothing on the Sphere

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ABSTRACT

The equivalence of taking an isotropic, moving, spatial average of a two-dimensional field on the sphere to multiplying the coefficients in its spherical harmonics representation with factors that depend only on the total wavenumber n is discussed. Equivalent spatial averaging operators for several such spectral filters are displayed.

1. Introduction

The spatial smoothing of global meteorological fields can be accomplished by using filters in either the physical or the spectral domain. With the advent of spectral methods in numerical weather prediction, spectral filtering has become popular because of its computational efficiency and its ease of application. For any function $f(\lambda, \theta)$ of longitude λ and colatitude θ (equal to 90-latitude in degrees), expressed as a sum of orthonormal spherical harmonics (defined in Jackson, 1975; Courant and Hilbert, 1953; Haltiner and Williams, 1980),

$$f(\lambda, \theta) = \sum_n \sum_m f_n^m Y_n^m(\lambda, \theta). \qquad (1)$$

This is done by determining the spectral coefficients f_n^m and reconstructing the sum

$$\bar{f}(\lambda, \theta) = \sum_n \sum_m S_n^m f_n^m Y_n^m(\lambda, \theta), \qquad (2)$$

where S_n^m is a specified, and usually simple, function of the zonal wavenumber m and total wavenumber n. Two trivial examples of such filters are

$$\text{triangular: } S_n^m = \begin{cases} 1 \text{ for } n \leq M \\ 0 \text{ otherwise,} \end{cases}$$

$$\text{rhomboidal: } S_n^m = \begin{cases} 1 \text{ for } |m| \leq M, \quad n - |m| \leq J \\ 0 \text{ otherwise.} \end{cases}$$

What one is doing in the physical domain with these filters is, however, not always obvious. If S_n^m is chosen to be a function of m alone, smoothing is clearly implied in the zonal direction only. On the other hand, S_n^m depending only on n intuitively suggests a more isotropic smoothing, although precisely of what form is not immediately apparent. One wonders if in this case $\bar{f}$ may be expressed as a weighted average of f over the neighborhood of every point (λ_0, θ_0), with an isotropic weighting function w that depends only on the distance from (λ_0, θ_0):

$$\bar{f}(\lambda_0, \theta_0) = \int_0^{2\pi} d\lambda' \int_{-1}^{1} d(\cos\theta') w(\theta') f(\lambda', \theta'), \qquad (3)$$

where the primes refer to quantities in a rotated frame $(\lambda, \theta) \to (\lambda', \theta')$ in which the point $P_0 = (\lambda_0, \theta_0)$ is the north pole. This note is concerned with showing that this does indeed hold rigorously, i.e., we can prove the following theorem:

For every spectral filter $S_n^m \equiv S_n$, there exists a weighting function $w(\theta)$ such that

$$\begin{aligned}\bar{f}(\lambda_0, \theta_0) &= \sum_n \sum_m S_n f_n^m Y_n^m(\lambda_0, \theta_0) \\ &= \int_0^{2\pi} d\lambda' \int_{-1}^{1} d(\cos\theta') w(\theta') f(\lambda', \theta')\end{aligned} \qquad (4)$$

and vice versa, where

$$w(\theta) = \sum_n w_n^0 Y_n^0(\theta) = \sum_n \left(\frac{2n+1}{4\pi}\right)^{1/2} S_n Y_n^0(\theta).$$

In addition to providing a physical interpretation for some of the commonly used filters, the theorem could perhaps prove useful in designing spectral filters with desired spatial characteristics on the sphere.

2. Proof

To determine the smoothed value $\bar{f}$ at a point $P_0 = (\lambda_0, \theta_0)$, it is convenient to rotate coordinate axes so that P_0 is the north pole in the rotated frame.

Then

$$\bar{f}(\lambda_0, \theta_0) = \bar{f}'(0, 0)$$
$$= \int d\lambda' \int d(\cos\theta') w(\theta') f(\lambda', \theta')$$
$$= \int d\lambda' \int d(\cos\theta') \{\sum_{n'} w_{n'}^0 Y_{n'}^0(\theta')\} \times \{\sum_n \sum_m f_n'^m Y_n^m(\lambda', \theta')\}$$
$$= \sum_{n'} \sum_n \sum_m w_{n'}^0 f_n'^m \iint d\lambda' d(\cos\theta') \times Y_{n'}^0(\theta') Y_n^m(\lambda', \theta')$$
$$= \sum_{n'} \sum_n \sum_m w_{n'}^0 f_n'^m \delta_{m0}\delta_{nn'}$$
$$= \sum_n w_n^0 f_n'^0, \quad (5)$$

where the orthogonal property of the normalized spherical harmonics (Jackson, 1975),

$$\iint d\lambda' d(\cos\theta') Y_{n'}^{*m'} Y_n^m = \begin{cases} 1 \text{ for } (m', n') = (m, n) \\ 0 \text{ for } (m', n') \neq (m, n), \end{cases}$$

has been used. The problem thus reduces to relating the $f_n'^0$ in the primed (rotated) frame to the f_n^m in the original frame. To this end, we first consider how the Y_n^m themselves transform under a rotation $\mathcal{R}(\lambda_0, \theta_0)$ which carries the north pole over to the point P_0. Remembering that $\mathcal{R}$ leaves the total wavenumber n unaltered (since $\mathcal{R}$ and ∇^2 commute and thus have a complete set of simultaneous eigenfunctions), we may write (Courant and Hilbert, 1953, Chap. VII, Section 7)

$$Y_n^m(\lambda', \theta') = \mathcal{R} Y_n^m(\lambda, \theta) = \sum_{m'} D_{m'm}^n(\lambda_0, \theta_0) Y_n^{m'}(\lambda, \theta), \quad (6)$$

where $\mathbf{D}_{m'm}^n$ is a unitary matrix of order $2n + 1$, i.e., one whose inverse $\mathbf{D}^{-1}$ is given by $(\mathbf{D}_{mm'}^n)^*$. We now consider $\hat{f}$, the spectrally smoothed value of f at P_0:

$$\hat{f} = \sum_{m,n} S_n f_n^m Y_n^m(\lambda_0, \theta_0)$$
$$= \sum_n S_n \sum_m f_n^m Y_n^m(\lambda_0, \theta_0)$$
$$= \sum_n S_n \sum_m f_n'^m Y_n^m(0', 0') \quad \text{(using the unitary of } \mathbf{D})$$
$$= \sum_n S_n f_n'^0 Y_n^0(0, 0).$$

Since $Y_n^0(0, 0) = P_n^0(1) = [(2n + 1)/(4\pi)]^{1/2} P_n(1) = [(2n + 1)/(4\pi)]^{1/2}$, where P_n and P_n^0 refer to the Legendre and associated Legendre polynomials of the first kind divided by $\sqrt{4\pi}$ respectively, this becomes

$$\hat{f} = \sum_n S_n \left(\frac{2n + 1}{4\pi}\right)^{1/2} f_n'^0. \quad (7)$$

Comparing with (5), we have $\bar{f} = \hat{f}$ if

$$w_n^0 = S_n \left(\frac{2n + 1}{4\pi}\right)^{1/2},$$

thus completing the proof.

3. Examples and discussion

The isotropic weight function w and the spectral filter S_n thus form a Legendre-transform pair

$$\left.\begin{aligned} w(\mu) &= \sum_{n=0}^{\infty} \left(\frac{2n + 1}{4\pi}\right) S_n P_n(\mu) \\ S_n &= 2\pi \int_{-1}^{1} w(\mu) P_n(\mu) d\mu \end{aligned}\right\}, \quad (8)$$

where $\mu = \cos\theta$. It should be stressed that only spectral filters S_n whose properties are determined completely by the total wavenumber n are equivalent to isotropic spatial averaging operators. This is not true of the rhomboidal-truncation filter mentioned in the Introduction, which could be viewed as one of the less appealing features of this type of truncation. The weight functions $w(\theta)$ for triangular truncations at M = 6, 12, 18, 24, 30 and 36 denoted by T6, T12, T18, T24, T30 and T36 respectively are shown in Fig. 1. The abscissa in the figure may be interpreted alternatively as distance $a\theta$ from P_0, the point at which the average is sought, where a is the radius of the earth. The value of w at, say, $\theta = 45°$ is thus the weight that one would give to values of the function f at points that are $\pi a/4 \approx 5000$ km away from P_0. It is apparent from the plots that the first zero of w occurs roughly at an angle of $180/M$ deg for each M, the smallest resolvable scale in a TM truncation. Note that w does not approach zero smoothly for any of the truncations shown but continues to oscillate with diminishing amplitude; however, the amplitude of the first minimum is about the same in the six cases, approaching the value −0.133 for higher M, which is by no means negligible. This Gibbs phenomenon can cause spurious highs and lows to appear in a field with sharp local gradients, such as the earth's topography, when represented by a truncated series of spherical harmonics (see Hoskins, 1980). To reduce these, Hoskins suggested using the filter $S_n = \exp\{-K[n(n + 1)]^2\}$ with $S_M = 0.1$. Figure 2 shows the w for this filter when $M = 24$. The Gibbs

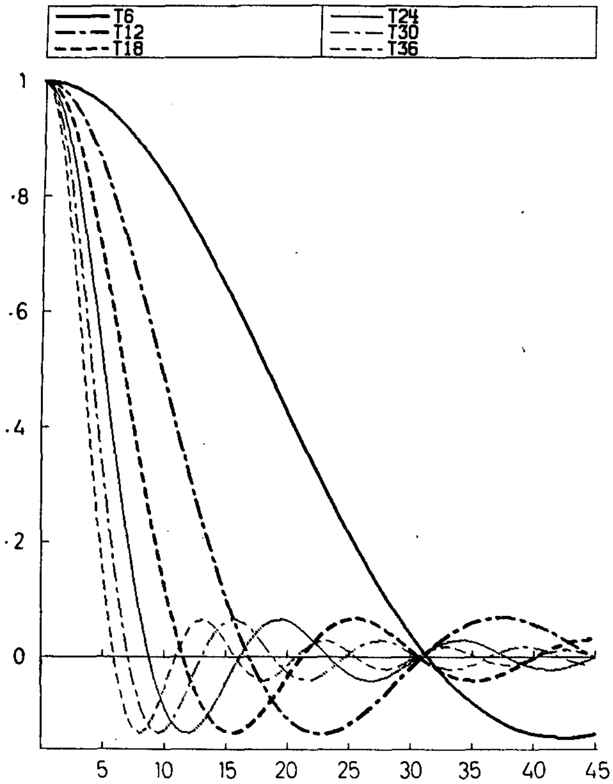

FIG. 1. Weight functions of the equivalent spatial averaging operators for several triangular truncations. The abscissa denotes angle in degrees. The graphs have been normalized with respect to their value at the origin.

phenomenon, although not completely eliminated, is much reduced compared to that of T24, also shown in the figure.

One may consider other filters of the form

$$S_n = \exp - \left(\frac{n(n+1)}{n_0(n_0+1)}\right)^r, \quad n \leqslant M, \tag{9}$$

to see if the Gibbs phenomenon could be further reduced. The Hoskins filter of Fig. 2 is of this form with $r = 2$ and $n_0 = 19.4$. This type of spectral filtering has the attractive feature of being equivalent to applying a ∇^{2r} diffusion over a certain interval of time. To see this, we consider the effect of a ∇^{2r}-type diffusion on the spectral coefficients of f:

$$\frac{\partial f}{\partial t} = -\nu(-1)^r \nabla^{2r} f,$$

where ν is a diffusion coefficient with dimensions of $m^{2r}\, s^{-1}$. We may write this as

$$\frac{\partial}{\partial t} \sum_{m,n} f_n^m Y_n^m = -\nu(-1)^r \sum_{m,n} f_n^m \nabla^{2r} Y_n^m,$$

$$\sum_{m,n} \left(\frac{df_n^m}{dt}\right) Y_n^m = \sum_{m,n} - \nu(-1)^r f_n^m (-1)^r \left\{\frac{n(n+1)}{a^2}\right\}^r Y_n^m,$$

where a is the radius of the earth.

From the linear independence of spherical harmonics, this becomes

$$\frac{df_n^m}{dt} = -\nu \left\{\frac{n(n+1)}{a^2}\right\}^r f_n^m;$$

i.e.,

$$f_n^m(t) = f_n^m(0) \exp[-\nu\{n(n+1)/a^2\}^r t].$$

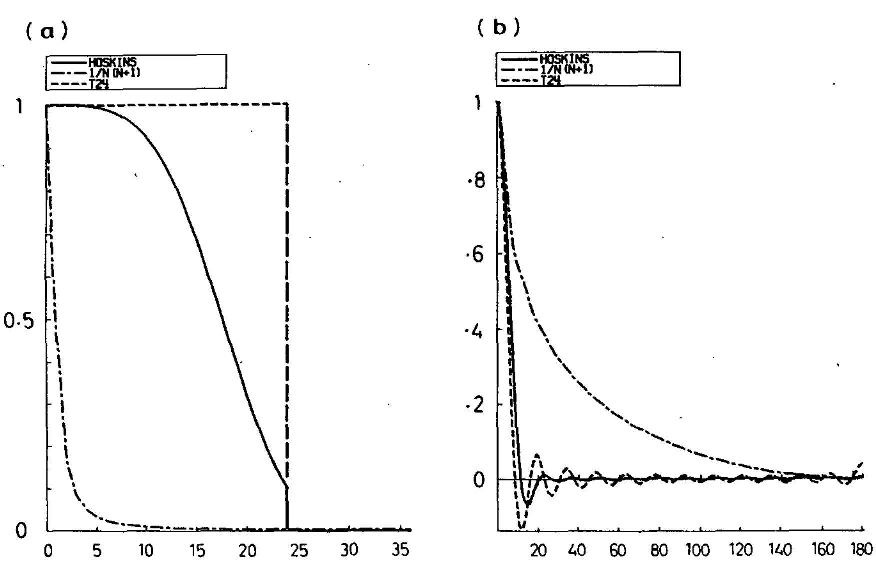

FIG. 2. Spectral filters (a) and their corresponding normalized weight functions (b) referred to in the text. The abscissa denotes total wavenumber n in (a); angle in degrees in (b).

After a time interval of

$$\tau = \frac{1}{\nu}\left[\frac{a^2}{n_0(n_0+1)}\right]^r,$$

we would have

$$f_n^m(\tau) = f_n^m(0)\exp\left\{-\left[\frac{n(n+1)}{n_0(n_0+1)}\right]^r\right\} = f_n^m(0)S_n.$$

Thus $r = 1, 2, 3$ would cover the commonly desired ∇^2, ∇^4, and ∇^6 cases of scale selectivity. Figure 3 shows the weight functions for these for $n_0 = 10$, 15 and 18 and $M = 24$ and 36. There is clearly a trade-off between scale selectivity and the Gibbs phenomenon. For higher values of r, the scale selectivity in S_n is more sharply defined, but the Gibbs phenomenon is worse, although it cannot become any worse than that shown in Fig. 1 because S_n tends to triangular truncation at $M = n_0$ in the limit of $r \to \infty$. For averaging purposes, however, it may sometimes be desirable to use weight functions that decay smoothly to zero, and therefore it might be better to use an $r = 1$ filter which has no ripples in w except when n_0 is quite close to M.

Thus, by choosing just the two parameters r and n_0 in (9), filters with a fairly wide range of behavior in both spectral and physical space may be generated and the Gibbs phenomenon brought under some

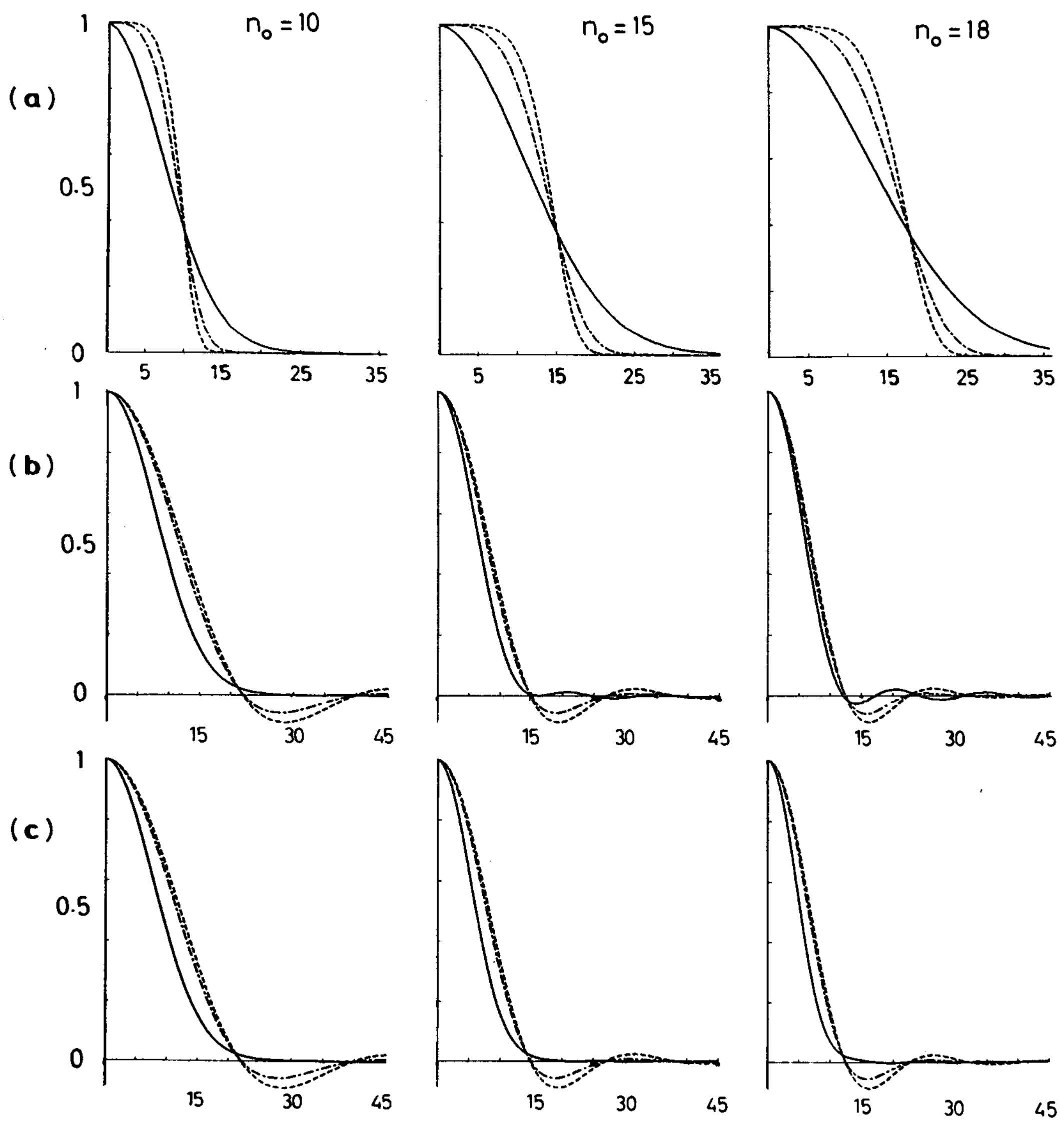

FIG. 3. (a) Plots of filters defined in (9) for n = 10, 15 and 18 and for r = 1 (solid line), 2 (dot–dashed line) and 3 (dashed line). The corresponding normalized weight functions are shown for (b) M = 24 and (c) M = 36. The abscissa denotes total wavenumber n in (a); angle in degrees in (b) and (c).

control. There are, of course, various other methods of controlling this phenomenon, notably those developed by Lanczos (reviewed by Duchon, 1979). The extension of Lanczos filtering to the two-dimensional spherical domain appears, however, to be neither straightforward nor elegant, in spite of the reduction (8) of the filtering problem to a one-dimensional one. This complexity is primarily caused by the cumbersome analytical expressions for the Legendre transforms of even the simplest functions.

Having ripples in w is, again, not necessarily bad. There are cases when one may desire w, for example, to have a Bessel-function form (Thompson, 1956) to reduce spurious small-scale noise in f. The equivalent

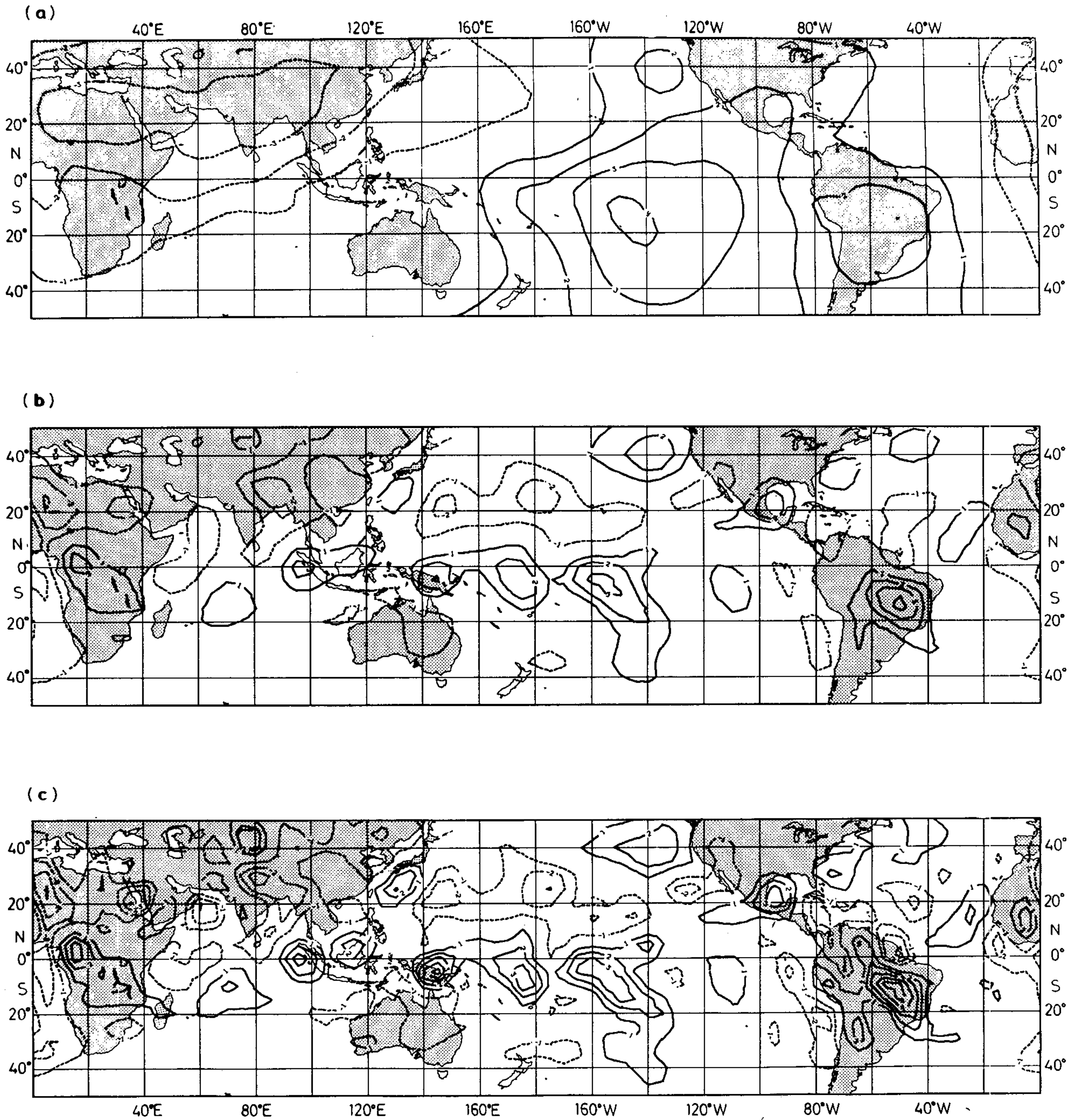

FIG. 4. The effect of smoothing the mean horizontal divergence of the wind at 150 mb during December 1982–February 1983 with the three filters shown in Fig. 2. (a) $1/[n(n+1)]$; (b) Hoskins; (c) T24. Negative contours are dashed and the zero contour is not shown. The contour interval is 5×10^{-8} s^{-1} in (a); 10^{-6} s^{-1} in (b) and (c). Note that the field in (a) is proportional to the velocity potential and that the contour interval is 1/20 of that in (b) and (c).

spectral filters S_n may then be readily computed from (8).

Finally, it may be pointed out that while some broadening and reduction in amplitude of the extrema of f is inevitable as a result of smoothing, it is possible to avoid shifting the positions of the extrema by using weight functions that do not have long tails, i.e., those that decay smoothly and rapidly to zero beyond a certain distance. A commonly used spectral filter for which this is not true is

$$S_n = 1/[n(n+1)], \quad n \leq M; \quad S_0 = 1,$$

also known as the ∇^{-2} filter because if

$$f = \nabla^2\psi, \quad \text{then} \quad \bar{f} = -\psi/a^2,$$

where ∇^2 is the two-dimensional Laplacian on the sphere. The filter and its corresponding weight function are shown in Fig. 2. The filter retains only the planetary scales, and thus loses much of the spatial variance of any field with significant smaller scales in it, such as the horizontal divergence $\nabla \cdot \mathbf{v}$ of the wind.

When applied to this field, it frequently not only attenuates the extrema strongly (see Fig. 4) but also shifts their positions, destroying much of the information contained in the original field. However, while $\bar{f}$ obtained with this filter may be a poor representation of f, it is an accurate representation of $\nabla^{-2}f$, and is therefore still capable of useful interpretation. Thus, for example, if f is a vorticity source, $\bar{f}$ may be interpreted as a streamfunction source.

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